

Introduction to risk modelling

Miloš Kopa

Department of Probability and Mathematical Statistics
Faculty of Mathematics and Physics
Charles University in Prague

kopa@karlin.mff.cuni.cz

What is a risk?

- To drive a car is a big risk....
- To believe to unknown people is a risk...
- To invest in the stock market is a risk...

For math/OR/MS/finance people:

- Risk is always connected with a random outcome
- Risk can be measured - theory of risk measures
- Risk can be managed - theory of decision making under risk

Risk measure - definition

Assume that a random outcome (element) is described by a random variable $X \in \Omega$ with known probability distribution P .

Definition

Risk measure is any mapping from Ω to \mathbb{R}

Examples of some (not very good) measures:

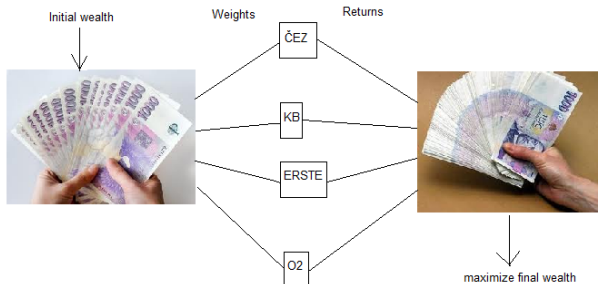
- Expected value: EX
- Variance: $E(X - EX)^2$
- Median: $\min\{t : P(X \leq t) \geq 0.5\}$

It takes into account the whole distribution P not only a particular realization.

Two approaches in the decision theory

- Decision making under risk - a probability distribution of the random variable is known
- Decision making under uncertainty - a probability distribution is **not know**
 - The theory of risk measures can not be applied
 - Worst-case analysis
 - Robustness
 - Not interesting for this talk

Example



Introduction of the most popular risk measures:

- variance
- mean absolute deviation (MAD) ... like variance but with L_1 norm
- semivariance
- Value at Risk (VaR)
- Conditional Value at Risk (CVaR) also known as Average VaR or Tail VaR or Expected Shortfall

Formulas for these measures under assumption of a particular probability distribution:

- Normal (Gaussian) distribution
- Student distribution (t-distribution)
- Log-normal distribution
- Discrete (empirical) distribution - scenario approach

Risk measures - general distribution

We define either for returns X or losses $L = -X$ - depends on the applications....

Definition

Variance for return X is defined as

$$\text{var}(X) = E(X - EX)^2.$$

Let $\alpha \in (0, 1)$ be the threshold (level of significancy). Then we define VaR_α as:

$$\text{VaR}_\alpha(L) = \inf \{I \in \mathbb{R}, P(L > I) \leq 1 - \alpha\}$$

Definition

Conditional Value at Risk for a loss L ($CVaR_\alpha(L)$) is defined as:

$$CVaR_\alpha(L) = \inf \left\{ a \in \mathbb{R}, a + \frac{1}{1-\alpha} E[\max(0, L - a)] \right\}.$$

Alternatively:

$$CVaR_\alpha(L) = E(L | L > VaR_\alpha(L))$$

Absolute deviation can be calculated as:

$$r_a(L) = E|L - EL|.$$

Semivariance can be calculated as:

$$r_s(L) = E \left[\max(0, L - EL)^2 \right]$$

Normal distribution

Assume that L has normal distribution $N(-\mu, \sigma^2)$ and let q_α be an α - quantile of normal distribution $N(0, 1)$. Then:

- VaR

$$\text{VaR}_\alpha(L) = -\mu + q_\alpha \sqrt{\sigma^2}$$

- cVaR

$$\text{cVaR}_\alpha(L) = -\mu + \frac{\exp\left\{-\frac{q_\alpha^2}{2}\right\}}{(1-\alpha)\sqrt{2\pi}} \sqrt{\sigma^2}$$

- absolute deviation

$$r_a(L) = \sqrt{\frac{2}{\pi}} \sqrt{\sigma^2}$$

- semivariance

$$r_s(L) = \frac{1}{2} \text{E} \left[(L - \text{EL})^2 \right] = \frac{1}{2} \sigma^2$$

Student distribution

Assume that L has student distribution with ν degrees of freedom. Then $EL = 0$, $\text{var}(L) = \frac{\nu}{\nu-2}$. Let $t_{\alpha,\nu}$ be an α - quantile of student distribution with ν degrees of freedom. Then:

- VaR

$$\text{VaR}_{\alpha}(L) = t_{\alpha,\nu}$$

- cVaR

$$\text{CVaR}_{\alpha}(L) = \frac{\Gamma\left(\frac{\nu-1}{2}\right) \sqrt{\nu} \left(1 + \frac{t_{\alpha,\nu}^2}{\nu}\right)^{-\frac{\nu-1}{2}}}{\Gamma\left(\frac{\nu-2}{2}\right) (1-\alpha) (\nu-2) \sqrt{\pi}}$$

where

$$\Gamma(y) = \int_0^{\infty} x^{y-1} e^{-x} dx$$

and $\Gamma(n) = (n-1)!$ for positive integer n

Student distribution

Assume that L has student distribution with ν degrees of freedom. Then $EL = 0$, $\text{var}(L) = \frac{\nu}{\nu-2}$. Let $t_{\alpha,\nu}$ be an α - quantile of student distribution with ν degrees of freedom. Then:

