

Utility functions and portfolio selection problem.

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1 The preference switching utility functions

There were used a number of methods in the theory of portfolio selection. Some of them were based on expected value, another on Markowitz model and finally, there is an approach based on utility functions. Using the utility function in the portfolio selection problem, the optimal portfolio has the maximal expected utility. The utility functions are very useful for modeling the investor's behavior, e.g. risk aversion (or seeking). On the other side it can be difficult to solve the portfolio selection problem for some types of utility functions.

Definition 1:

A function $u : I \rightarrow R$ is called utility function if u is continuous and non-decreasing in the interval $I \subseteq R$.

The basic analysis of utility functions of Arrow and Pratt offer an intuitive way of looking at absolute and relative risk aversion coefficients. The Arrow-Pratt coefficient of absolute risk aversion, also called absolute risk averse (ARA) function, is defined as

$$r(x) = -\frac{u''(x)}{u'(x)} \quad (1)$$

for $x \in I$ and increasing, twice differentiable utility function u in I .

We assume that investor (decision maker) has utility function u and initial wealth x . Let ε be a gamble with distribution P . The investor is called risk averse at wealth level x if:

$$Eu(x + \varepsilon) < u(x + E\varepsilon).$$

It is easily seen that $r(x) > 0$ for every risk averse investor at wealth level x whenever $u(x)$ is twice differentiable utility function (see Ingersoll [2] for more details). According to Pratt [6], a value $\pi(x, P)$ satisfying

$$u(x + E\varepsilon - \pi(x, P)) = Eu(x + \varepsilon),$$

is called a risk premium. We consider only the situations where $Eu(x + \varepsilon)$ exists and is finite. The risk averse decision maker would be indifferent between receiving a risk ε and receiving the non-random amount $E\varepsilon - \pi(x, P)$.

Let us consider $\pi(x, P)$ for a risk ε with small variance σ_ε^2 . Then can be proved the approximation (see Pratt [6]):

$$\pi(x, P) \approx \frac{1}{2}\sigma_\varepsilon^2 r(x + E\varepsilon). \quad (2)$$

According to computation of (2) it is clear that ARA function is a measure of investor's local risk aversion.

The Rubinstein's risk aversion measure is an example of measure of global risk aversion and is defined for normal distributed risk $\varrho'x$ as:

$$r_g(x_0) = -\frac{x_0 E[u''(w)]}{E[u'(w)]} \quad (3)$$

where $w = x_0 \varrho'x$. See Kallberg, Ziemba [3] for more details. This measure is less tractable and less familiar then the ARA function. Expect for a few special cases, $r_g(x_0)$ does not have a simple form, but on the other side investors with the same Rubinstein's risk aversion measure have the same optimal portfolios (see Kallberg, Ziemba [3] for proof or more details). The characterization of utility functions by the measure of risk aversion is suitable only for increasing, twice differentiable functions in I . Another approach is based on preference switching between the two gambles and generally we need not to assume that the utility functions are increasing and twice differentiable. We will go into this topic in the rest of this section.

Definition 2:

Assume that $I \subseteq R$ is an interval and $u: I \rightarrow R$. Then $u(x)$ is a n -switch function (with notation $u_n^a(x)$) if for any two gambles ε, ω can be I divided to at most $n + 1$ disjoint, open intervals I_i , $i = 1, 2, \dots, n + 1$ satisfying two following conditions:

$$(i) \quad \bigcup_{i=1}^{n+1} \bar{I}_i = I$$

where \bar{I}_i is the closure of the interval I_i ,

(ii) The following formula holds for every $i = 1, 2, \dots, n$ and $x_i \in I_i$:

$$\{Eu(x_{i+1} + \varepsilon) - Eu(x_{i+1} + \omega)\} \cdot \{Eu(x_i + \varepsilon) - Eu(x_i + \omega)\} < 0 \quad (4)$$

whenever all expected values exist.

This definition means that n -switch function has the property that, as initial wealth increases, we switch preferences between any two gambles at most n times. It is clear that n -switch functions are also $(n+k)$ -switch functions for $k \in \mathbb{N}$. There exist also functions which has the property that for every n exist two gambles such that we switch preferences between this two gambles at least n times. Such functions will be called infinite-switch functions. In general, of course, the n -switch function needn't be continuous and nondecreasing in I . The suffix a in the notation of n -switch function denotes additive approach for the final wealth. The multiplicative approach will be denoted by the suffix m . We have defined the n -switch functions only for additive approach where the final wealth is in the form: $x + \varepsilon$. Using the forms εx , $x\omega$ instead of $x + \varepsilon$, $x + \omega$ in definition 2, we obtain the n -switch function in multiplicative approach with the notation $u_n^m(x)$. From now on we call it additive n -switch functions and multiplicative n -switch functions. The special type of the n -switch functions is zero-switch function.

Definition 3:

Let $I \subseteq \mathbb{R}$ is an interval and $u: I \rightarrow \mathbb{R}$. If any two gambles ε, ω satisfy at least one of the following conditions:

$$(i) \quad Eu(x + \varepsilon) \leq Eu(x + \omega) \quad \text{for all } x \in I$$

$$(ii) \quad Eu(x + \varepsilon) \geq Eu(x + \omega) \quad \text{for all } x \in I$$

then $u(x)$ is called additive zero-switch function with the notation $u_0^a(x)$, whenever all expected value exist.

By analogy, if any two gambles ε, ω satisfy at least one of the following conditions:

$$(i) \quad Eu(x\varepsilon) \leq Eu(x\omega) \quad \text{for all } x \in I$$

$$(ii) \quad Eu(x\varepsilon) \geq Eu(x\omega) \quad \text{for all } x \in I$$

then $u(x)$ is called multiplicative zero-switch function denoted by $u_0^m(x)$, whenever all expected value exist.

Example 1:

Let us analyze the utility function: $u(x) = -e^{-x}$ and specify the number of preference switching in additive approach. We can consider $I = R$ for this function. Let ε, ω are any two gambles. There is no loss of generality in assuming:

$$Eu(\varepsilon) \leq Eu(\omega)$$

for all $x \in R$. It is easily seen that:

$$\begin{aligned} Eu(\varepsilon) \leq Eu(\omega) &\Leftrightarrow Ee^{-\varepsilon} \geq Ee^{-\omega} \Leftrightarrow -e^{-x}Ee^{-\varepsilon} \leq -e^{-x}Ee^{-\omega} \\ &\Leftrightarrow Eu(x + \varepsilon) \leq Eu(x + \omega) \end{aligned}$$

for all $x \in R$. Therefore $u(x) = -e^{-x}$ is additive zero-switch function. Suppose that $\varepsilon = \pm 1$ with the same probability and $\omega = \pm 2$ with the same probability. We can see the expected values $Eu(x + \varepsilon)$ (green colour) and $Eu(x + \omega)$ (red colour) in Fig. 1.

