

Optimal mean - risk portfolios under NSD efficiency constraints

Miloš Kopa

- Mean–risk models

$$\max_{\lambda \in \Lambda} \quad m(\lambda' \varrho) - \nu r(\lambda' \varrho)$$

or

$$\begin{aligned} \min_{\lambda \in \Lambda} \quad & r(\lambda' \varrho) \\ \text{s.t.} \quad & m(\lambda' \varrho) \geq \mu \end{aligned}$$

- ϱ is a random vector of assets returns
- maximizing mean $m(\lambda' \varrho)$ & minimizing risk $r(\lambda' \varrho)$
- risk measure (variance, semi variance,..., VaR, CVaR)
- risk or return parameter (ν, μ)

If we do not know the parameters – efficiency frontier

- Utility functions approach

$$\max_{\lambda \in \Lambda} Eu(\lambda' \varrho)$$

- u is an utility function (non-decreasing function)
- maximizing expected utility
- choice of utility function - risk attitude

If no information about utility function is known – stochastic dominance
– portfolio efficiency with respect to stochastic dominance

- Crucial question of portfolio efficiency (in the sense of optimality):
Is a given portfolio a maximizer of expected utility for at least one considered utility function?
- Crucial question of portfolio efficiency (in the sense of admissibility):
Does there exist a better portfolio (having higher expected utility) than a given portfolio for all considered investors (utility functions)

We focus on the optimality approach: K. and Post (2009), Post and K. (2013).

How mean-risk efficiency corresponds to stochastic dominance efficiency?

Ogryczak & Ruszczyński (2002): risk measures consistent with the second order stochastic dominance:

$$A \succeq_{SSD} B \Rightarrow \rho(A) \leq \rho(B)$$

We focus on:

- utility based models - only partial knowledge of the utility function - stochastic dominance
- Nth order stochastic dominance relations
- portfolio efficiency with respect to Nth order stochastic dominance ($N \geq 2$)
- risk-averse decision makers: mean-risk efficiency combined with Nth order stochastic dominance efficiency

We consider a random vector $\mathbf{r} = (r_1, r_2, \dots, r_N)'$ of returns of N assets with a discrete probability distribution described by T equiprobable scenarios. The returns of the assets for the various scenarios are given by

$$X = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^T \end{pmatrix}$$

where $\mathbf{x}^t = (x_1^t, x_2^t, \dots, x_N^t)$ is the t -th row of matrix X representing the assets returns along t -th scenario. We assume that the decision maker may also combine the alternatives into a portfolio. We will use $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)^T$ for a vector of portfolio weights and $X\boldsymbol{\lambda}$ represents returns of portfolio $\boldsymbol{\lambda}$. The portfolio possibilities are given by a simplex

$$\Lambda = \{\boldsymbol{\lambda} \in R^N | \mathbf{1}'\boldsymbol{\lambda} = 1, \lambda_j \geq 0, j = 1, 2, \dots, N\}.$$

Second order stochastic dominance

The random return of the tested portfolio τ is denoted by $\mathbf{r}'\tau$.

Definition

Portfolio λ *dominates* portfolio τ with respect to *second-order stochastic dominance* ($\lambda \succ_{SSD} \tau$) if $Eu(\mathbf{r}'\lambda) \geq Eu(\mathbf{r}'\tau)$ for all non-decreasing and concave utility functions with strict inequality for at least one such utility function.

Second order stochastic dominance

Interpretation: $\lambda \succ_{SSD} \tau$ if

- No non-satiable and risk averse decision maker prefers portfolio τ to portfolio λ and at least one prefers λ to τ .
- $F_{r'\lambda}^{(2)}(y) \leq F_{r'\tau}^{(2)}(y) \quad \forall y \in \mathbb{R}$ with strict inequality for at least one $y \in \mathbb{R}$ where $F_{r'\lambda}^{(2)}(y)$ is a twice cumulated probability distribution
 $F_{r'\lambda}^{(2)}(y) = \int_{-\infty}^y F_{r'\lambda}(x) dx$.
- $F_{r'\lambda}^{-2}(y) \leq F_{r'\tau}^{-2}(y) \quad \forall y \in [0, 1]$ with strict inequality for at least one $y \in [0, 1]$, where $F_{r'\lambda}^{-2}$ is a cumulated quantile function.
- $\text{CVaR}_\alpha(-r'\lambda) \leq \text{CVaR}_\alpha(-r'\tau) \quad \forall \alpha \in [0, 1]$ with strict inequality for at least one $\alpha \in [0, 1]$, where

