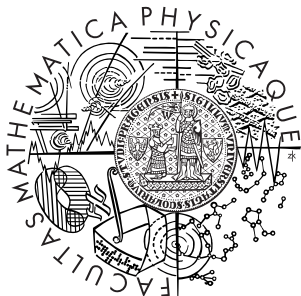


# Convergence of approximate solutions in mean-risk models



Miloš Kopa & Václav Kozmík

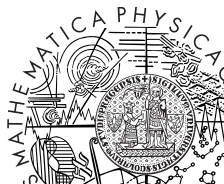
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# Introduction

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- Mean-risk models
  - aim to find optimal portfolio of assets
  - analytical solutions for continuous distributions
  - solutions using generated scenarios
    - predetermined, for instance by historical data
    - generated with the assumption of continuous distribution
    - generated according to few moment estimators
- comparison of the approaches mentioned above
  - convergence and its properties
  - different continuous distributions
  - different risk measures
- computational part
  - data processing and generating the scenarios
  - optimization tasks in GAMS



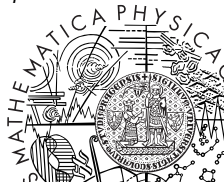
# Efficient portfolios

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- we consider portfolio based on  $N$  assets
- weights of the assets  $\mathbf{w}$ ,  $\sum_{i=1}^N w_i = 1$
- expected returns  $u_{\mathbf{w}}$  (always using expectation)
- different risk measures  $r_{\mathbf{w}}$
- minimal required returns  $u_e$

## Definition

*Portfolio of given  $N$  assets with weights  $\mathbf{w}$  is (mean-risk) efficient, if there are no other weights  $w_1^*, \dots, w_N^*$  such that  $\sum_{i=1}^N w_i^* = 1$  and  $u_{\mathbf{w}^*} \geq u_{\mathbf{w}}$  and  $r_{\mathbf{w}^*} \leq r_{\mathbf{w}}$ .*



# Classical optimization task

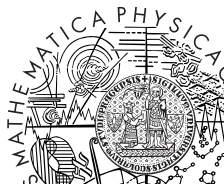
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Efficient portfolios can be obtained while solving following task:

$$\begin{aligned} \min_{\mathbf{w}} \quad & r_{\mathbf{w}} \\ \text{s. t.} \quad & u_{\mathbf{w}} \geq u_e \\ & \sum_{i=1}^N w_i = 1 \\ & w_i \in \mathbb{R}, i = 1, \dots, N. \end{aligned}$$

Non-negativity condition:

$$w_1, \dots, w_N \geq 0.$$



# Risk measures

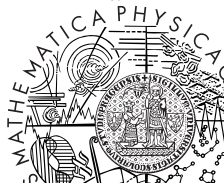
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- variance
- VaR
- cVaR
- absolute deviation
- semivariance

## Definition

*Let  $\alpha \in (0, 1)$  be the threshold and  $L$  random variable which represents the loss from holding the portfolio. Then we define  $VaR_\alpha$  as:*

$$VaR_\alpha(L) = \inf \{I \in \mathbb{R}, P(L > I) \leq 1 - \alpha\}$$



# Risk measures

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## Definition

$cVaR_\alpha$  is defined as:

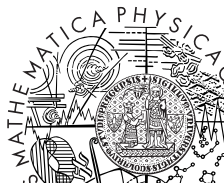
$$cVaR_\alpha(L) = \inf \left\{ a \in \mathbb{R}, a + \frac{1}{1-\alpha} E[\max(0, L - a)] \right\}.$$

*Absolute deviation can be calculated as:*

$$r_a(L) = E|L - EL|.$$

*Semivariance can be calculated as::*

$$r_s(L) = E \left[ \max(0, L - EL)^2 \right]$$



# Elliptical distributions

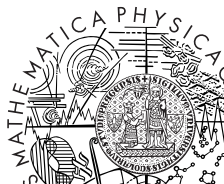
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- generalization of normal distribution
- include normal distribution, Student distribution, logistic elliptical distribution and others
- symmetrical around the mean
- simple analysis of linear combinations

## Theorem

Let  $\mathbf{X} \sim \mathbf{E}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ ,  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$ . Then it holds:

$$\mathbf{AX} + \mathbf{b} \sim \mathbf{E}\left(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T, \psi\right).$$