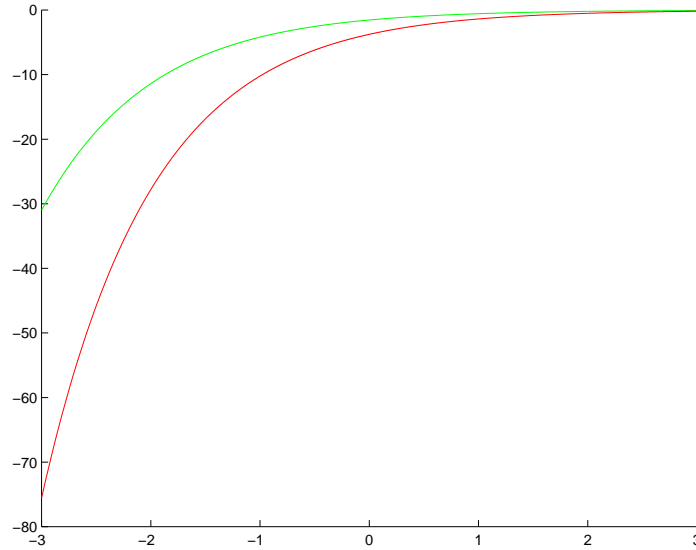


Figure 1: Expected values for exponential utility function.

We follow Pedersen, Satchel[4] in assuming that $I = R$ for additive n-switch function. In the class of infinitely differentiable functions we will find only two examples of additive zero-switch utility functions. If $u_0^a(x)$ is infinitely differentiable and ε, ω are any distinct gambles with suitably small spread then by Taylor's formula we may write:

$$Eu(x + \varepsilon) - Eu(x + \omega) = \sum_{k=1}^{\infty} m_k u^{(k)}(x), \quad \text{where } m_k = \frac{E(\varepsilon^k) - E(\omega^k)}{k!}$$

and $u^{(k)}(x)$ is the k th derivative of $u(x)$. We choose the gambles ε, ω such that $m_k = 0$ for all natural $k > 2$. Let a_1, a_2 satisfy the conditions: $a_1 a_2 < 0$, $a_1, a_2 \in R$ then the system of equations

$$\begin{aligned} m_1 u'(x_1) + m_2 u''(x_1) &= a_1 \\ m_1 u'(x_2) + m_2 u''(x_2) &= a_2 \end{aligned}$$

cannot hold for any x_1, x_2 . Failing which, there exist preference switching between ε and ω therefore $u(x)$ cannot be the additive zero-switch function. Thus the necessary condition for a additive zero-switch function to satisfy is that the first two derivatives be collinear i.e. for some coefficient c we have:

$$u''(x) + cu'(x) = 0 \tag{5}$$

for all $x \in R$. The differential equation (5) has two solutions:

$$u_0^a(x) = ae^{cx} + b \quad ac \geq 0, b \in R, \tag{6}$$

$$u_0^a(x) = ax + b \quad a \geq 0, b \in R. \tag{7}$$

such that $u_0^a(x)$ is also the utility function. Summarizing, we have the only two functions which can be the additive zero-switch utility functions. It is easily seen that linear function (7) is additive zero-switch utility function. It is also easy to check that exponential function (6) is utility function and similarly to example 1 we will prove that (6) has no preference switching:

$$\begin{aligned} Eu(\varepsilon) < Eu(\omega) &\Leftrightarrow Eae^{c\varepsilon} < Eae^{c\omega} \Leftrightarrow e^{cx} Eae^{c\varepsilon} < e^{cx} Eae^{c\omega} \\ &\Leftrightarrow Eu(x + \varepsilon) < Eu(x + \omega) \quad \text{for all } x \in R. \end{aligned}$$

We have thus proved the following result.

Theorem 1:

The only zero-switch infinitely differentiable utility functions are:

$$\begin{aligned} u_0^a(x) &= ae^{cx} + b \quad ac \geq 0, b \in R, \\ u_0^a(x) &= ax + b \quad a \geq 0, b \in R. \end{aligned}$$

We recall that zero-switch utility functions are feasible for a decision maker which prefer one game to another independent of wealth level.

In the same manner we can evolve the form of the infinitely differentiable additive one-switch utility functions. Analysis similar to that in the proof of Theorem 1 (see Bell [1]) shows that a necessary condition for an additive one-switch function to satisfy is that the first three derivatives be collinear. What is left is to find the function satisfying the differential equation:

$$u'''(x) + ku''(x) + lu'(x) = 0 \quad (8)$$

for some coefficients k, l . It is easy to check that the solution of (8) is:

$$(i) \quad \text{for } k = 0, l = 0 : u_1^a(x) = ax^2 + bx + c \quad (9)$$

$$(ii) \quad \text{for } k = 0, l \neq 0 : u_1^a(x) = (ax + b)e^{cx} + d \quad (10)$$

$$(iii) \quad \text{for } k \neq 0, l = 0 : u_1^a(x) = ax + be^{cx} + d \quad (11)$$

$$(iv) \quad \text{for } k \neq 0, l \neq 0 : u_1^a(x) = ae^{bx} + ce^{dx} + f \quad (12)$$

where $a, b, c, d, f \in R$. In general, the solution of (8) could involve imaginary coefficients and the functions which alternates sign indefinitely. These can easily be dismissed in the context of utility functions. Summarizing, we have the only functions which can be the additive one-switch functions. It remains to show that all the functions (9)-(12) have at most one preference switching.

(i) If $u(x) = ax^2 + bx + c$ and ε, ω are any gambles, then

$$\begin{aligned} Eu(x + \varepsilon) &= ax^2 + 2axE\varepsilon + aE\varepsilon^2 + bx + bE\varepsilon + c \\ Eu(x + \omega) &= ax^2 + 2axE\omega + aE\omega^2 + bx + bE\omega + c \\ Eu(x + \varepsilon) - Eu(x + \omega) &= (2ax + b)E(\varepsilon - \omega) + aE(\varepsilon^2 - \omega^2), \end{aligned}$$

so the difference of expected utility is linear or constant function for any two gambles, therefore preference between these gambles will switch at most once and (9) is an additive one-switch function.

