

Multiperiod risk premiums

Miloš Kopa

Dept. of Probability and Mathematical Statistics, Faculty of Mathematics and Physics,
Charles University in Prague, Czech Republic

kopa@karlin.mff.cuni.cz

Univariate risk premium

The traditional Arrow-Pratt measure of (absolute) risk aversion:

$$R_u(x) = -u''(x)/u'(x)$$

Risk aversion:

$$Eu(w + x) < u(w + E(x)) \quad (1)$$

An investor is risk averse $\Leftrightarrow R_u(x) > 0$

The risk averse decision maker is willing to pay a risk premium π to eliminate the risk x .

$$u(w + E(x) - \pi(w, P_x)) = Eu(w + x) \quad (2)$$

$$\pi(w, P_x) > 0 \Leftrightarrow R_u(x) > 0 \Leftrightarrow u \text{ is concave}$$

Univariate risk premium with random initial wealth

$$Eu_w(w + E(x) - \pi(w, P_x)) = Eu_{x,w}(w + x) \quad (3)$$

$$\pi(w, P_x) > 0 \not\Rightarrow R_u(x) > 0$$

Multivariate risk premium I

Suppose a decision maker with utility function $u(\mathbf{w})$ and initial wealth $\mathbf{w} = (w_1, w_2, \dots, w_n)'$.

Assume that $u(\mathbf{w})$ is continuous and increasing in all variables.

The multivariate risk premium π :

$$u(\mathbf{w} + E\mathbf{x} - \pi) = E_{\mathbf{x}} u(\mathbf{w} + \mathbf{x}) \quad (4)$$

The risk aversion at level \mathbf{w} :

$$u(\mathbf{w} + E\mathbf{x}) > E_{\mathbf{x}} u(\mathbf{w} + \mathbf{x}) \quad (5)$$

for any given gamble \mathbf{x} .

Multivariate risk premium II

If u is concave then there exists a nonnegative risk premium for any gamble \mathbf{x} .

If there exists a nonnegative risk premium for any gamble \mathbf{x} then an investor is risk averse.

The absolute risk aversion matrix measure:

$$R(\mathbf{x}) = \frac{-u_{ij}(\mathbf{x})}{u_i(\mathbf{x})}$$

Multivariate risk premium with random initial wealth

The decision maker is indifferent between two random vectors: $(\mathbf{w} - \boldsymbol{\pi})$ and $(\mathbf{w} + \mathbf{x})$:

$$E_{\mathbf{w}} u(\mathbf{w} + E\mathbf{x} - \boldsymbol{\pi}) = E_{\mathbf{w}, \mathbf{x}} u(\mathbf{w} + \mathbf{x}). \quad (6)$$

The decision maker is *risk averse at level \mathbf{w} with respect to gamble \mathbf{x}* if:

$$E_{\mathbf{w}} u(\mathbf{w} + E\mathbf{x}) > E_{\mathbf{w}, \mathbf{x}} u(\mathbf{w} + \mathbf{x}). \quad (7)$$

Example 1

Let $(w_1, x_1, w_2, x_2) = (\frac{1}{2}, 1, 0, -\frac{1}{2})$ or $(1, -1, \frac{1}{2}, \frac{1}{2})$ with the same probabilities.

Consider $u(\mathbf{w}) = \log(w_1 + w_2)$.

$$E_{\mathbf{w}, \mathbf{x}} u(\mathbf{w} + \mathbf{x}) > E_{\mathbf{w}} u(\mathbf{w} + E\mathbf{x}).$$

Moreover if $(w_1, x_1, w_2, x_2) = (\frac{1}{2}, 1, \frac{1}{2}, -\frac{3}{2})$ or $(1, -1, 0, \frac{3}{2})$ with the same probabilities then

$$E_{\mathbf{w}, \mathbf{x}} u(\mathbf{w} + \mathbf{x}) < E_{\mathbf{w}} u(\mathbf{w} + E\mathbf{x}).$$

The *risk premium in the i -th direction* is a solution of (6) with the property that j -th component of π satisfies

$$\begin{aligned} \pi_j &= 0 & j &\neq i \\ &= \hat{\pi}_i & j &= i. \end{aligned}$$

