

Stability of Optimal Portfolios

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Portfolio selection problem

Suppose that an investor wishes to allocate his wealth among assets $i = 1, \dots, n$ and he chooses $\mathbf{x} = (x_1, \dots, x_n)'$ to maximize the expected utility of final wealth. This model will be formulated as:

$$\begin{aligned} \max \quad & Eu(x_0 + \boldsymbol{\varrho}'\mathbf{x}) \\ \text{subject to: } & \mathbf{1}'\mathbf{x} = x_0 \\ & x_i \geq 0 \end{aligned} \tag{1}$$

x_0 ... the initial wealth

$\boldsymbol{\varrho}$... the random vector of returns for unit of wealth

\mathbf{x} ... the investment strategy

u ... the utility function

Portfolio selection problem - multiplicative approach

Assuming multiplicative approach, we could also formulate the problem as:

$$\begin{aligned} \max \quad & Eu(\varrho' \mathbf{x} x_0) \\ \text{subject to : } & \mathbf{1}' \mathbf{x} = 1 \\ & x_i \geq 0, \end{aligned} \tag{2}$$

The basic analysis of utility functions of Arrow and Pratt offer an intuitive way of looking at absolute and relative risk aversion coefficients. The Arrow-Pratt coefficient of *absolute risk aversion*, also called *absolute risk aversion (ARA) measure*, is defined as

$$r(x) = -\frac{u''(x)}{u'(x)} \quad (3)$$

for increasing, twice differentiable utility function u .

Stability of optimal portfolios with Rubinstein measure

Kallberg, Ziemba (1983) proved that investors with the same Rubinstein measure of global risk aversion, defined as:

$$r_g(x_0) = -\frac{x_0 E[u''(w)]}{E[u'(w)]} \quad (4)$$

where $w = x_0 \boldsymbol{\varrho}' \mathbf{x}$, have the same optimal investment strategies, i.e. the same optimal solutions of (2), under the additional assumption that $\boldsymbol{\varrho}' \mathbf{x}$ is normally distributed.

Assumptions

- (i) There exists such an interval $\langle a, b \rangle$ that $P(x_0 + \boldsymbol{q}'\mathbf{x} \in \langle a, b \rangle) = 1$ for any choice of $x_i \geq 0, i = 1, \dots, n$, satisfying: $\mathbf{1}'\mathbf{x} = x_0$.
- (ii) Utility functions $u(x), u_1(x), u_2(x), \dots$ are increasing, twice differentiable on $\langle a, b \rangle$ and $r(x), r_1(x), r_2(x), \dots$ are corresponding ARA measures.
- (iii) $\lim_{n \rightarrow \infty} r_n(x) = r(x) \quad \forall x \in \langle a, b \rangle$,
- (iv) $u''(x), u_k''(x), k = 1, 2, \dots$ are continuous and negative in interval $\langle a, b \rangle$.

Notation:

$$\begin{aligned}X &= \{\mathbf{x} = (x_1, x_2, \dots, x_n) : \mathbf{1}'\mathbf{x} = x_0, x_i \geq 0, i = 1, 2, \dots, n\} \\X^k &= \arg \max_{\mathbf{x} \in X} Eu_k(x_0 + \boldsymbol{\varrho}'\mathbf{x}) \\X^* &= \arg \max_{\mathbf{x} \in X} Eu(x_0 + \boldsymbol{\varrho}'\mathbf{x}).\end{aligned}$$

Let us denote by \mathbf{x}^k the element of X^k .

Utility functions: $u_1(x), \quad u_2(x), \quad u_3(x), \quad \dots \quad u(x)$

ARA measures: $r_1(x), \quad r_2(x), \quad r_3(x), \quad \longrightarrow \quad r(x)$

Optimal solutions: $\mathbf{x}^1, \quad \mathbf{x}^2, \quad \mathbf{x}^3, \quad \dots \quad \mathbf{x}^*$

Sets of opt. solutions: $X^1, \quad X^2, \quad X^3, \quad \dots \quad X^*$

Proposition 1

Let assumptions (i) - (iii) hold. Then

$$\lim_{k \rightarrow \infty} Eu(x_0 + \varrho' \mathbf{x}^k) - Eu(x_0 + \varrho' \mathbf{x}^*) = 0,$$

$$\lim_{k \rightarrow \infty} Eu_k(x_0 + \varrho' \mathbf{x}^k) - Eu_k(x_0 + \varrho' \mathbf{x}^*) = 0, \quad k = 1, 2, \dots,$$

where $\mathbf{x}^k \in X^k$ and $\mathbf{x}^* \in X^*$.

