

Theory of coherent risk measures and deviation measures

Miloš Kopa

Department of Probability and Mathematical Statistics
Faculty of Mathematics and Physics
Charles University in Prague
Czech Republic

Previous lecture

Measure of risk assigns a real number to any random variable L (loss).

Favorite risk measures:

- variance: $\text{var}(L) = \mathbb{E}(L - \mathbb{E}L)^2$
- standard deviation $sd(L) = (\mathbb{E}(L - \mathbb{E}L)^2)^{\frac{1}{2}}$
- semivariance: $r_s(L) = \mathbb{E} \left[\max(0, L - \mathbb{E}L)^2 \right]$
- mean absolute deviation: $r_a(L) = \mathbb{E}|L - \mathbb{E}L|$
- mean absolute semideviation: $r_{as}(L) = \mathbb{E}[\max(0, L - \mathbb{E}L)]$
- Value at Risk (VaR):
$$\text{VaR}_\alpha(L) = \inf \{ l \in \mathbb{R}, \mathbb{P}(L > l) \leq 1 - \alpha \}$$
- Conditional Value at Risk (CVaR):
$$\text{CVaR}_\alpha(L) = \inf \left\{ a \in \mathbb{R}, a + \frac{1}{1-\alpha} \mathbb{E}[\max(0, L - a)] \right\},$$

alternatively:
$$\text{CVaR}_\alpha(L) = \beta \mathbb{E}(L | L > \text{VaR}_\alpha(L)) + (1 - \beta) \text{VaR}_\alpha(L) \text{ with}$$

some $\beta \in [0, 1]$

Questions for today

Risk measures:

- What are the “reasonable” properties that should have all “good” risk measures?
- Which of the considered measures has the properties?
- Is it possible to generalize a very well-known and popular standard deviation (variance)?
- What is the dual expression of measures with these properties?

Multiobjective optimization:

- How to formulate an optimization problem when multiple objective are considered?
- What are the best solutions of such problems?
- How to find all these best solutions?

Coherent risk measures

CRM: $\mathcal{R} : \mathcal{L}_2(\Omega) \rightarrow (-\infty, \infty]$ that satisfies

- (R1) Translation equivariance: $\mathcal{R}(L + C) = \mathcal{R}(L) + C$ for all L and constants C ,
- (R2) Positive homogeneity: $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda L) = \lambda \mathcal{R}(L)$ for all L and all $\lambda > 0$,
- (R3) Subadditivity: $\mathcal{R}(L + M) \leq \mathcal{R}(L) + \mathcal{R}(M)$ for all L and M ,
- (R4) Monotonicity: $\mathcal{R}(L) \geq \mathcal{R}(M)$ when $L \geq M$.

Coherent risk measures

CRM: $\mathcal{R} : \mathcal{L}_2(\Omega) \rightarrow (-\infty, \infty]$ that satisfies

- (R1) Translation equivariance: $\mathcal{R}(L + C) = \mathcal{R}(L) + C$ for all L and constants C ,
- (R2) Positive homogeneity: $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda L) = \lambda \mathcal{R}(L)$ for all L and all $\lambda > 0$,
- (R3) Subadditivity: $\mathcal{R}(L + M) \leq \mathcal{R}(L) + \mathcal{R}(M)$ for all L and M ,
- (R4) Monotonicity: $\mathcal{R}(L) \geq \mathcal{R}(M)$ when $L \geq M$.

Strictly expectation bounded risk measures satisfy (R1), (R2), (R3), and

- (R5) $\mathcal{R}(L) > \mathbb{E}[L]$ for all nonconstant L , whereas $\mathcal{R}(L) = \mathbb{E}[L]$ for constant L .

Coherent risk measures

CRM: $\mathcal{R} : \mathcal{L}_2(\Omega) \rightarrow (-\infty, \infty]$ that satisfies

- (R1) Translation equivariance: $\mathcal{R}(L + C) = \mathcal{R}(L) + C$ for all L and constants C ,
- (R2) Positive homogeneity: $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda L) = \lambda \mathcal{R}(L)$ for all L and all $\lambda > 0$,
- (R3) Subadditivity: $\mathcal{R}(L + M) \leq \mathcal{R}(L) + \mathcal{R}(M)$ for all L and M ,
- (R4) Monotonicity: $\mathcal{R}(L) \geq \mathcal{R}(M)$ when $L \geq M$.

Strictly expectation bounded risk measures satisfy (R1), (R2), (R3), and

- (R5) $\mathcal{R}(L) > \mathbb{E}[L]$ for all nonconstant L , whereas $\mathcal{R}(L) = \mathbb{E}[L]$ for constant L .

Note: (R2)&(R3) implies convexity of \mathcal{R} : for each $a \in [0, 1]$ we have:

$$\mathcal{R}(aL + (1-a)M) \leq \mathcal{R}(aL) + \mathcal{R}((1-a)M) = a\mathcal{R}(L) + (1-a)\mathcal{R}(M)$$

Other classes of risk measures and functionals: Follmer and Schied (2002), Pflug and Romisch (2007).

Coherent properties for the popular risk measures

- variance: none of (R1)-(R5)
- standard deviation: (R2)
- semivariance: none of (R1)-(R5)
- mean absolute deviation: (R2), (R3)
- mean absolute semideviation: (R2), (R3)
- Value at Risk (VaR): (R1),(R2), (R4)
- Conditional Value at Risk (CVaR): (R1)-(R5)

Dual representation of coherent risk measures

Consider a measurable space (Ω, \mathcal{A}) and the set \mathcal{P} of all probability measures on the space.

Definition

A set $\mathcal{Q} \subset \mathcal{P}$ is called a risk envelope if for each $Q \in \mathcal{Q}$ one has: $Q \geq 0$ and $\mathbb{E}Q = 1$.