# Variance

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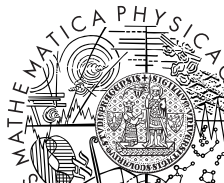
We get classical optimization task which can be used for all elliptical distributions:

$$\min_{\mathbf{w}} \mathbf{w}^T \mathbf{V} \mathbf{w}$$

$$\text{s. t. } \mathbf{w}^T \boldsymbol{\mu} \geq u_e$$

$$\sum_{i=1}^N w_i = 1$$

$$w_i \in \mathbb{R}, i = 1, \dots, N$$





# Normal distribution

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- VaR

$$VaR_{\alpha}(L) = -\mathbf{w}^T \boldsymbol{\mu} + q_{\alpha} \sqrt{\mathbf{w}^T \mathbf{V} \mathbf{w}}$$

- cVaR

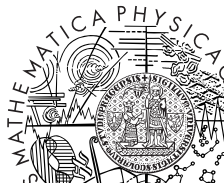
$$cVaR_{\alpha}(L) = -\mathbf{w}^T \boldsymbol{\mu} + \frac{\exp\left\{-\frac{q_{\alpha}^2}{2}\right\}}{(1-\alpha)\sqrt{2\pi}} \sqrt{\mathbf{w}^T \mathbf{V} \mathbf{w}}$$

- absolute deviation

$$r_a(L) = \sqrt{\frac{2}{\pi}} \sqrt{\mathbf{w}^T \mathbf{V} \mathbf{w}}$$

- semivariance

$$r_s(L) = \frac{1}{2} \mathbb{E} \left[ (L - \mathbb{E}L)^2 \right]$$



# Student distribution

- VaR

$$VaR_{\alpha}(L) = VaR_{\alpha}(L) = -\mathbf{w}^T \boldsymbol{\mu} + t_{\alpha, \nu} \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$$

- cVaR

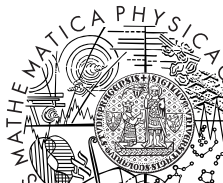
$$cVaR_{\alpha}(L) = -\mathbf{w}^T \boldsymbol{\mu} + \frac{\Gamma\left(\frac{\nu-1}{2}\right) \sqrt{\nu} \left(1 + \frac{t_{\alpha, \nu}^2}{\nu}\right)^{-\frac{\nu-1}{2}}}{\Gamma\left(\frac{\nu-2}{2}\right) (1-\alpha) (\nu-2) \sqrt{\pi}} \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$$

- absolute deviation

$$r_a(L) = \frac{2\sqrt{\nu}\Gamma\left(\frac{\nu+1}{2}\right)}{(\nu-1)\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$$

- semivariance

$$r_s(L) = \frac{1}{2} \mathbb{E} \left[ (L - \mathbb{E}L)^2 \right]$$

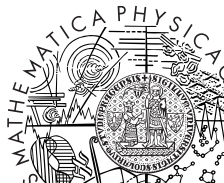


## Variance - scenarios

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- suppose we have  $M$  scenarios of possible stock prices
- we can use mean and variance-covariance estimators  $\hat{\mathbf{I}}$  and  $\hat{\mathbf{V}}$  to minimize variance
- allows us to process estimators before running the optimization task and therefore is quick

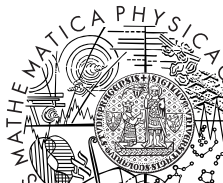
$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^T \hat{\mathbf{V}} \mathbf{w} \\ \text{s. t.} \quad & \mathbf{w}^T \hat{\mathbf{I}} \geq u_e \\ & \sum_{i=1}^N w_i = 1 \\ & w_i \in \mathbb{R}, i = 1, \dots, N. \end{aligned}$$



## VaR - scenarios

- general case could be nonconvex
- we reformulate the task using integer programming
- still hardly computable -  $M$  binary variables

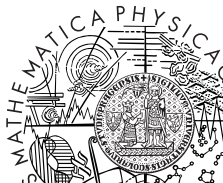
$$\begin{aligned} \min_{\nu, \mathbf{w}, \delta^j} \quad & \nu \\ \text{s. t.} \quad & -\mathbf{w}^T \mathbf{p}^j \leq \nu + K \delta^j, j = 1, \dots, M \\ & \sum_{j=1}^M \delta^j = \lfloor (1 - \alpha) M \rfloor \\ & \delta^j \in \{0, 1\}, j = 1, \dots, M \\ & \frac{1}{M} \sum_{j=1}^M \mathbf{w}^T \mathbf{p}^j \geq u_e \dots \end{aligned}$$