$$\begin{aligned} \text{CVaR}_\alpha(-r'\lambda) = \min_{v \in \mathbb{R}, z_t \in \mathbb{R}^+} \quad & v + \frac{1}{1-\alpha} \sum_{t=1}^S p_t z_t \\ \text{s.t.} \quad & z_t \geq -\mathbf{x}^t \lambda - v, \quad t = 1, 2, \dots, S \end{aligned} \quad (1)$$

Definition

A given portfolio τ is *SSD efficient* if there exists a strictly concave utility function u such that:

$$Eu(\mathbf{r}'\tau) \geq Eu(\mathbf{r}'\lambda) \quad \forall \lambda \in \Lambda. \quad (2)$$

Otherwise, portfolio τ is *SSD inefficient*.

SSD portfolio efficiency test (Post 2003)

Theorem

Let $y^t, t = 1, 2, \dots, T$ be the t – th smallest return of portfolio τ and

$$\theta^* = \min_{\theta, \beta_t} \theta \quad (3)$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{t=1}^T \beta_t (y^t - x_n^t) + T\theta \geq 0 \quad n = 1, 2, \dots, N \\ & \beta_t - \beta_{t+1} \geq 0 \quad t = 1, 2, \dots, T-1 \\ & \beta_t \geq 0 \quad t = 1, 2, \dots, T-1 \\ & \beta_T = 1. \end{aligned}$$

A given portfolio τ is SSD efficient if and only if $\theta^* \leq 0$.

If the tested portfolio is SSD efficient then an admissible utility function can be constructed from marginal utility levels β_t^* identified by (3) as the optimal solutions. On the other hand, the optimal coefficients β_t^* have no economic meaning for SSD inefficient portfolio.

N -th order stochastic dominance

Let U_N be the set of N times differentiable utility functions such that: $(-1)^k u^{(k)} \leq 0$ for all $k = 1, 2, \dots, N$.

Definition

Portfolio λ dominates portfolio τ with respect to N -th order stochastic dominance ($\lambda \succ_{NSD} \tau$) if $Eu(\mathbf{r}'\lambda) \geq Eu(\mathbf{r}'\tau)$ for all utility functions $u \in U_N$ with strict inequality for at least one such utility function.

The general definition of NSD efficiency for $N \geq 2$ can be seen as an extension of SSD efficiency and, following Post and K. (2013), we formulate it in the “NSD optimality” form. We allow for non-equal probabilities of scenarios (p_1, \dots, p_T) .

Definition

A given portfolio τ is NSD efficient ($N \geq 2$), if there exists at least one utility function $u \in U_N$ such that $Eu(\mathbf{r}'\tau) - Eu(\mathbf{r}'\lambda) \geq 0$ for all $\lambda \in \Lambda$ with strict inequality for at least one $\lambda \in \Lambda$.

Necessary and sufficient condition for NSD efficiency

Using KKT condition in problem $\max_{\lambda \in \Lambda} \sum_{t=1}^T p_t u(\mathbf{x}^t \lambda)$, Post and K. (2013) derived the following NSD efficiency test:

