

# Illustrative examples – The newsboy problem

Newsboy sells newspapers for the cost  $c$  each. Before he starts selling, he has to buy the daily supply at the cost  $p$  a paper. The demand is random and the unsold newspapers are returned without refund at the end of the day. How many newspapers should he buy?

In the framework of stochastic programming, one assumes that the demand is random and the verbal description of the newsboy problem leads to the familiar mathematical formulation

$$\max_{x \geq 0} [(c - p)x - cE_P(x - \omega)^+] \quad (1)$$

where  $c > p > 0$  and  $E_P$  denotes expectation with respect to a known probability distribution  $P$  of the random (nonnegative) demand  $\omega$ . The optimal decision is then

$$x(P) = u_{1-\alpha}(P) \quad (2)$$

where  $\alpha = p/c$  and  $u_{1-\alpha}(P)$  is the 100  $(1 - \alpha)\%$  quantile of probability distribution  $P$ .

# Newsboy problem cont.

In practice, the newsboy does not know probability distribution  $P$ . He may base his decision on [historical records](#), or on [a few expert forecasts - scenarios](#), he may use [worst-case analysis](#)

For instance, his decision based on independent identically distributed observed past realizations  $u^\nu$  of  $\omega$ ,  $\nu = 1, \dots, N$ , can be obtained as

$$\arg \max_{x \geq 0} [(c - p)x - \frac{c}{N} \sum_{\nu=1}^N (x - u^\nu)^+] \quad (3)$$

with the optimal solution  $x(P^N)$ , the  $100(1 - \alpha)\%$  quantile of empirical distribution  $P^N$ . [Sample from a censored distribution!](#)

Applicability of this procedure depends on the available sample size  $N$ , in particular for  $\alpha$  near to 0 or 1. With  $0 < \alpha < 1$ , empirical quantiles are asymptotically normal under quite general assumptions and the quantile process can be bootstrapped to obtain an estimate of the variance; consequently, for  $N$  large enough, asymptotical confidence intervals for  $x(P)$  can be constructed. Any additional knowledge about the rules that influence the changes of demand can be in principle incorporated into this procedure.

# Newsboy problem cont.

Alternatively, newsboy can confine himself to a **parametric family** of probability distributions, estimate its parameters from the sample and apply formula (2) for the obtained probability distribution  $P^\#$ . If he was right in his choice of the family (and this assumption seems to be the stumbling block of the approach), parametric analysis plus statistical inference or the worst case analysis with respect to the parameter values can be used to obtain a relationship between the true  $x(P)$  and the obtained  $x(P^\#)$ .

Known (or estimated) range and moments of  $P$  in connection with the **minimax approach** can be used to obtain lower and upper bounds  $L$  and  $U$  on the optimal value of the objective function in (1): one considers a family of probability distributions, say  $\mathcal{P}$  described by the moments values and solves the problem for the “worst” and the “best” distribution of the family.  $\rightarrow$  bounds

$$L(\mathcal{P}) = \max_{x \geq 0} \min_{P \in \mathcal{P}} [(c - p)x - cE_P(x - \omega)^+] \quad (4)$$

$$U(\mathcal{P}) = \max_{x \geq 0} \max_{P \in \mathcal{P}} [(c - p)x - cE_P(x - \omega)^+] \quad (5)$$

such that

# Newsboy problem cont.

$$L(\mathcal{P}) \leq \max_{x \geq 0} [(c - p)x - cE_P(x - \omega)^+] \leq U(\mathcal{P}) \quad \forall P \in \mathcal{P}$$

and, wrt. family  $\mathcal{P}$ , they are tight. The result depends, of course, on the choice of  $\mathcal{P}$ . If (for the chosen family  $\mathcal{P}$ ) the difference between  $L(\mathcal{P})$  and  $U(\mathcal{P})$  is too large, the newsboy should try to collect an additional information about the distribution of demand.

Without any historical records, the newsboy might base his decision on **experts' estimates** of “low” and “high” demand (we can relate these values to the given range of  $P$ ) augmented perhaps by subjective probabilities of these outcomes or by a qualitative information such as ranking probabilities of the outcomes. In the former case, he solves (1) for the corresponding discrete distribution  $P$  and, naturally, he gets interested in the robustness of the obtained decision, its sensitivity on the occurrence of another outcome (scenario), etc. In the later case, the available qualitative information can be used to define the family  $\mathcal{P}$  needed for the worst case analysis.