## cVaR - scenarios

- linear programming task, can be solved quickly

$$\begin{aligned} \min_{a, \mathbf{w}, z^j} \quad & a + \frac{1}{(1-\alpha)M} \sum_{j=1}^M z^j \\ \text{s. t.} \quad & z^j \geq -\mathbf{w}^T \boldsymbol{\psi}^j - a, j = 1, \dots, M \\ & z^j \geq 0, j = 1, \dots, M \\ & \frac{1}{M} \sum_{j=1}^M \mathbf{w}^T \boldsymbol{\psi}^j \geq u_e \\ & \sum_{i=1}^N w_i = 1 \\ & w_i \in \mathbb{R}, i = 1, \dots, N. \end{aligned}$$



# Absolute deviation - scenarios

- linear programming task

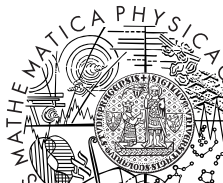
$$\min_{\mathbf{w}, z^j} \frac{1}{M} \sum_{j=1}^M z^j$$

$$\text{s. t. } \mathbf{w}^T \mathbf{l}^j - \frac{1}{M} \sum_{i=1}^M \mathbf{w}^T \mathbf{l}^i \leq z^j, j = 1, \dots, M$$

$$- \mathbf{w}^T \mathbf{l}^j + \frac{1}{M} \sum_{i=1}^M \mathbf{w}^T \mathbf{l}^i \leq z^j, j = 1, \dots, M$$

$$\frac{1}{M} \sum_{j=1}^M \mathbf{w}^T \mathbf{l}^j \geq u_e$$

...



# Semivariance - scenarios

- quadratic programming task

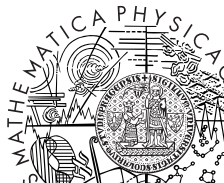
$$\min_{\mathbf{w}, z^j} \frac{1}{M} \sum_{j=1}^M (z^j)^2$$

$$\text{s. t. } z^j \geq -\mathbf{w}^T \mathbf{l}^j + \frac{1}{M} \sum_{i=1}^M \mathbf{w}^T \mathbf{l}^i, j = 1, \dots, M$$

$$z^j \geq 0, j = 1, \dots, M$$

$$\frac{1}{M} \sum_{j=1}^M \mathbf{w}^T \mathbf{l}^j \geq u_e$$

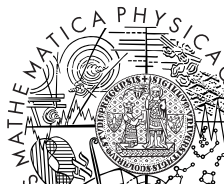
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# Computational part

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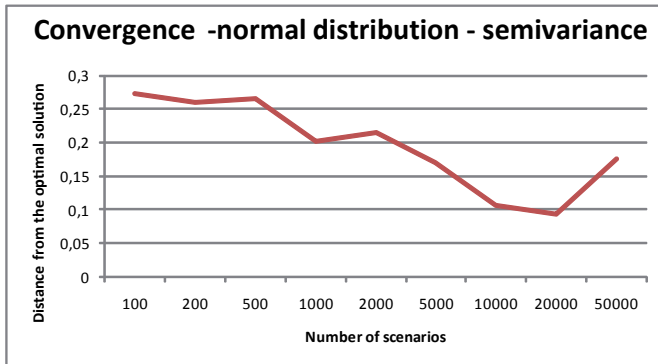
- own software in C++ & GAMS
- software configuration
  - 50 iterations of generating scenarios, average of the solutions found
  - scenarios are always generated from scratch
  - non-negativity condition
  - ML estimators of distribution parameters
  - threshold 95%
  - different minimal expected returns - small, medium, high
- maximal number of scenarios
  - for VaR approx. 1 000 scenarios
  - other risk measures up to 50 000 scenarios
- data used
  - stock market indices
  - Japan, USA, Great Britain, Czech Rep. and Germany
  - from 15.9.2008 to 18.9.2009