(ii) If $u(x) = (ax+b)e^{cx} + d$ and ε, ω are any gambles let $\alpha = E(\varepsilon e^{c\varepsilon} - \omega e^{c\omega})$, $\beta = E(e^{c\varepsilon} - e^{c\omega})$ then

$$Eu(x + \varepsilon) - Eu(x + \omega) = \beta(ax + b)e^{cx} + \alpha ae^{cx}.$$

It has at most one solution $x = -\frac{\alpha a + \beta b}{a\beta}$. Therefore, (10) is an additive one-switch function.

(iii) If $u(x) = ax + be^{cx} + d$ and ε, ω are any gambles let $\alpha = E(\varepsilon - \omega)$, $\beta = E(e^{c\varepsilon} - e^{c\omega})$ then

$$Eu(x + \varepsilon) - Eu(x + \omega) = a\alpha + b\beta e^{cx},$$

which is a monotone function. Therefore, (11) is an additive one-switch function.

(iv) If $u(x) = ae^{bx} + ce^{dx} + f$ and ε, ω are any gambles let $\alpha = E(e^{b\varepsilon} - e^{b\omega})$, $\beta = E(e^{d\varepsilon} - e^{d\omega})$ then

$$Eu(x + \varepsilon) - Eu(x + \omega) = \alpha ae^{bx} + \beta ce^{dx} = e^{bx}(a\alpha + \beta ce^{(d-b)x}).$$

This equation has at most one solution $x = \frac{1}{d-b} \log \left(\frac{-a\alpha}{\beta c} \right)$. Therefore, (12) is an additive one-switch function.

We have thus proved the following theorem.

Theorem 2:

Infinitely differentiable function satisfies the additive one-switch rule if and only if it belongs to one of the following families:

$$\begin{aligned} u_1^a(x) &= ax^2 + bx + c \\ u_1^a(x) &= (ax + b)e^{cx} + d \\ u_1^a(x) &= ax + be^{cx} + d \\ u_1^a(x) &= ae^{bx} + ce^{dx} + f \end{aligned}$$

where a, b, c, d, f are some real parameters.

It is not at all clear wheather one-switching is a natural property for an investor. The additive one-switch function can be used in this situation: we prefer lottery tickets to traded funds when poor but most likely reverse this preference as we become richer.

From the analyses of utility function based on ARA measure we can choose the utility functions with positive and decreasing ARA measure. They are suitable for risk averse decision maker such that, as his wealth increases, his risk aversion decreases. For more details we refer the reader to Pratt [6]. We will follow Bell [1] in assuming that decision maker:

- (a) prefers more money to less,
- (b) wishes to obey the axioms of expected utility,
- (c) is risk averse at all wealth levels,
- (d) is decreasingly risk averse at all wealth levels,
- (e) will approach risk neutrality for small gambles when extremely rich.

The assumption (e) show the natural property that extremely rich investor has very small ARA measure for the small risk. An investor is risk neutral at wealth level x if $u(x + E\varepsilon) = Eu(x + \varepsilon)$ for all gambles ε . We may write the assumption (e) also as

$$(e') \quad \lim_{x \rightarrow \infty} r(x) = 0.$$

For a deeper discussion of these assumptions we refer the reader to Pratt [6].

Theorem 3:

Let assumptions (a)-(e) hold. Then the only feasible utility function which is infinitely differentiable everywhere, is

$$u_1^a(x) = ax + be^{cx}$$

for positive parameter a and negative parameters b, c .

Proof: according to Bell [1]

We will analysed all four additive one-switch functions from Theorem 2. The following table contains its ARA measures and derivatives of it.

$u_1^a(x)$	$r(x)$	$r'(x)$
$ax^2 + bx + c$	$-\frac{2a}{2ax+b}$	$\frac{4a^2}{(2ax+b)^2}$
$(ax + b)e^{cx} + d$	$-c - \frac{ac}{acx+bc+a}$	$\frac{(ac)^2}{(acx+bc+a)^2}$
$ax + be^{cx} + d$	$-\frac{bc^2e^{cx}}{a+bce^{cx}}$	$-\frac{abc^3e^{2cx}}{(a+bce^{cx})^2}$
$ae^{bx} + ce^{dx} + f$	$-\frac{ab^2e^{bx}+cd^2e^{dx}}{abe^{bx}+cde^{dx}}$	$-\frac{e^{bx+dx}abcd(b-d)^2}{(abe^{bx}+cde^{dx})^2}$

Table 1: ARA measures of additive one-switch functions.

The first two functions in Table 1 do not have decreasing ARA measure for any choice of parameters. The sumex functions (12) are increasing, risk averse and decreasingly risk averse so long as each of a, b, c, d is negative but

$$\lim_{x \rightarrow \infty} r(x) = \min(-b, -d) \neq 0,$$

which contradicts the assumption (e).

If $u(x) = ax + be^{cx} + d$ then $r' < 0$ whenever $abc > 0$. For $u' > 0$ we require $a > 0$ and $bc > 0$. For $r > 0$ we require $b < 0$. Hence the conditions $a > 0$, $b < 0$, $c < 0$ are necessary and sufficient for $ax + be^{cx}$ to satisfy the conditions of the proposition. Q.E.D

We proceed to consider the HARA functions, defined by

$$r(x) = \frac{1}{ax + b},$$

where $r(x)$ is ARA measure of HARA functions. According to Pedersen, Satchel [4], it is easy to show that HARA functions are additive zero-switch functions whenever $a = 0$ and multiplikative zero-switch functions whenever $b = 0$.

From now on we leave the assumption of differentiability. The following theorem give us an example of additive zero-switch utility function which is not infinitely differentiable everywhere.

Theorem 4:

Utility function:

$$\begin{aligned} u(x) &= \lambda x & x > x_h \\ u(x) &= \lambda x - (x_h - x)^\alpha & x \leq x_h \end{aligned}$$

for positive λ , $1 < \alpha < 2$, is an additive zero-switch utility function.