Direction risk premiums

Direction risk premiums $\hat{\pi}_1, \hat{\pi}_2$ in Example 1 are:

$$\begin{aligned}E_{\mathbf{w}, \mathbf{x}} u(\mathbf{w} + \mathbf{x}) &= E_{\mathbf{w}} u(\mathbf{w} - \boldsymbol{\pi}); \quad \boldsymbol{\pi} = (\hat{\pi}_1, 0)' \\0 &= \frac{1}{2} \log \left(\frac{1}{2} - \hat{\pi}_1 \right) + \frac{1}{2} \log \left(\frac{3}{2} \right) \\ \hat{\pi}_1 &= -\frac{1}{6}\end{aligned}$$

$$\begin{aligned}E_{\mathbf{w}, \mathbf{x}} u(\mathbf{w} + \mathbf{x}) &= E_{\mathbf{w}} u(\mathbf{w} - \boldsymbol{\pi}); \quad \boldsymbol{\pi} = (0, \hat{\pi}_2)' \\0 &= \frac{1}{2} \log \left(\frac{1}{2} \right) + \frac{1}{2} \log \left(\frac{3}{2} - \hat{\pi}_2 \right) \\ \hat{\pi}_2 &= -\frac{1}{2}\end{aligned}$$

Concavity of u does not guarantee nonnegativity of directional risk premiums.

Multiperiod risk premium

Let $u(\mathbf{w})$ be an increasing utility function. We interpret the arguments of \mathbf{w} as the random amounts of cash measured at times $1, \dots, n$. It is the vector of initial wealth in each period.

We would like to define i -th element of multiperiod risk premium Π such that the decision maker is indifferent between accepting the gamble x_i and paying $\Pi_i - E\mathbf{x}$ in i -th time period.

The i -th element of multiperiod risk premium depends on the initial wealth at time i and on the probabilistic distribution of \mathbf{x} . The initial wealth w_i depends on w_{i-1} and on the decision of investor at time $i-1$, whether he accepted gamble x_{i-1} or paid $\Pi_{i-1} - Ex_{i-1}$.

We assume that $E\mathbf{x} = 0$.

Multiperiod problem

Finally, we assume that a history of decisions does not depend on \mathbf{x} and all possible histories of decisions are described by the following scenarios where an investor has only two possibilities in each time period: to accept the gamble or to pay risk premium.

$$\begin{array}{l}
 \langle \text{accept } x_1 \langle \begin{array}{l} \text{accept } x_2 \langle \dots \text{accept } x_{n-1} \\ \text{pay } \Pi_2 \langle \dots \text{pay } \Pi_{n-1} \end{array} \\
 \text{pay } \Pi_1 \langle \begin{array}{l} \text{accept } x_2 \langle \dots \vdots \\ \text{pay } \Pi_2 \langle \dots \text{accept } x_{n-1} \\ \text{pay } \Pi_{n-1} \end{array}
 \end{array}$$

If the decision maker accepts a gamble in i -th time period then let $k_i^s = 1$ else $k_i^s = 0$. The scenario s is represented by vector

$$K^s = (k_1^s, k_2^s, \dots, k_{n-1}^s)$$

Multiperiod problem

The initial wealth in j -th time period along scenario s :

$$w_j^s = w_1 + \sum_{i=1}^{j-1} [k_i^s x_i - (1 - k_i^s) \Pi_i]. \quad (8)$$

In a formal way, we would like to define multiperiod risk premium by the system of equations:

$$E_x u(\mathbf{w}^s + \mathbf{x}) = E_x u(\mathbf{w}^s - \mathbf{\Pi}) \quad \forall s \in S. \quad (9)$$

However, this system of 2^{n-1} equations and n variables does not usually have a solution unless $n \leq 2$.

Multiperiod risk premium

Given \mathbf{x} , let

$$f^s(\boldsymbol{\Pi}) = |E_{\mathbf{x}} u(\mathbf{w}^s + \mathbf{x}) - E_{\mathbf{x}} u(\mathbf{w}^s - \boldsymbol{\Pi})|$$

We are interested to find $\boldsymbol{\Pi}$ which minimizes $f^s(\boldsymbol{\Pi})$ jointly for all $s \in S$.

We are looking for a vector which minimizes the maximal value of $f^s(\boldsymbol{\Pi})$:

$$\min_{\boldsymbol{\Pi}} \max_{s \in S} f^s(\boldsymbol{\Pi})$$

Multiperiod risk premium

An equivalent form:

$$\begin{aligned} & \min_{\Pi} d \\ \text{s.t.} \quad & f^s(\Pi) \leq d \quad \forall s \in S. \end{aligned} \tag{10}$$

The *multiperiod risk premium* is a solution of the problem:

$$\begin{aligned} & \min_{\Pi} d \\ \text{s.t.} \quad & -d \leq E_{\mathbf{x}} u(\mathbf{w}^s + \mathbf{x}) - E_{\mathbf{x}} u(\mathbf{w}^s - \Pi) \leq d \quad \forall s \in S. \end{aligned} \tag{11}$$