# Newsboy problem cont.

Finally, the newsboy may prefer **ANOTHER MODEL**

$$\max_{x \geq 0} (c - p)x$$

subject to a reliability type constraint

$$P(x \leq \omega) \geq 1 - \varepsilon.$$

– individual probability constraint which may be written as

$$F(x) \leq \varepsilon \text{ or } x \leq u_\varepsilon(P)$$

with  $u_\varepsilon(P)$  the 100 $\varepsilon$ % quantile of  $P$ . The newsboy chooses the value of  $\varepsilon \in (0, 1)$  and according the problem formulation, it will be a value close to 0. The optimal decision is

$$x(P) = u_\varepsilon(P).$$

Notice, that in both cases, the resulting optimal decision can be obtained by solving simple optimization problem

$$\max_{x \geq 0} (c - p)x \text{ subject to } x \leq \hat{\omega}$$

obtained by replacing the random demand by  $\hat{\omega}$  — a quantile of its probability distribution  $P$ ; NOT expectation  $E_P \omega$ !

# The flower-girl problem

The flower-girl sells roses at  $c$   
buys them in the morning at  $p$

$x_1$  has to be bought on the first day

$\omega_t$ ,  $t = 1, 2, \dots$  (random) demand on the  $t$ -th day

$x_t(\cdot)$ ,  $t = 2, 3 \dots$  order on the  $t$ -th day

Flowers can be sold only for two consecutive days,  
then they are thrown away.

The aim is to maximize the expected profit for the considered number of consecutive days.

## Extension of the newsboy problem.

Simple case of the flower-girl problem:

The flower-girl sells roses only during the weekend,

orders the amount  $x_1$  on Friday evening,

observes the demand  $\omega_1$  on Saturday,

stores the unsold roses (without any additional costs)

and, possibly, orders  $x_2(\omega_1)$  new roses on Saturday evening.

The demand  $\omega_2$  on Sunday then determines the amount of unsold roses to be thrown away.

# The flower-girl problem cont.

Obviously,

$$x_1 - s_2(\omega_1) \leq \omega_1$$

$$x_2(\omega_1) + s_2(\omega_1) - z(\omega_2) \leq \omega_2$$

For  $\omega_1, \omega_2$  known, the objective function is

$$(c - p)(x_1 + x_2(\omega_1)) - cz(\omega_2)$$

Then the choice  $x_1 = \omega_1$  and  $x_2 = \omega_2$  guarantees the maximal profit of  $(c - p)(\omega_1 + \omega_2)$ .

Similar problems — **additional information revealed later on, cannot be used for the first stage decision.**

Private investor, Dedicated bond portfolio problem, Bond portfolio management, etc.

## MULTISTAGE STOCHASTIC PROGRAMS

# Simple case – Two-stage Stochastic Linear Programs

Random objective function  $f(x, \omega)$  comes from a two-stage SLP:

$$\min_{x_1 \in \mathcal{X}_1} E_P f(x, \omega) = c_1^\top x_1 + E_P \varphi_1(x_1, \omega) \quad (6)$$

where for given  $x_1 \in \mathcal{X}_1$  and  $\omega \in \Omega$ ,

$$\varphi_1(x_1, \omega) = \min\{c_2(\omega)^\top x_2(\omega) \mid M(\omega)x_2(\omega) = b(\omega) - A(\omega)x_1, x_2(\omega) \geq 0\}. \quad (7)$$

Common assumption:  $\mathcal{X}_1 \neq \emptyset$ , convex polyhedral and  $c_1$  is deterministic.  
(Nonnegativity of 2nd-stage variables  $x_2(\omega)$  can be replaced by requirement that  $x_2(\omega)$  belong to convex polyhedral cone  $\mathcal{X}_2$ .)

Special instances:

- $M$  is deterministic — fixed recourse
- fixed recourse &  $M(x, A, b) := \{y \geq 0, My = z\} \neq \emptyset$  — complete fixed recourse
- simple recourse — complete fixed recourse with  $M = [I, -I]$