Proof: according to Pedersen, Satchel [4]

We consider two gambles ε , ω and initial wealth $x \in R$. Let $\tau = x_h - x$. Then clearly

$$\begin{aligned} &Eu(x + \varepsilon) > Eu(x + \omega) \\ \Leftrightarrow &P(x + \varepsilon > x_h)E[\lambda(x + \varepsilon) \mid x + \varepsilon > x_h] \\ &+ P(x + \varepsilon \leq x_h)E[\lambda(x + \varepsilon) - (x_h - x + \varepsilon)^\alpha \mid x + \varepsilon \leq x_h] \\ &> P(x + \omega > x_h)E[\lambda(x + \omega) \mid x + \omega > x_h] \\ &+ P(x + \omega \leq x_h)E[\lambda(x + \omega) - (x_h - x + \omega)^\alpha \mid x + \omega \leq x_h] \\ \Leftrightarrow &\lambda E\varepsilon - P(\varepsilon \leq \tau)E[(\tau - \varepsilon)^\alpha \mid \varepsilon \leq \tau] > \lambda E\omega - P(\omega \leq \tau)E[(\tau - \omega)^\alpha \mid \omega \leq \tau] \end{aligned}$$

and this is independent of x . Hence, preferences can not switch as wealth increases, and the proof is complete. Q.E.D

This utility function is feasible for a decision maker with initial wealth $x < x_h$ such that if his wealth level after the gamble is higher then x_h he

reach his goal, so he can be risk neutral, else he must pay a penalty, so he can be risk averse. We turn back to the function:

$$u_1^a(x) = ax - be^{-cx} \quad (13)$$

for positive a, b, c , which can be used for an investor with at most one preference switching. It can be difficult to solve the maximazing expected utility problem using (13) so this is its disadvantage. Pflug, Swietanowski [5] considered the piecewise linear function:

$$u(x) = \begin{cases} a - (1 - v)(a - x) & x \leq a \\ x & a < x < b \\ b + v(x - b) & b \leq x \end{cases} \quad (14)$$

for $0 < v < 1$ and it is simpler to find the solution of this problem using (14). It is clear that if investor's wealth level is between a and b then his utility is equal to his wealth and each dollar over b increases his utility by v . Respectively, each dollar below a decreases his utility by $1 + v$. The value a can be some safe limit i.e. the decision maker's wealth level after the gamble must be at least a and if not he must pay the penalty. The value b can be the decision maker's goal and if his wealth level is higher then he must pay more taxes or has some other costs regarding it (legislative limit). We follow this idea and we consider the similar function as (14):

$$\begin{aligned} u(x) &= ax + b & x > x_h & \quad x_h = \frac{d - b}{a - c} \\ &= cx + d & x \leq x_h & \quad a > c > 0, b, d \in R \end{aligned} \quad (15)$$

which can be the approximation of (13) as we can see in Fig. 2.

However, it is not possible to characterized the function (15) by ARA measure because of nondifferentiability in x_h . Only if we reduce to the case $x < x_h$ and assume only the gamble satisfying $P(x + \varepsilon < x_h) = 1$, or if we assume that decision maker's wealth level before and after the gamble is higher then x_h , then the decision maker is risk neutral. But in general, it is not feasible assumption. Also we can't use this function for risk averse decision maker with decreasing measure of risk aversion. Finally, (15) is infinite-switch utility function, so (15) is not suitable for decision maker, which switch preferences at most once. The reason of it is also in nondifferentiability in x_h . It is easily seen that the same statements hold also for (14), which was used in Pflug, Swietanowski [5].

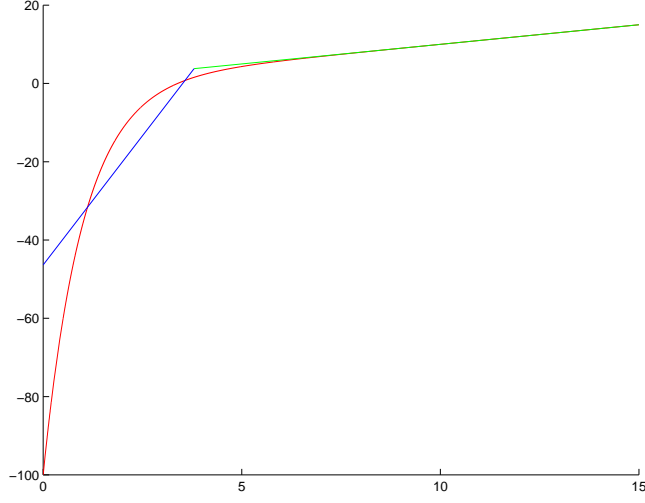


Figure 2: Approximation of additive one-switch utility function.

We will prove the infinite-switching only for (15) and for (14) it follows. Without loss of generality we will do it with the assumptions: $b = 0$, $d = 0$, $x_h = 0$, because these parameters can change only the location of (15).

Theorem 5:

The function:

$$\begin{aligned} u(x) &= bx & x \geq 0 \\ &= ax & x < 0 \end{aligned} \quad a > b > 0 \quad (16)$$

is an additive infinite-switch utility function.

Proof:

It is easily seen that $u(x)$ is utility function. Let n be any natural number. Let ε be a gamble with uniform distribution $U(0, 1)$ and ω be a gamble with discrete distribution: $P(\omega = x_k) = \frac{1}{n}$, $k = 1, 2, \dots, n$, where:

$$x_k = \frac{k - \frac{1}{2} - \frac{1}{n^\alpha}}{n}, \quad \alpha > \frac{\log \frac{8a}{a-b}}{\log(n)} + 1. \quad (17)$$

This equidistant construction of distribution and the choice of α guarantee the final estimation of

$$Eu(-x_k + \omega) - Eu(-x_k + \varepsilon).$$

Let us compute expected utility for ω in each of the intervals $\langle -x_{k+1}, -x_k \rangle$, $k = 1, \dots, n-1$.

$$\begin{aligned}
Eu(x + \omega) &= \sum_{i=1}^k \frac{a}{n}(x + x_i) + \sum_{i=k+1}^n \frac{b}{n}(x + x_i) \\
&= \sum_{i=1}^k \frac{a}{n} \left(x + \frac{i - \frac{1}{2} - \frac{1}{n^\alpha}}{n} \right) + \sum_{i=k+1}^n \frac{b}{n} \left(x + \frac{i - \frac{1}{2} - \frac{1}{n^\alpha}}{n} \right) \\
&= \frac{(a-b)k}{n}x + bx + \frac{(a-b)k^2}{2n^2} + \frac{b}{2} - \frac{b}{n^{\alpha+1}} - \frac{(a-b)k}{n^{\alpha+2}}
\end{aligned}$$