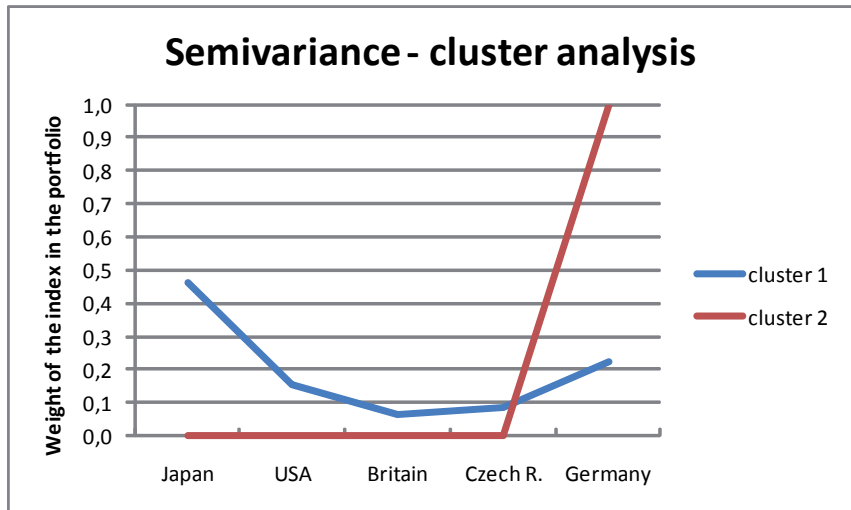
# Convergence issues

- we can experience difficulties with large number of scenarios

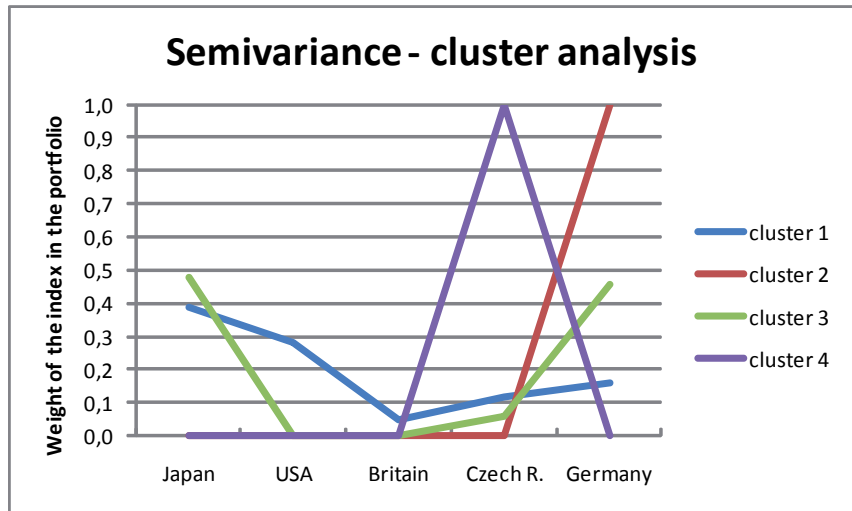


- solutions from repeated iterations form clusters

## Convergence issues - normal distribution



## Convergence issues - Student distribution



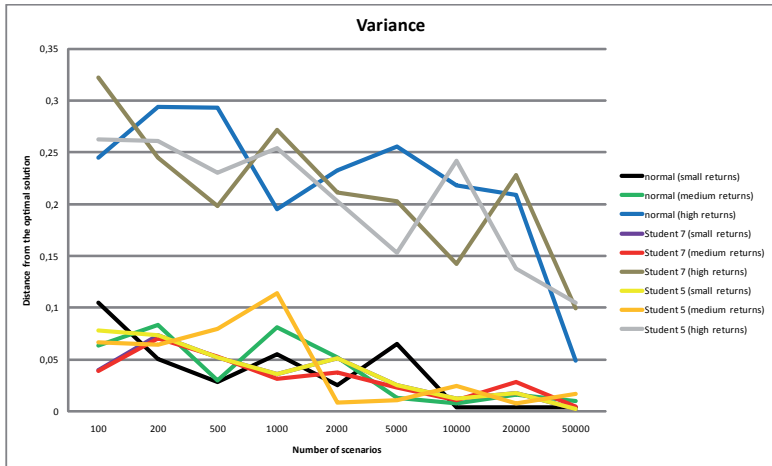
# Cluster analysis

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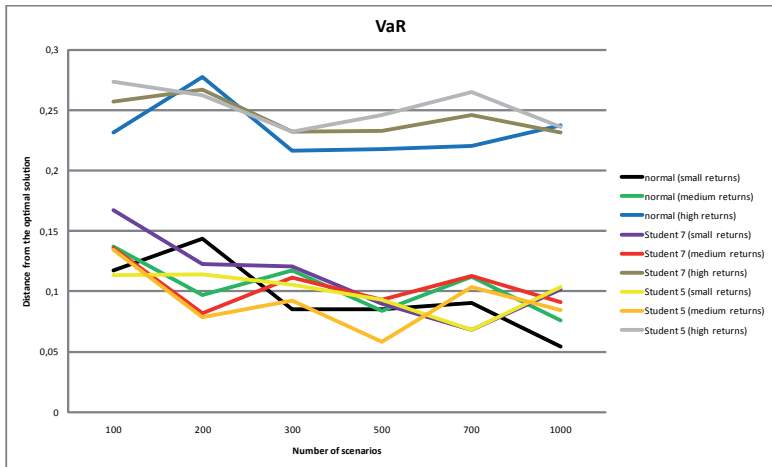
- we use the k-means procedure
- given the number of clusters we choose the mean of largest one as the optimal solution
- how to choose the optimal number of clusters
  - we use Bayes Information Criterion (BIC) modified for multidimensional case
  - if the information criterion decreases while adding more clusters we use the last largest cluster which had more than half of the observations included
- needed mostly for elliptical distributions and semivariance



