

Multiple Objectives

Very often, plausible economic decisions cannot be chosen only according to one criterion, such as the maximal profit, production efficiency or yield. In production planning, environmental criteria have to be taken into account, in macroeconomical problems regional aspects such as the local unemployment level play an essential role. Very different, even conflicting goals can be set for a short time horizon and for the long one, etc. The mentioned disparate criteria will hardly be satisfied by a uniformly optimal decision.

Investment decisions should hedge against risks of various kinds, such as liquidity, volatility or currency risks.

Problems of the above kind belong under *multi-objective programming*. We shall briefly introduce main approaches for the case of continuous decision variables and illustrate them in the context of portfolio optimization and risk management.

The problem:

“minimize” $K \geq 2$ functions f_1, \dots, f_K , $f_k : R^n \rightarrow R^1$ on a closed set \mathcal{X}
briefly

“min” $\mathbf{f}(\mathbf{x})$ on \mathcal{X} .

Ideal solution $\tilde{\mathbf{x}} \in \mathcal{X}$ of the multi-objective programming problem:

$$\tilde{\mathbf{x}} \in \bigcap_{k=1}^K \arg \min_{\mathbf{x} \in \mathcal{X}} f_k(\mathbf{x}). \quad (1)$$

Ideal solutions exist only rarely.

Definition. Solution $\hat{\mathbf{x}} \in \mathcal{X}$ is an **efficient solution** of multi-objective problem if there is no element $\mathbf{x} \in \mathcal{X}$ for which $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\hat{\mathbf{x}})$ and $\mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\hat{\mathbf{x}})$.

How to get efficient solutions?

Theorem. Let \mathcal{X} be compact, $f_k, k = 1, \dots, K$, continuous on \mathcal{X} and $h : R^K \rightarrow R^1$ arbitrary continuous function nondecreasing in its arguments. Then at least one solution belonging to

$$\mathcal{X}_h^* := \arg \min_{\mathbf{x} \in \mathcal{X}} h(f_1(\mathbf{x}), \dots, f_K(\mathbf{x}))$$

is efficient for the multi-objective problem (1).

Comment

Optimal solution and numerical tractability of the optimization problem depend substantially on the choice of function h .

It would not be necessary to deal with multi-objective programming if the choice of h was straightforward.

Special simple choice of function h

$$h(\mathbf{z}) = \sum_{k=1}^K t_k z_k \text{ with vector parameter } \mathbf{t} \in R_+^K, \mathbf{t} \neq 0.$$

Using all possible choices of weights \mathbf{t} , one may generate *all efficient solutions* of the solved multi-objective problem:

Theorem. For vector parameter $\mathbf{t} \in R_+^K$, let $\bar{\mathbf{x}}$ be an optimal solution of

$$\min_{\mathbf{x} \in \mathcal{X}} \sum_{k=1}^K t_k f_k(\mathbf{x}). \quad (2)$$

Assume that parameter vector \mathbf{t} is positive OR that $\bar{\mathbf{x}}$ is unique optimal solution of (2). Then $\bar{\mathbf{x}}$ is efficient solution of the multi-objective program (1).

For *convex* f_k , $\forall k$ and for \mathcal{X} nonempty, convex, compact \implies for arbitrary efficient solution $\bar{\mathbf{x}}$ of (1) there exists $\mathbf{t} \in R_+^K$ so that $\bar{\mathbf{x}}$ is optimal solution of (2).

Stronger result holds true for *linear* functions f_k , $\forall k$ and a convex polyhedral set \mathcal{X} :

$\hat{\mathbf{x}}$ is an efficient solution of the multi-objective problem (1) \iff it is an optimal solution of (2) for a *positive* parameter vector $\mathbf{t} \in R_+^K$.

\exists other approaches that provide efficient solutions

ϵ -Constrained Approach

Select one of considered objective functions, say, f_1 , choose a threshold vector $\epsilon \in R^{K-1}$ and solve the classical optimization problem

$$\text{minimize } f_1(\mathbf{x}) \text{ subject to } \mathbf{x} \in \mathcal{X} \text{ and } f_k(\mathbf{x}) \leq \epsilon_k, k = 2, \dots, K. \quad (3)$$

If the set $\mathcal{X}_\epsilon := \{\mathbf{x} \in \mathcal{X} | f_k(\mathbf{x}) \leq \epsilon_k, k = 2, \dots, K\} \neq \emptyset$, then

- (i) the unique optimal solution $\bar{\mathbf{x}}$ of (3) is an efficient solution of (1).
- (ii) Let $\hat{\mathbf{x}}$ be an efficient solution of (1). Then there exists $\epsilon \in R^{K-1}$ such that $\hat{\mathbf{x}}$ is an optimal solution of (3).