If $x < -x_n$ then

$$\begin{aligned}
Eu(x + \omega) &= \sum_{i=1}^n \frac{a}{n}(x + x_i) \\
&= ax + a \left(\frac{n+1}{2n} - \frac{1}{2n} - \frac{1}{n^{\alpha+1}} \right) \\
&= ax + a \left(\frac{1}{2} - \frac{1}{n^{\alpha+1}} \right).
\end{aligned}$$

If $x > -x_1$ then

$$\begin{aligned}
Eu(x + \omega) &= \sum_{i=1}^n \frac{b}{n}(x + x_i) \\
&= bx + b \left(\frac{n+1}{2n} - \frac{1}{2n} - \frac{1}{n^{\alpha+1}} \right) \\
&= bx + b \left(\frac{1}{2} - \frac{1}{n^{\alpha+1}} \right).
\end{aligned}$$

Similarly, let us compute it for the gamble ε .

$$\begin{aligned}
Eu(x + \varepsilon) = \int_0^1 u(x + \varepsilon) d\varepsilon &= ax + \frac{a}{2} && \text{when } x \leq -1 \\
&= bx + \frac{b}{2} && \text{when } x \geq 0 \\
&= \int_0^{-x} a(x + \varepsilon) d\varepsilon + \int_{-x}^1 b(x + \varepsilon) d\varepsilon \\
&= \frac{b-a}{2}x^2 + bx + \frac{b}{2} && \text{when } x \in (-1, 0).
\end{aligned}$$

Combining these computations we obtain

$$Eu(x + \varepsilon) > Eu(x + \omega) \text{ where } x < -1 \text{ or } x > 0. \quad (18)$$

We proceed to show that $Eu(x_k + \varepsilon) < Eu(x_k + \omega)$, where $k = 1, 2, \dots, n$, and for $x = -\frac{k}{n}$ we obtain $Eu(x + \varepsilon) > Eu(x + \omega)$. It is clear that $-\frac{k}{n} \in \langle -x_{k+1}, -x_k \rangle$ for any $k = 1, 2, \dots, n-1$.

$$\begin{aligned}
& Eu(-x_k + \omega) - Eu(-x_k + \varepsilon) \\
&= \frac{(a-b)k}{n} \left(-\frac{k - \frac{1}{2} - \frac{1}{n^\alpha}}{n} \right) + \frac{(a-b)k^2}{2n^2} - \frac{b}{n^{\alpha+1}} - \frac{(a-b)k}{n^{\alpha+2}} \\
&\quad + \frac{a-b}{2} \left(-\frac{k - \frac{1}{2} - \frac{1}{n^\alpha}}{n} \right)^2 \\
&= -\frac{b}{n^{\alpha+1}} + \frac{a-b}{8n^2} + \frac{a-b}{2n^{2\alpha+1}} - \frac{(a-b)k}{n^{\alpha+2}} + \frac{a-b}{2n^{\alpha+1}} \\
&= \frac{1}{n^{\alpha+1}} \left(-b + \frac{a-b}{8}n^{\alpha-1} + \frac{a-b}{2n^\alpha} - \frac{(a-b)k}{n} + \frac{a-b}{2} \right) \\
&> \frac{1}{n^{\alpha+1}} \left(-a + \frac{a-b}{8}n^{\alpha-1} \right) > 0
\end{aligned}$$

using the assumption (17) for α .

$$\begin{aligned}
& Eu\left(-\frac{k}{n} + \omega\right) - Eu\left(-\frac{k}{n} + \varepsilon\right) \\
&= -\frac{(a-b)k^2}{n^2} + \frac{(a-b)k^2}{2n^2} - \frac{b}{n^{\alpha+1}} - \frac{(a-b)k}{n^{\alpha+2}} + \frac{(a-b)k^2}{2n^2} < 0
\end{aligned}$$

Summarizing, we have proved that there are switched preferences at least twice in any interval $\langle -x_{k+1}, -x_k \rangle$ and combining it with (18), we obtain at least one preference switching in each of intervals $(-\infty, -x_n)$, $(-x_1, \infty)$. Since for any n we have found such gambles ε and ω that there is at least $2n$ preference switching between these gambles, thus $u(x)$ is an additive infinite-switch utility function. Q.E.D

In Fig. 3 we can see an example of expected values $Eu(x + \varepsilon)$ (dashed black colour), $Eu(x + \omega)$ (green colour) for a special choice: $n = 3$, $a = 10$, $b = 0.5$, $\alpha = 3$, where ε, ω are the gambles using in the proof of Theorem 5 with at least $2n$ preference switching. In the right side of this figure we can see the difference of these expected values: $Eu(x + \varepsilon) - Eu(x + \omega)$, where is more easily seen the preference switching.

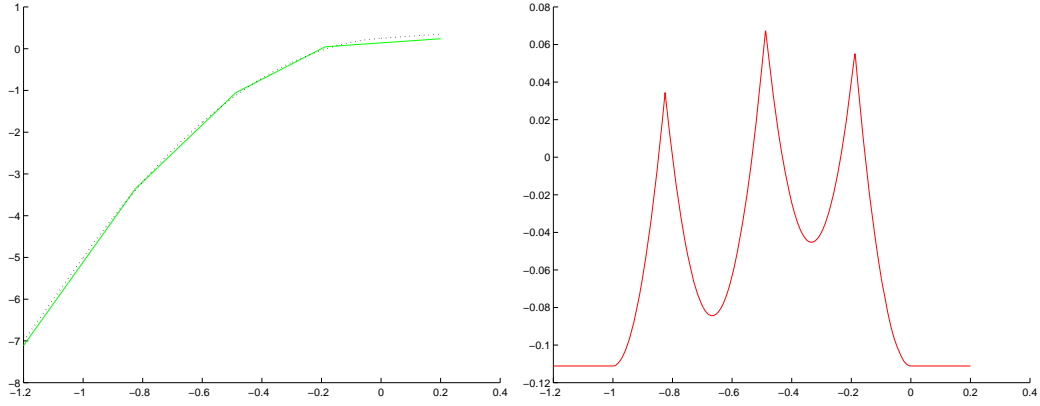


Figure 3: The preference switching.