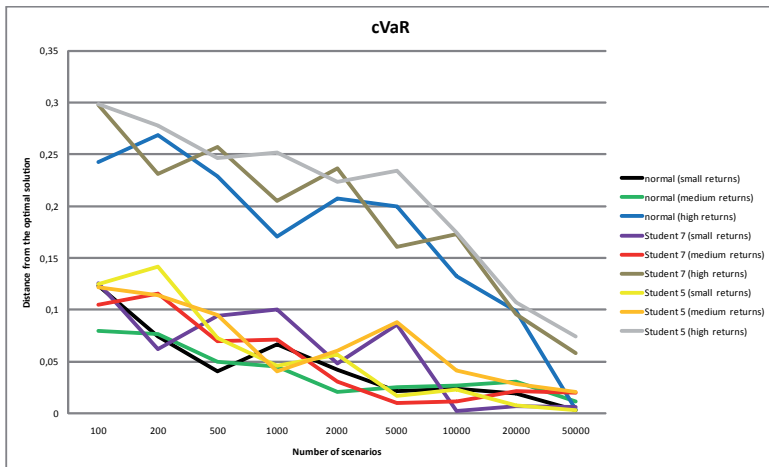
# Variance



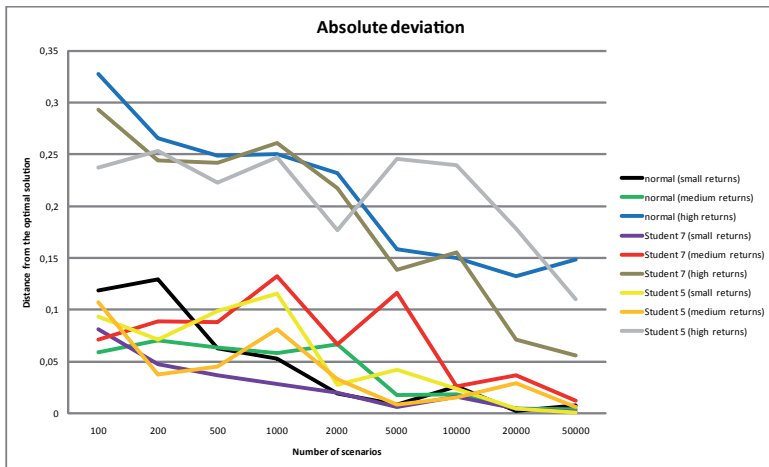
# VaR



# cVaR

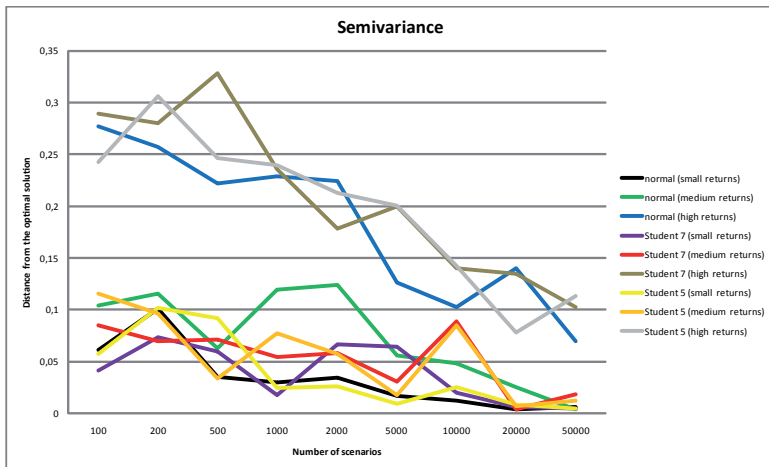


# Absolute deviation

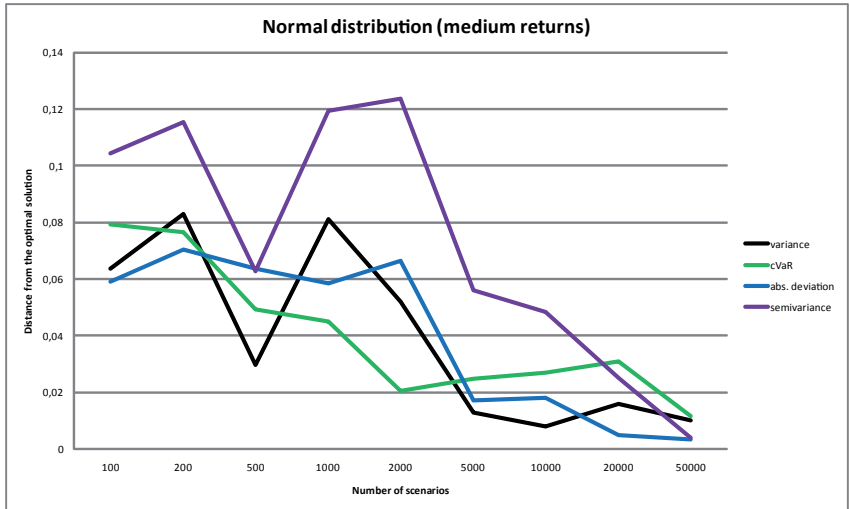




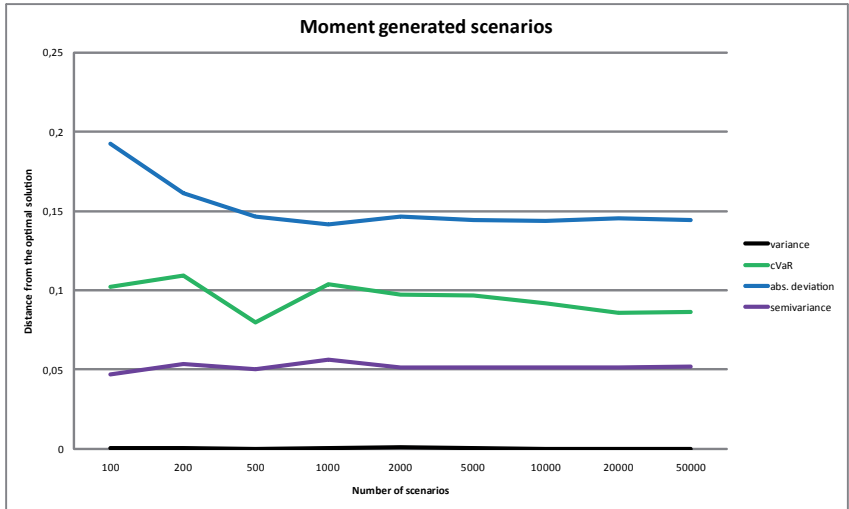
# Semivariance



# Normal distribution



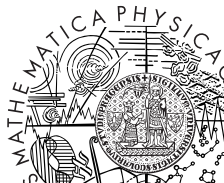
# Moment generated scenarios



# Conclusion

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- the convergence could not be achieved without proper analysis of repeated experiments
- using cluster analysis we achieved convergence for all risk measures
- the convergence properties depend on the minimal expected returns
  - even though we have smaller set of feasible solutions the convergence properties could be worse
- moment generated scenarios should be used only for a small sample of initial scenarios
- best convergence properties
  - small differences between distributions
  - variance or cVaR - fast computation



# Conclusion

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Thank you for your attention!

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