Proposition 2

Let assumptions (i) - (iv) hold. Then from any sequence $\mathbf{x}^1, \mathbf{x}^2, \dots$, where $\mathbf{x}^k \in X^k$, $k = 1, 2, \dots$, a subsequence $\mathbf{x}^{k_1}, \mathbf{x}^{k_2}, \dots$ can be extracted such that

$$\varrho' \mathbf{x}^{k_n} \xrightarrow{k_n \rightarrow \infty} \varrho' \mathbf{x}^* \quad \text{a.s.} \quad \text{and} \quad \mathbf{x}^* \in X^*.$$

To simplify notation, set

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \left(\frac{\partial f(\mathbf{x})}{\partial x_1}, \frac{\partial f(\mathbf{x})}{\partial x_2}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n} \right).$$

By Taylor's formula, we have:

$$-Eu(x_0 + \varrho' \mathbf{x}^k) = -Eu(x_0 + \varrho' \mathbf{x}^*) + A + B \quad (5)$$

where

$$A = -\frac{\partial Eu(x_0 + \varrho' \mathbf{x}^*)}{\partial \mathbf{x}} (\mathbf{x}^k - \mathbf{x}^*)$$

Since \mathbf{x}^* is an optimal solution of (1), we obtain

$$A = -(\lambda^* \cdot \mathbf{1} - \boldsymbol{\eta}^*)' (\mathbf{x}^k - \mathbf{x}^*) = \boldsymbol{\eta}^{*'} \mathbf{x}^k \geq 0. \quad (6)$$

By assumption (iv), $\xi > 0$ exists such that

$$B \geq \frac{\xi}{2} E [\varrho'(\mathbf{x}^k - \mathbf{x}^*)]^2 \quad (7)$$

$$E [\varrho'(\mathbf{x}^k - \mathbf{x}^*)]^2 \xrightarrow{k \rightarrow \infty} 0. \quad (8)$$

Corollary 3

Let assumptions (i) - (iv) hold. Assume that $\mathbf{x}^1, \mathbf{x}^2, \dots$ where $\mathbf{x}^k \in X^k$, $k = 1, 2, \dots$, is a Cauchy sequence. Then

$$\boldsymbol{\varrho}'\mathbf{x}^k \xrightarrow{k \rightarrow \infty} \boldsymbol{\varrho}'\mathbf{x}^* \quad \text{a.s.} \quad \text{and} \quad \mathbf{x}^* \in X^*.$$

Set

$$\begin{aligned} Y &= \{\mathbf{y} \in R^n : \mathbf{1}'\mathbf{y} = 0, \mathbf{y} \neq \mathbf{0}\}, \\ \mathcal{P} &= \{\boldsymbol{\varrho} : \exists \delta > 0 : P(\boldsymbol{\varrho} = \mathbf{0}) \leq 1 - \delta \\ &\quad \text{and } P(\boldsymbol{\varrho}'\mathbf{y} = 0) \leq 1 - \delta \quad \forall \mathbf{y} \in Y\}. \end{aligned}$$

In the remainder of this paper we assume that:

$$P(\boldsymbol{\varrho} = \mathbf{0}) < 1.$$

Let

$$\begin{aligned}\overline{Y}_{\boldsymbol{\varrho}} &= \{\mathbf{y} \in Y : P(\boldsymbol{\varrho}'\mathbf{y} = 0) = 1\} && \text{for } \boldsymbol{\varrho} \notin \mathcal{P} \\ &= \emptyset && \text{for } \boldsymbol{\varrho} \in \mathcal{P}.\end{aligned}$$

Proposition 4

Let assumptions (i) - (iv) hold. Let $\varrho \in \mathcal{P}$. Then

- (a) portfolio selection problem (1) has a unique solution using $u(x), u_k(x), k = 1, 2, \dots$
- (b) from the sequence $\mathbf{x}^1, \mathbf{x}^2, \dots$, where $\mathbf{x}^k \in X^k$, $k = 1, 2, \dots$, a Cauchy subsequence $\mathbf{x}^{l_1}, \mathbf{x}^{l_2}, \dots$ can be extracted such that

$$\mathbf{x}^{l_n} \xrightarrow{l_n \rightarrow \infty} \mathbf{x}^* \quad \text{and} \quad \mathbf{x}^* \in X^*.$$

Proposition 5

Let assumptions (i) - (iv) hold. Let X^* be a singleton. Then from the sequence $\mathbf{x}^1, \mathbf{x}^2, \dots$, where $\mathbf{x}^k \in X^k$, $k = 1, 2, \dots$, a Cauchy subsequence $\mathbf{x}^{l_1}, \mathbf{x}^{l_2}, \dots$ can be extracted such that

$$\mathbf{x}^{l_n} \xrightarrow{l_n \rightarrow \infty} \mathbf{x}^* \quad \text{and} \quad \mathbf{x}^* \in X^*.$$

Lemma 6: Structure of the optimality sets

Assume that $\mathbf{x}^* \in X^*$, $\mathbf{x}^k \in X^k$, $k = 1, 2, \dots$, are fixed.

Let $Z^k = \{\mathbf{z} \in R^n : \mathbf{z} = \mathbf{x}^k + \mathbf{y}; \mathbf{y} \in \overline{Y}_{\underline{\rho}}\}$, $k = 1, 2, \dots$,

$$Z^* = \{\mathbf{z} \in R^n : \mathbf{z} = \mathbf{x}^* + \mathbf{y}; \mathbf{y} \in \overline{Y}_{\underline{\rho}}\}.$$

Then $X^k = Z^k \cap X$, $k = 1, 2, \dots$ and $X^* = Z^* \cap X$.

Theorem 7

Let assumptions (i) - (iv) hold. Then

$$\limsup_{k \rightarrow \infty} d_h(X^k, X^*) = 0.$$

where *Hausdorf distance* between two sets, A and B is defined as:

$$d_h(A, B) = \max\{\max_{a \in A} d(a, B), \max_{b \in B} d(b, A)\}$$

$$\text{where } d(p, Q) = \min_{q \in Q} d(p, q)$$

and $d(p, q)$ is the Euclidean distance from p to q .

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