Mixed Approach is related to a different treatment of distinct objective functions: some of them are put into constraints as in the ϵ -constrained approach whereas weights $t_k > 0$ are assigned to the remaining objective functions.

Goal Programming

The main idea: Get solution from \mathcal{X} for which the outcome measured by vector $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_K(\mathbf{x}))^\top$ is as close as possible to the K -vector of the best attainable outcomes $f_k^* = \min_{\mathbf{x} \in \mathcal{X}} f_k(\mathbf{x})$, $k = 1, \dots, K$.

The distances are defined in the space of *function values*, a subset of R^K , and one is free to use any of (weighted) L_p -distances, $1 \leq p \leq \infty$.

Hence, one solves a minimization problem

$$\min_{\mathbf{x} \in \mathcal{X}} \|\mathbf{T}(\mathbf{f}^* - \mathbf{f}(\mathbf{x}))\|_p \quad (4)$$

with a diagonal matrix $\mathbf{T} = \text{diag}\{t_1, \dots, t_K\}$, $t_k > 0 \forall k$.

The following statements are easy to prove:

(i) Let $\bar{\mathbf{x}}$ be an optimal solution of (4) with $1 \leq p < \infty$. Then $\bar{\mathbf{x}}$ is an efficient solution of (1).

(ii) For $p = \infty$, at least one of optimal solutions of the minimax problem

$$\min_{\mathbf{x} \in \mathcal{X}} \max_k t_k |f_k^* - f_k(\mathbf{x})|$$

is efficient for (1).

Techniques of multi-objective programming allow to exclude “bad”, non-efficient solutions of the multi-objective problems (1) provided that the selected criteria can be quantified and that the considered objective functions are a priori given the same importance.

In interactive numerical approaches the user is allowed to change the values of parameters (weights and thresholds) to achieve an acceptable balance between the criteria. There are various other problems, e.g., an appropriate treatment of hierarchically ordered objective functions or the case of a finite list of feasible alternatives.

Applications in finance and in modeling stochastic decision problems.

BOND PORTFOLIO IMMUNIZATION

Select the cheapest set of fixed-income securities, bonds, to pay future liability. “Future value of portfolio at time T must cover value of liabilities.”

Use present values — total discounted cash flow of portfolio, discounted value of debt.

Denote J set of available bonds j , x_j amount of bond j ,

PV_j their present values, c_j acquisition prices,

PV_L present value of debt, l_j , u_j limits on amount of bonds in portfolio.

—→

simple linear program

$$\min \mathbf{c}^\top \mathbf{x} \text{ subject to } \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}, \sum_j PV_j x_j \geq PV_L. \quad (5)$$

Discount factors are not fixed — main source of uncertainties for default free bonds.

Present Value and Duration

Assume for simplicity default free bonds with fixed cash flows $f_j\tau$ (coupons and nominal value) at time instances τ . For flat yield curve, present value at time t just after the coupon was paid equals the total cashflow in subsequent time instances discounted to t :

$$PV_{jt}(r) = \sum_{\tau=t+1}^T \frac{f_j\tau}{(1+r)^\tau}. \quad (6)$$

Formula (6) may be extended for the accrued interest and also modified to capture the time evolution of interest rates. Their changes — main risk factor of the sector of risk free securities.

The suggestion was to quantify this risk by **duration**; out of many definitions of duration,

dollar duration D_{jt} of bond j at time t is derivative with respect to r of the value $PV_{jt}(r)$ and it quantifies (up to the derivatives of the first order) the change in (6) due to a *small parallel shift* of the yield curve.

Equal duration of portfolio and of liabilities \longrightarrow immunized portfolio.

Extend linear program (5) for duration constraint

$$\sum_j D_j x_j = D_L.$$

Generalize formula (6):

Given short term interest rates r_h valid for time interval $(t, t + 1]$ formula (6) changes to

$$PV_{jt}(\mathbf{r}) = \sum_{\tau=t+1}^T f_{j\tau} \prod_{h=t}^{\tau-1} (1 + r_h)^{-1}. \quad (7)$$

Interest rates scenarios

T -dimensional vectors \mathbf{r} of interest rates are random, their finitely many realizations — scenarios are indexed as \mathbf{r}^s , $s = 1, \dots, S$ with probabilities $p_s > 0 \forall s$, $\sum_s p_s = 1$.

The same indices for present values and durations.

For example, present value of portfolio \mathbf{x} under scenario s is $PV^s(\mathbf{x}) = \sum_j PV_{jt}(\mathbf{r}^s)x_j$, etc.