Finally, we will compare the advantages of the functions (13) and (15) in the following table.

$u_1^a(x) = ax - be^{-cx} \quad a, b, c > 0$	$u(x) = \begin{cases} ax + b & x > x_h \\ cx + d & x \leq x_h \end{cases}$ $x_h = \frac{d-b}{a-c}, \quad a > c > 0, \quad b, d \in R$
- absolute risk averse	-maximizing utility
-decreasing absolute risk aversion	problem can be formulated as linear programming model
-additive one-switch utility function	

Table 2: The comparison of two functions.

The additive zero-switch and one-switch utility functions can be used in multistage stochastic programming also in the situation, when we must make a decision in the time when we don't know exactly the initial wealth level.

2 Utility function in portfolio selection problem

Suppose that, an investor wishes to allocate his wealth among assets $i = 1, \dots, n$ and he chooses the $\mathbf{x} = (x_1, \dots, x_n)'$ to maximize the expected utility of final wealth. This model will be formulated as:

$$\begin{aligned} \max \quad & Eu(x_0 + \boldsymbol{\varrho}'\mathbf{x}) \\ \text{subject to: } \quad & \mathbf{1}'\mathbf{x} = x_0 \\ & x_i \geq 0 \end{aligned} \tag{19}$$

x_0 ... the initial wealth

$\boldsymbol{\varrho}$... the distribution of returns for unit of wealth

\mathbf{x} ... the investment strategy

u ... the utility function

Since $\boldsymbol{\varrho}'\mathbf{x}$ is a gamble, we consider the additive approach in (19). Assuming multiplicative approach, we can formulate this model as:

$$\begin{aligned} \max \quad & Eu(\boldsymbol{\varrho}'\mathbf{x}x_0) \\ \text{subject to: } \quad & \mathbf{1}'\mathbf{x} = 1 \\ & x_i \geq 0, \end{aligned} \tag{20}$$

where ϱ_i isn't only the return on investment i , but it is the sum of unit of wealth and return on investment i for unit of wealth. Of course, it is assumed that expected values in (19) and (20) exist. As in Sec. 1 we will prefer the additive approach with focus on model (19).

It is easily seen that any two functions $u(x)$ and $v(x) = au(x) + b$ have the same optimal solution of (19) whenever $a \neq 0$, $a, b \in R$. It is easy to check that these two functions have the same ARA measure as long as it exists. In Kallberg, Ziemba [3]) was empirically examined the extent to which investors with "similar" ARA measures have "similar" optimal portfolios. The fundamental result of Kallberg, Ziemba [3]) is, as we mention in Sec. 1, that investors with the same Rubinstein measure have the same optimal portfolios. In the following theorem we will exactly formulate the similar result for ARA measure. To do this, we need to assume that a gamble have bounded distribution, i.e. there exists such an interval that

$$P(x_0 + \boldsymbol{\varrho}'\mathbf{x} \in \langle a, b \rangle) = 1.$$

However, it is natural assumption because the ARA measure is a measure of local risk aversion. For a deeper discussion of differences between local and global risk aversion we refer the reader to Pratt [6].

Theorem 6:

Let $\boldsymbol{\varrho} = (\varrho_1, \varrho_2, \dots, \varrho_n)'$ be the returns on investments with bounded distribution. Let x_0 be the investor's initial wealth level. Let interval $\langle a, b \rangle$ be constructed such that $P(x_0 + \boldsymbol{\varrho}'\mathbf{x} \in \langle a, b \rangle) = 1$ for any choice of $x_i \geq 0$, $i = 1, \dots, n$, satisfying: $\mathbf{1}'\mathbf{x} = x_0$. Let $r_1(x)$, $r_2(x)$ be ARA measures of twice differentiable, increasing utility functions $u_1(x)$, $u_2(x)$ in $\langle a, b \rangle$. Let δ be positive. If

$$|r_1(x) - r_2(x)| < \delta \quad (21)$$

for all $x \in \langle a, b \rangle$ then

$$Eu_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^1) - Eu_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^2) < [u_1(b) - u_1(a)](e^{2\delta(b-a)} - 1), \quad (22)$$

where $\mathbf{x}^1, \mathbf{x}^2$ are the optimal solutions of (19) for the functions $u_1(x)$, $u_2(x)$.

Proof:

According to (21) we have

$$-\delta < \frac{u_2''(x)}{u_2'(x)} - \frac{u_1''(x)}{u_1'(x)} < \delta$$

for all $x \in \langle a, b \rangle$. Integrating it from a to any $y \in \langle a, b \rangle$ we obtain

$$-\delta(y - a) < \log u_2'(y) - \log u_2'(a) - \log u_1'(y) + \log u_1'(a) < \delta(y - a).$$

Set $v_1(x) = \frac{u_1(x)}{u_1'(a)}$; $v_2(x) = \frac{u_2(x)}{u_2'(a)}$ and combining it with $y \leq b$ we get

$$-\delta(b - a) < \log \frac{v_2'(y)}{v_1'(y)} < \delta(b - a)$$

and in equivalent form

$$e^{-\delta(b-a)} v_1'(y) < v_2'(y) < e^{\delta(b-a)} v_1'(y).$$

After one more integration from a to any $x \in \langle a, b \rangle$ we have

$$e^{-\delta(b-a)} [v_1(x) - v_1(a)] < v_2(x) - v_2(a) < e^{\delta(b-a)} [v_1(x) - v_1(a)]$$

and by substitution $w_1(x) = v_1(x) - v_1(a)$; $w_2(x) = v_2(x) - v_2(a)$ we obtain

$$e^{-\delta(b-a)}w_1(x) < w_2(x) < e^{\delta(b-a)}w_1(x). \quad (23)$$

By the substitutions $w_1(x) = \frac{u_1(x)-u_1(a)}{u'_1(a)}$; $w_2(x) = \frac{u_2(x)-u_2(a)}{u'_2(a)}$, it is easy to check that \mathbf{x}^1 , \mathbf{x}^2 are optimal solutions of (19) also for utility functions $w_1(x)$, $w_2(x)$. Combining (23) and optimality of \mathbf{x}^1 , \mathbf{x}^2 we can estimate the difference of expected utilities between these optimal portfolios

$$\begin{aligned} 0 &\leq E[w_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^1) - w_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^2)] \\ &< E[w_2(x_0 + \boldsymbol{\varrho}'\mathbf{x}^1)e^{\delta(b-a)} - w_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^2)] \\ &< E[w_2(x_0 + \boldsymbol{\varrho}'\mathbf{x}^2)e^{\delta(b-a)} - w_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^2)] \\ &< (e^{2\delta(b-a)} - 1)Ew_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^2) < (e^{2\delta(b-a)} - 1)w_1(b), \end{aligned}$$

using $w'_1(x) > 0$ in the last inequality. It follows immediately that

$$\begin{aligned} E[w_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^1) - w_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^2)] &= E\left[\frac{u_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^1) - u_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^2)}{u'_1(a)}\right] \\ w_1(b) &= \frac{u_1(b) - u_1(a)}{u'_1(a)}. \end{aligned}$$