Multiperiod risk aversion

The decision maker is *multiperiod risk averse at wealth level \mathbf{w} with respect to gamble \mathbf{x}* if

$$E_{\mathbf{x}}u(\mathbf{w}^s + \mathbf{x}) < E_{\mathbf{x}}u(\mathbf{w}^s) \quad \forall s \in S. \quad (12)$$

We define *i -th directional multiperiod risk premium $\hat{\Pi}_i$* as a solution of the following problem:

$$\begin{aligned} \min_{\Pi} \quad & d \\ \text{s.t.} \quad & -d \leq E_{\mathbf{x}}u(\mathbf{w}^s + \mathbf{x}) - E_{\mathbf{x}}u(\mathbf{w}^s - \Pi) \leq d \quad \forall s \in S \\ & \Pi_j = 0 \quad j \neq i. \end{aligned} \quad (13)$$

Theorem 1:

If the decision maker is risk averse at wealth level \mathbf{w} with respect to gamble \mathbf{x} then all directional risk premiums are positive.

Sketch of the proof:

Choose $i \in \{1, 2, \dots, n\}$. Let Π_i^s be a solution of equation:

$E_{\mathbf{x}} u(\mathbf{w}^s + \mathbf{x}) = E_{\mathbf{x}} u(\mathbf{w}^s - \Pi)$ under conditions: $\Pi_j = 0$ for all $j \neq i$.

Assumption of risk aversion at wealth level \mathbf{w} with respect to gamble \mathbf{x} is equivalent to positivity of Π_i^s for all $s \in S$. Let

$$\bar{\Pi}_i = \min_{s \in S} \Pi_i^s.$$

It is easy to show that

$$f^s(\Pi) = |E_{\mathbf{x}} u(\mathbf{w}^s + \mathbf{x}) - E_{\mathbf{x}} u(\mathbf{w}^s - \Pi)|$$

is a decreasing function in variable Π_i on $(-\infty, \bar{\Pi}_i)$ for all $s \in S$ under conditions: $\Pi_j = 0$ for all $j \neq i$.

Therefore $\hat{\Pi}_i \geq \bar{\Pi}_i > 0$.

Let A be the set of considered time periods in multiperiod risk premium construction. If $i \in A$ then let $y_i = -\Pi_i$ else $y_i = x_i$. We will denote by S_A the subset of S which consist of scenarios with the property: if $i \in \{1, 2, \dots, n\} \setminus A$ then $k_i^s = 1$.

Partial multiperiod risk premium

The *partial multiperiod risk premium* Π^A is a solution of the problem:

$$\begin{aligned} \min_{\Pi^A} \quad & d \\ \text{s.t.} \quad & -d \leq E_{\mathbf{x}} u(\mathbf{w}^s + \mathbf{x}) - E_{\mathbf{x}} u(\mathbf{w}^s + \mathbf{y}) \leq d \quad \forall s \in S_A \\ & y_i = -\Pi_i^A \quad i \in A \\ & y_i = x_i \quad i \notin A. \end{aligned} \tag{14}$$

Example 2

Consider $u(w_1, w_2, w_3) = \log(w_1 + w_2 + w_3)$. Let x_1, x_2, x_3 be an independent random variables: $x_i = \pm \frac{1}{2}$ with the same probabilities, $i = 1, 2, 3$. Finally, set $w_1 = 2$.

It is clear that S consists of four scenarios: $s_1 \sim (1, 1)$, $s_2 \sim (1, 0)$, $s_3 \sim (0, 1)$ and $s_4 \sim (0, 0)$.

The multiperiod risk premium is:

$$\Pi = (1.252, 1.27, -2.319) \text{ and } d^* = 6.10^{-4}.$$

Example 2

The direction risk premiums:

$$\hat{\Pi}_1 = 0.1367 \quad \text{and} \quad d^* = 0.0124$$

$$\hat{\Pi}_2 = 0.1368 \quad \text{and} \quad d^* = 0.0124$$

$$\hat{\Pi}_3 = 0.1368 \quad \text{and} \quad d^* = 0.0124$$

The partial multiperiod risk premium:

we assume that the insurance possibility does not exist in the second period, i.e. $A = \{1, 3\}$. Thus $\mathbf{y} = (-\Pi_1, x_2, -\Pi_3)$.

$$\Pi_1 = 1.638 \quad \text{and} \quad \Pi_3 = -1.5$$

Modification for random initial wealth

We define the *multi-period risk premium for random w_1* as a solution of the problem:

$$\begin{aligned} & \min_{\Pi} \quad d \\ \text{s.t.} \quad & -d \leq E_{\mathbf{x}, w_1} u(\mathbf{w}^s + \mathbf{x}) - E_{\mathbf{x}, w_1} u(\mathbf{w}^s - \Pi) \leq d \quad \forall s \in S. \end{aligned}$$

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