Assume that scenarios are ordered in the ascending order according to returns of portfolio τ , i.e. $\mathbf{x}^t \tau \leq \mathbf{x}^{t+1} \tau, t = 1, 2, \dots, T-1$. Let

$$\theta^*(\tau) = \min_{\beta_n, \gamma_k, \theta} \theta \quad (4)$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{t=1}^T (\mathbf{x}^t \tau - \mathbf{x}_j^t) p_t \left(\sum_{n=1}^{N-2} n \beta_n (\mathbf{x}^t \tau - \mathbf{x}^T \tau)^{n-1} + \right. \\ & \left. (N-1) \sum_{k=t}^T \gamma_k (\mathbf{x}^t \tau - \mathbf{x}^k \tau)^{N-2} \right) + \theta \geq 0, \quad j = 1, \dots, M \end{aligned}$$

$$(-1)^n \beta_n \leq 0, \quad n = 1, \dots, N-2$$

$$(-1)^{N-1} \gamma_k \leq 0, \quad k = 1, 2, \dots, T$$

$$\sum_{t=1}^T \left(\sum_{n=1}^{N-2} n \beta_n (\mathbf{x}^t \tau - \mathbf{x}^T \tau)^{n-1} + (N-1) \sum_{k=t}^T \gamma_k (\mathbf{x}^t \tau - \mathbf{x}^k \tau)^{N-2} \right) p_t = 1.$$

A portfolio τ is NSD efficient $\Leftrightarrow \theta^*(\tau)$ given by (4) is equal to zero.

Ordering of returns

To be able to use the necessary and sufficient condition for NSD efficiency one needs to order the returns of any portfolio. We may do it, for example, using so-called permutations matrix $P = \{p_{i,k}\}_{i,k=1}^T$, that is, a 0-1 matrix that satisfies:

$$\sum_{i=1}^T p_{i,k} = \sum_{k=1}^T p_{i,k} = 1, \quad p_{i,k} \in \{0, 1\}, \quad i, k = 1, \dots, T.$$

Then for any portfolio returns $\mathbf{x}^t \boldsymbol{\tau}$, $t = 1, 2, \dots, T$, a permutation matrix P exists such that:

$$(\mathbf{X}\boldsymbol{\tau})^{[t]} = \sum_{k=1}^T p_{t,k} \mathbf{x}^k \boldsymbol{\tau}$$

that is, $P\mathbf{X}\boldsymbol{\tau}$ is a vector of ordered returns of portfolio $\boldsymbol{\tau}$ from the smallest one.

$$\min_{\tau \in \Lambda} \sigma^2(\mathbf{r}\tau) = \frac{1}{T} \sum_{t=1}^T \left(\mathbf{x}^t \tau - \frac{1}{T} \sum_{s=1}^T \mathbf{x}^s \tau \right)^2$$

s.t.

$$\frac{1}{T} \sum_{t=1}^T \mathbf{x}^t \tau \geq m$$

$$y^t = \sum_{k=1}^T p_{t,k} \mathbf{x}^k \tau, \quad t = 1, 2, \dots, T$$

$$\sum_{i=1}^T p_{i,k} = \sum_{k=1}^T p_{i,k} = 1, \quad p_{i,k} \in \{0, 1\}, \quad i, k = 1, \dots, T$$

$$y^{t+1} \geq y^t, \quad t = 1, 2, \dots, T-1$$

$$\begin{aligned} & \sum_{t=1}^T (y^t - \sum_{k=1}^T p_{t,k} x_j^k) p_t \left(\sum_{n=1}^{N-2} n \beta_n (y^t - y^T)^{n-1} + \right. \\ & \left. (N-1) \sum_{k=t}^T \gamma_k (y^t - y^k)^{N-2} \right) \geq 0, \quad j = 1, \dots, M \\ & (-1)^n \beta_n \leq 0, \quad n = 1, \dots, N-2 \\ & (-1)^{N-1} \gamma_k \leq 0, \quad k = 1, 2, \dots, T \\ & \sum_{t=1}^T \left(\sum_{n=1}^{N-2} n \beta_n (y^t - y^T)^{n-1} + (N-1) \sum_{k=t}^T \gamma_k (y^t - y^k)^{N-2} \right) p_t = 1. \end{aligned}$$

Alternative models

One can easily use another measure of risk instead of variance, for example for equiprobable scenarios:

- semivariance:

$$\sigma^2(\mathbf{r}'\boldsymbol{\tau}) = \frac{1}{T} \sum_{t=1}^T \left((\mathbf{x}^t \boldsymbol{\tau} - \frac{1}{T} \sum_{s=1}^T (\mathbf{x}^s \boldsymbol{\tau}))^- \right)^2$$