Substituting it into last inequality we obtain

$$E\frac{[u_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^1) - u_1(x_0 + \boldsymbol{\varrho}'\mathbf{x}^2)]}{u'_1(a)} < \frac{u_1(b) - u_1(a)}{u'_1(a)}(e^{2\delta(b-a)} - 1),$$

which completes the proof. Q.E.D

The above theorem yields information about the maximal expected utility error of the optimal portfolio. This error is caused by using some utility function $u_2(x)$ instead of the true investor's utility function $u_1(x)$, whenever the difference between ARA measures of these two utility functions is smaller than some fixed δ for all $x \in \langle a, b \rangle$.

Example 2:

Let us consider the decision maker with utility function $u_1(x) = -e^{-x}$ and initial wealth level $x_0 = 4$. Set $\boldsymbol{\varrho}' = (1, 2)'$ with probability 0.6 and $\boldsymbol{\varrho}' = (3, 1)'$

with probability 0.4. Let us evaluate the optimal solution of (19), which can be rewritten as

$$\begin{aligned} & \max E[-e^{-4-\varrho_1 x_1 - \varrho_2 x_2}] \\ & \text{subject to } x_1 + x_2 = 4 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{aligned} \quad (24)$$

The equivalent formulation can be

$$\max f(y) = -0.6e^{-4-y-2(4-y)} - 0.4e^{-4-3y-(4-y)}$$

subject to $0 \leq y \leq 4$. The function $f(y)$ is concave in $\langle 0, 4 \rangle$ and $f'(y) = 0$ whenever

$$y = \frac{4 - \log(0.75)}{3}.$$

Therefore, the solution of (24) is

$$x_1^1 = \frac{4 - \log(0.75)}{3}, \quad x_2^1 = \frac{8 + \log(0.75)}{3}. \quad (25)$$

It is easily seen that the ARA measure of $u_1(x) = -e^{-x}$ is constant: $r_1(x) = 1$. Let us consider the utility function $u_2(x) = -(x+a)^b$ for the same problem. Let us choose the parameters $a > 0$, $b < 0$ satisfying (21), so the differences between ARA measures of $u_1(x)$ and $u_2(x)$ are smaller than the fixed δ in the interval $\langle 8, 16 \rangle$. It is easy to check that this interval holds the condition of Theorem 6. Since

$$r_2(x) = \frac{-b+1}{x+a}$$

is a decreasing function in $\langle 8, 16 \rangle$, we can evaluate the parameters a, b as the solution of

$$\begin{aligned} -\frac{b-1}{8+a} &= 1 + \delta \\ -\frac{b-1}{16+a} &= 1. \end{aligned} \quad (26)$$

We obtain

$$a = -8 + \frac{8}{\delta}, \quad b = -7 - \frac{8}{\delta}. \quad (27)$$

We can rewrite (19) using $u_2(x)$ as

$$\begin{aligned} & \max E[-(4 + \varrho_1 x_1 + \varrho_2 x_2 + a)^b] \\ & \text{subject to } x_1 + x_2 = 4, \\ & x_1 \geq 0, x_2 \geq 0, \end{aligned} \quad (28)$$

which can be also formulated as

$$\begin{aligned} \max g(y) &= -0.6(4 + y + 2(4 - y) + a)^b - 0.4(4 + 3y + (4 - y) + a)^b \\ \text{subject to } & 0 \leq y \leq 4. \end{aligned}$$

Thanks to negative b , $g(y)$ is concave in $\langle 0, 4 \rangle$. Hence $g'(y) = 0$ whenever

$$y = \frac{12(0.75)^{\frac{1}{b-1}} - 8 + a \left[(0.75)^{\frac{1}{b-1}} - 1 \right]}{2 + (0.75)^{\frac{1}{b-1}}},$$

the optimal solution of (28) is

$$x_1^2 = \frac{12(0.75)^{\frac{1}{b-1}} - 8 + a \left[(0.75)^{\frac{1}{b-1}} - 1 \right]}{2 + (0.75)^{\frac{1}{b-1}}} \quad (29)$$

$$x_2^2 = \frac{-8(0.75)^{\frac{1}{b-1}} + 16 - a \left[(0.75)^{\frac{1}{b-1}} - 1 \right]}{2 + (0.75)^{\frac{1}{b-1}}}, \quad (30)$$

or one of the following: $(x_1^2, x_2^2) = (0, 4)$, $(x_1^2, x_2^2) = (4, 0)$ depending on a, b . According to (27), letting $\delta \rightarrow 0$, we have $E[u_1(x_0 + \boldsymbol{\varrho}' \mathbf{x}^1) - u_1(x_0 + \boldsymbol{\varrho}' \mathbf{x}^2)] \rightarrow 0$ for optimal solutions (25), (29), (30) as a consequence of Theorem 6. We will prove it without using Theorem 6 in the following computation. Hence $x_2 = 4 - x_1$ it suffices to show that

$$\lim_{\delta \rightarrow 0} x_1^2 = x_1^1.$$

The first equation of this computation follows from (27) and (29), the last one from (25).

$$\begin{aligned} \lim_{\delta \rightarrow 0} x_1^2 &= \lim_{b \rightarrow -\infty} \frac{12(0.75)^{\frac{1}{b-1}} - 8 - (16 + b - 1) \left[(0.75)^{\frac{1}{b-1}} - 1 \right]}{2 + (0.75)^{\frac{1}{b-1}}} \\ &= \lim_{b \rightarrow -\infty} \frac{-4(0.75)^{\frac{1}{b-1}} + 8 - (b - 1) \left[(0.75)^{\frac{1}{b-1}} - 1 \right]}{2 + (0.75)^{\frac{1}{b-1}}} \\ &= \frac{4}{3} - \lim_{b \rightarrow -\infty} \frac{1}{2 + (0.75)^{\frac{1}{b-1}}} \frac{(0.75)^{\frac{1}{b-1}} - 1}{\frac{1}{b-1}} \\ &= \frac{4}{3} - \lim_{c \rightarrow 0} \frac{1}{2 + 0.75^c} \frac{0.75^c - 1}{c} \\ &= \frac{4}{3} - \frac{1}{3} \log(0.75) \\ &= x_1^1, \end{aligned}$$

which completes the example.