Denote v^s the minimal cost of portfolio if scenario s occurs — the optimal value of the LP (5) extended for duration constraint,

$PV^s(\mathbf{x})$, PV_L^s present values of portfolio \mathbf{x} and of the liabilities under scenario s

$D_j^s(\mathbf{x})$, D_L^s durations of portfolio \mathbf{x} and of liabilities under scenario s

The decision must not depend on the unknown future development of interest rates, so called *nonanticipativity condition*.

Tracking model

To hedge against all considered evolutions \mathbf{r}^s of interest rates as much as possible,

use the *goal programming approach*.

The resulting model (cf. Dembo) is

$$\text{minimize } \sum_{s=1}^S p_s [\|\mathbf{c}^\top \mathbf{x} - v^s\| + \|PV^s(\mathbf{x}) - PV_L^s\| + \|D^s(\mathbf{x}) - D_L^s\|]$$

subject to $\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$

Additional constraints, e.g. a budget constraint, can be included.

Problem of a private investor I.

Investor wishes to raise enough money for his child college education N years from now by investing w into some of I considered investments.

Tuition goal is g ,

exceeding g after N years \longrightarrow additional income of $q\%$ of the excess,

not meeting the goal \longrightarrow borrowing at the rate $r > q$.

Investor plans to *revise* his investment at certain time instances prior to N using additional information that will be available in future.

Time instances (and the corresponding time periods) are indexed by $t = 1, \dots, T - 1$, and horizon N corresponds to $t = T$.

Main uncertainty: returns $\rho_i(t, \omega)$ on investments i in each period t depend on an underlying *random element* ω and are observable at the end period t . Investment decisions $x_i(t, \omega)$ made at the beginning of period t can be only based on the already observed part of the trajectory of ω , i.e., they are *nonanticipative* of future outcomes.

\implies at the beginning of 1st period, investment decisions $x_i(1, \omega) = x_i(1)$ are fixed for all ω belonging to a probability space (Ω, \mathcal{F}, P) .

Problem of a private investor II.

Assume first that trajectory of random data $\omega = (\omega_1, \dots, \omega_T)$ is known.

\implies known values of the returns $\rho_i(t) \forall i, t$ and the investment problem can be formulated as

$$\text{maximize } qy^+ - ry^-$$

subject to

$$\sum_i x_i(1) = w, \quad x_i(t) \geq 0 \quad \forall i, t$$

$$\sum_i \rho_i(t)x_i(t) - \sum_i x_i(t+1) = 0, \quad t = 1, \dots, T-1$$

$$\sum_i \rho_i(T)x_i(T) - y^+ + y^- = g$$

variables $y^+ \geq 0, y^- \geq 0$ denote surplus resp. deficit.

Different investment decisions for different scenarios — EXAMPLE

Problem of a private investor III.

With $T = 2$ and with *discrete distribution* P concentrated on a finite number of atoms (scenarios) $\omega^s, s = 1, \dots, S$ with probabilities $p_s > 0 \forall s, \sum_s p_s = 1$ we denote $x_i(2, \omega^s), y^+(\omega^s), y^-(\omega^s)$ *scenario dependent second-stage decision variables* and solve the following **two-stage stochastic program**:

$$\text{maximize} \quad \sum_s p_s [qy^+(\omega^s) - ry^-(\omega^s)]$$

subject to

$$\sum_i x_i(1) = w, \quad x_i(1) \geq 0 \quad \forall i$$

$$\sum_i \rho_i(1, \omega^s) x_i(1) - \sum_i x_i(2, \omega^s) = 0 \quad \forall s$$

$$\sum_i \rho_i(2, \omega^s) x_i(2, \omega^s) - y^+(\omega^s) + y^-(\omega^s) = g \quad \forall s$$

$$x_i(2, \omega^s) \geq 0 \forall i, s, \quad y^+(\omega^s) \geq 0, y^-(\omega^s) \geq 0 \quad \forall s.$$

To solve this problem (now an ordinary linear program) one has to fix the values of w, g, r, q and the returns $\rho_i(t, \omega^s)$ for all investments, all scenarios and for $t = 1, 2$.

The obtained 1st stage decision does not depend on scenarios; this allows to *hedge* (through optimally chosen second-stage decisions $x_i(2, \omega^s), y^+(\omega^s), y^-(\omega^s)$) against the considered future returns so that the expected value of the final outcome is maximal.

Notice that the problem is always feasible; this is due to the assumed unlimited possibilities of borrowing.

Investment strategies may be explicitly constrained by cash limitations, nonnegativity conditions and structural constraints; nonanticipativity enters implicitly.