- Value at Risk:

$$\begin{aligned} \text{VaR}_\alpha(-\mathbf{r}'\boldsymbol{\tau}) &= \min_{\gamma, \delta_t} \gamma \\ \text{s.t. } &\gamma + M\delta_t \geq -\mathbf{x}^t \boldsymbol{\tau}, \quad t = 1, \dots, T \\ &\sum_{t=1}^T \delta_t = \lfloor (1 - \alpha)T \rfloor, \quad \delta_t \in \{0, 1\}, \quad t = 1, \dots, T \end{aligned}$$

- Conditional Value at Risk:

$$\begin{aligned} \text{CVaR}_\alpha(-\mathbf{r}'\boldsymbol{\tau}) &= \min_{v \in \mathbb{R}, z_t \in \mathbb{R}^+} v + \frac{1}{(1 - \alpha)T} \sum_{t=1}^T z_t \\ \text{s.t. } &z_t \geq -\mathbf{x}^t \boldsymbol{\tau} - v, \quad t = 1, 2, \dots, T \end{aligned}$$

- US stock market data from the Kenneth French library.
- We consider a standard set of 10 active benchmark stock portfolios as the base assets. They are formed, and annually rebalanced, based on individual stocks market capitalization of equity, each representing a decile of the cross-section of stocks in a given year. The first decile stocks (the smallest size) are called "small" and the last decile stocks are called "large".
- We include US Treasury bill as a riskless asset.
- We use data on annual excess returns (in %) from 1982 to 2011 (30 observations).
- Hence we have $n=11$ base assets and $T=30$ scenarios.

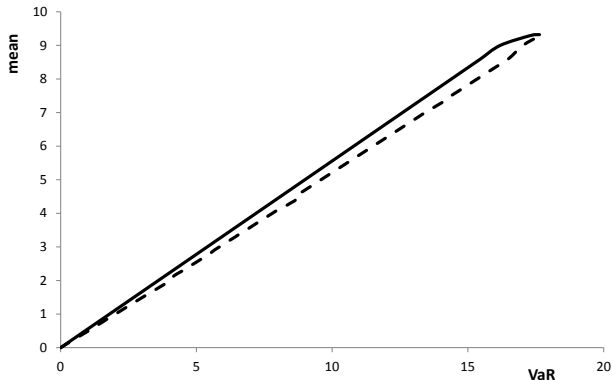
Empirical study - descriptive statistics

	mean	st. deviation	min	max	skewness	c. kurtosis
Small	8.43	26.66	-44.67	90.27	0.65	1.88
2nd decile	8.06	22.44	-37.92	60.56	0.04	-0.07
3rd decile	8.58	19.56	-34.80	49.95	-0.20	-0.21
4th decile	7.83	18.64	-30.75	47.68	-0.17	-0.15
5th decile	8.97	19.50	-36.86	45.49	-0.19	0.00
6th decile	8.75	17.01	-29.90	40.97	-0.20	-0.15
7th decile	9.32	18.44	-42.48	43.68	-0.49	0.95
8th decile	8.69	17.77	-40.89	39.67	-0.57	0.92
9th decile	8.62	17.12	-43.38	37.90	-0.82	1.67
Large	7.18	16.57	-36.56	32.47	-0.74	0.48

Table 1: Base assets 1982-2011 descriptive statistics

Empirical study - results

Fig. 1: Mean-VaR efficiency frontiers with additional SSD efficiency constraints (dashed line) and without SSD efficiency constraints (solid line)



- We formulated a new type of optimization problems which “combines” two most common approaches to portfolio efficiency
- The new problem can be seen as a generalization of mean-risk models
- The new idea is to add constraints which reduce the feasibility set to the NSD efficient portfolios
- One can use several different risk measures and orders of stochastic dominance, including DARA SD (Post, Fang & K. 2014)
- The disadvantage: computational complexity
- Another disadvantage: the optimal portfolio is very sensitive to changes in probability distribution of returns → some stability analysis is needed, for example stress testing using contamination techniques as proposed in Dupačová & K. (2012, 2014)

Main related recent references

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