We can assume that the investor with utility function $u_1(x)$ at high wealth level x_0 is indifferent between the expected utility $Eu_1(x_0 + \mathbf{g}'\mathbf{x}^1)$ and $Eu_1(x_0 + \mathbf{g}'\mathbf{x}^1) - \varphi$ for the suitable small φ . In this situation we can use $u_2(x)$ instead of $u_1(x)$ in (19), because if the utility functions $u_1(x)$, $u_2(x)$ satisfy the conditions of Theorem 6, especially (21), then the investor will be indifferent between the optimal portfolio, which is evaluated for the true utility function, and the optimal portfolio for $u_2(x)$. The reason for this substitution is simply that (19) can be more easily solved using $u_2(x)$, e.g. using HARA functions. Thus our goal will be to find the suitable utility function $u_2(x)$. We first construct the feasible interval $\langle a, b \rangle$. Then, according to (22), we choose such δ that

$$\varphi = [u_1(b) - u_1(a)](e^{2\delta(b-a)} - 1). \quad (31)$$

Finally, we will find the utility function satisfying (21) in $\langle a, b \rangle$. We illustrate this method in the following example.

Example 3:

Let us consider the decision maker with utility function $u_1(x) = -e^{\frac{1}{x+1}}$ and initial wealth level $x_0 = 4$. Set $\mathbf{g}' = (1, 2)'$ with probability 0.6 and $\mathbf{g}' = (3, 1)'$ with probability 0.4. It is clear that the feasible interval can be $\langle 8, 16 \rangle$. Let us choose such $u_2(x) = -(x+a)^b$, $a \geq 0$, $b < 0$ that the differences between the ARA measures of $u_1(x)$ and $u_2(x)$ are smaller than $\varphi = 0.03$ for all $x \in \langle 8, 16 \rangle$. We first compute δ from (31)

$$\begin{aligned} \varphi &= [u_1(b) - u_1(a)](e^{2\delta(b-a)} - 1) \\ 0.03 &= 0.057(e^{16\delta} - 1) \\ \delta &= 0.0264. \end{aligned}$$

Then we can evaluate the ARA measures

$$\begin{aligned} r_1(x) &= -\frac{u_1''(x)}{u_1'(x)} = -\frac{-e^{\frac{1}{x+1}} \frac{2x+3}{(x+1)^4}}{\frac{e^{\frac{1}{x+1}}}{(x+1)^2}} = \frac{2}{x+1} + \frac{1}{(x+1)^2} \\ r_2(x) &= -\frac{u_2''(x)}{u_2'(x)} = -\frac{-b(b-1)(x+a)^{b-2}}{-b(x+a)^{b-1}} = -\frac{b-1}{x+a}. \end{aligned}$$

The task is now to find the parameters a , b such that $|r_1(x) - r_2(x)| < \delta$ for all $x \in \langle 8, 16 \rangle$. Since $r_1(x)$, $r_2(x)$ are decreasing and convex functions in $\langle 8, 16 \rangle$, we try to find a , b as a solution of the following system of equations.

$$\begin{aligned} r_1(8) - r_2(8) &= \delta \\ r_1(16) - r_2(16) &= -\delta \end{aligned}$$

After substituting we obtain

$$\begin{aligned}\frac{19}{81} + \frac{b-1}{8+a} &= 0.0264 \\ \frac{35}{289} + \frac{b-1}{16+a} &= -0.0264.\end{aligned}$$

and the solution is

$$a \doteq 11.4534, \quad b \doteq -3.0496. \quad (32)$$

In Fig. 4 we can see the course of function

$$f(x) = r_1(x) - r_2(x) = \frac{2}{x+1} + \frac{1}{(x+1)^2} + \frac{-4.0496}{x+11.4534}$$

in the interval $\langle 8, 16 \rangle$ and the boundary limit δ .

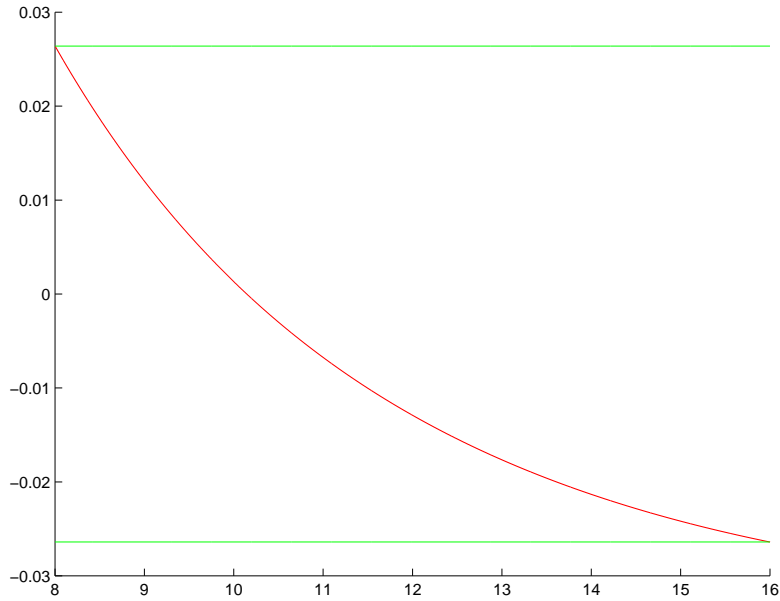


Figure 4: The difference of the ARA measures.

Thus the differences of the ARA measures are small enough in $\langle 8, 16 \rangle$ and thus $u_2(x) = -(x + 11.4534)^{-3.0496}$ is the utility function we wanted to find.

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