- absolute deviation

$$r_a(L) = \frac{2\sqrt{\nu}\Gamma\left(\frac{\nu+1}{2}\right)}{(\nu-1)\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}$$

- semivariance

$$r_s(L) = \frac{1}{2}E\left[(L - EL)^2\right] = 0.5t_{\alpha,\nu}$$

Assume that $X = \exp(M)$ and M has normal distribution $N(\mu, \sigma^2)$
Then $EX = \exp\{\mu + 0.5\sigma^2\}$,
 $\text{var}(X) = (\exp\{\sigma^2\} - 1)\exp\{2\mu + \sigma^2\}$. Let q_α be an α - quantile
of normal distribution $N(0, 1)$ Then:

- VaR

$$\text{VaR}_\alpha(L) = \exp\{\mu + \sigma q_{1-\alpha}\}$$

- CVaR

$$\text{CVaR}_\alpha(L) = \frac{1}{1-\alpha} \exp\left\{\mu + \frac{\sigma^2}{2}\right\} \Phi(q_{1-\alpha} - \sigma)$$

where Φ is cdf of normal distribution $N(0, 1)$.

Log-normal distribution

Assume that $X = \exp(M)$ and M has normal distribution $N(\mu, \sigma^2)$
Then $EX = \exp\{\mu + 0.5\sigma^2\}$,
 $\text{var}(X) = (\exp\{\sigma^2\} - 1)\exp\{2\mu + \sigma^2\}$. Let q_α be an α - quantile
of normal distribution $N(0, 1)$ Then:

- absolute deviation

$$r_a(L) = \exp\left\{\mu + \frac{\sigma^2}{2}\right\} \left(4\Phi\left(\frac{\sigma}{2}\right) - 2\right)$$

- semivariance

$$r_s(L) = \exp\{2\mu + \sigma^2\} \left(\exp\{\sigma^2\} \Phi\left(-\frac{3\sigma}{2}\right) - 2 + 3\Phi\left(\frac{\sigma}{2}\right)\right)$$

Variance - scenarios

Suppose that L has a discrete distribution given by M equiprobable scenarios l_j . Let

$$\bar{l} = \frac{1}{M} \sum_{j=1}^M l_j.$$

Then:

$$\text{var}(L) = \frac{1}{M} \sum_{j=1}^M (l_j - \bar{l})^2$$

Easy to calculate - questionable interpretation....

Suppose that L has a discrete distribution given by M equiprobable scenarios l_j , such that $l^{[1]} \leq l^{[2]} \leq \dots \leq l^{[M]}$. Then for $\alpha \in \left(\frac{n}{M}, \frac{n+1}{M}\right]$, $n = 1, 2, \dots, M-1$

$$\text{VaR}_\alpha(L) = l^{[n+1]}$$

Alternative, if pre-ordering is not possible:

$$\begin{aligned} \text{VaR}_\alpha(L) = \min_{\nu, \delta_j} \nu \\ \text{s. t. } l_j \leq \nu + K\delta_j, j = 1, \dots, M \\ \sum_{j=1}^M \delta_j = \lfloor (1 - \alpha) M \rfloor \\ \delta_j \in \{0, 1\}, j = 1, \dots, M \end{aligned}$$

where $\lfloor y \rfloor = \max\{n \in \mathbb{N} : n \leq y\}$ and K is sufficiently large, for example $K = \max_j \{l_j\} + 1$

- If pre-ordering is possible (approximately):

$$CVaR_{\alpha}(L) = \frac{1}{M} \sum_{l_j \geq VaR_{\alpha}(L)} l_j$$

- If not: linear programming task, can be solved also quite quickly

$$\begin{aligned} \min_{a, z_j} \quad & a + \frac{1}{(1 - \alpha) M} \sum_{j=1}^M z_j \\ \text{s. t.} \quad & z_j \geq l_j - a, j = 1, \dots, M \\ & z_j \geq 0, j = 1, \dots, M \end{aligned}$$

Absolute deviation - scenarios

- Directly

$$r_a(L) = \frac{1}{M} \sum_j |l_j - \bar{l}|$$

- Alternatively: linear programming task, can be solved also quite quickly

$$\min_{z_j, y_j} \frac{1}{M} \sum_{j=1}^M (z_j + y_j)$$

$$\begin{aligned} \text{s. t. } & l_j - \bar{l} \leq y_j, \quad j = 1, \dots, M \\ & -l_j + \bar{l} \leq z_j, \quad j = 1, \dots, M \\ & z_j \geq 0, \quad j = 1, \dots, M \\ & y_j \geq 0, \quad j = 1, \dots, M \end{aligned}$$

- If pre-ordering is possible:

$$r_s(L) = \frac{1}{M} \sum_{l_j \geq \bar{l}} (l_j - \bar{l})^2$$

- If not: quadratic programming task

$$\min_{z_j} \frac{1}{M} \sum_{j=1}^M (z_j)^2$$

$$\text{s. t. } z_j \geq l_j - \bar{l}, j = 1, \dots, M$$

$$z_j \geq 0, j = 1, \dots, M$$

- what are the reasonable properties for risk measures? (coherency)
- dual expression of risk measures - some worst case expectations
- how to employ risk measures in decision making under risk?
- efficient portfolios
- risk aversion via utility functions - stochastic dominance
- robustness and stress testing
- Empirical studies