Theorem

\mathcal{R} is a coherent risk measure if and only if there exists a risk envelope \mathcal{Q} such that:

$$\mathcal{R}(L) = \max_{Q \in \mathcal{Q}} E(QL)$$

and \mathcal{Q} can be chosen as a convex set.

Interpretation: A coherent risk measure can be understood as a worst-case expectation with respect to some class of probability distributions on (Ω, \mathcal{A}) . It means for some distribution P' . If the probability distribution of L is P then $Q = \frac{dP'}{dP}$.

Example - Risk envelope for CVaR

To simplify the situation consider a measurable space with M atoms (discrete distributions). Moreover let L has a discrete uniform distribution on the space - atoms are equiprobable, i.e. discrete distribution with M equiprobable scenarios l_j , $j = 1, 2, \dots, M$. Assume that $M(1 - \alpha)$ is an integer number. Then:

$$\begin{aligned} \text{CVaR}_\alpha(L) &= \min_{a, z_j} a + \frac{1}{(1 - \alpha) M} \sum_{j=1}^M z_j \\ \text{s. t. } z_j &\geq l_j - a, j = 1, \dots, M \\ z_j &\geq 0, j = 1, \dots, M \end{aligned}$$

Example - Risk envelope for CVaR

And dual problem:

$$\begin{aligned} \text{CVaR}_\alpha(L) &= \max_{y_j} \sum_{j=1}^M y_j l_j \\ \text{s. t. } \sum_{j=1}^M y_j &= 1 \\ y_j &\leq \frac{1}{(1-\alpha)M} \\ y_j &\geq 0, j = 1, \dots, M \end{aligned}$$

Note that optimal solution: $y_j^* = 0$ for $j = 1, 2, \dots, M\alpha$

$y_j^* = \frac{1}{(1-\alpha)M}$ for $j = M\alpha + 1, \dots, M$. Hence the risk envelope for CVaR is:

$$\mathcal{Q} = \left\{ Q : EQ = 1, 0 \leq Q \leq \frac{1}{(1-\alpha)} \right\}$$

A **return measure** is defined as a functional $\mathcal{E}(L) = -\mathcal{R}(L)$ for a coherent risk measure \mathcal{R} . It is obvious that the expectation belongs to this class.

Rockafellar, Uryasev and Zabarankin (2006A, 2006B): GDM are introduced as an extension of *standard deviation* but they need not to be symmetric with respect to *upside* $X - \mathbb{E}X$ and *downside* $\mathbb{E}X - X$ of a random variable X .

Rockafellar, Uryasev and Zabarankin (2006A, 2006B): GDM are introduced as an extension of *standard deviation* but they need not to be symmetric with respect to *upside* $X - \mathbb{E}X$ and *downside* $\mathbb{E}X - X$ of a random variable X .

Any functional $\mathcal{D} : \mathcal{L}_2(\Omega) \rightarrow [0, \infty]$ is called a general deviation measure if it satisfies

- (D1) $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all X and constants C ,
- (D2) $\mathcal{D}(0) = 0$, and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and all $\lambda > 0$,
- (D3) $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all X and Y ,
- (D4) $\mathcal{D}(X) \geq 0$ for all X , with $\mathcal{D}(X) > 0$ for nonconstant X .

(D2) & (D3) \Rightarrow convexity

- **Standard deviation**

$$\mathcal{D}(X) = \sigma(X) = \sqrt{\mathbb{E} \|X - \mathbb{E}X\|_2}$$

- **Mean absolute deviation**

$$\mathcal{D}(X) = \mathbb{E}[|X - \mathbb{E}X|].$$

- **Mean absolute lower and upper semideviation**

$$\mathcal{D}_-(X) = \mathbb{E}[\min(0, X - \mathbb{E}X)], \quad \mathcal{D}_+(X) = \mathbb{E}[\max(0, X - \mathbb{E}X)].$$

- **Worst-case deviation**

$$\mathcal{D}(X) = \sup_{\omega \in \Omega} |X(\omega) - \mathbb{E}X|.$$

- See Rockafellar et al (2006 A, 2006 B) for another examples.

Mean absolute deviation from $(1 - \alpha)$ -th quantile

CVaR deviation

For any $\alpha \in (0, 1)$ a finite, continuous, lower range dominated deviation measure

$$\mathcal{D}_\alpha(X) = \text{CVaR}_\alpha(X - \mathbb{E}X). \quad (1)$$

The deviation is also called **weighted mean absolute deviation from the $(1 - \alpha)$ -th quantile**, see Ogryczak, Ruszczyński (2002), because it can be expressed as

$$\mathcal{D}_\alpha(X) = \min_{\xi \in \mathbb{R}} \frac{1}{1 - \alpha} \mathbb{E}[\max\{(1 - \alpha)(X - \xi), \alpha(\xi - X)\}] \quad (2)$$

with the minimum attained at any $(1 - \alpha)$ -th quantile. In relation with CVaR minimization formula, see Pflug (2000), Rockafellar and Uryasev (2000, 2002).

According to Proposition 4 in Rockafellar et al (2006 A):

- if $\mathcal{D} = \lambda \mathcal{D}_0$ for $\lambda > 0$ and a deviation measure \mathcal{D}_0 , then \mathcal{D} is a deviation measure.
- if $\mathcal{D}_1, \dots, \mathcal{D}_K$ are deviation measures, then
 - $\mathcal{D} = \max\{\mathcal{D}_1, \dots, \mathcal{D}_K\}$ is also deviation measure.
 - $\mathcal{D} = \lambda_1 \mathcal{D}_1 + \dots + \lambda_K \mathcal{D}_K$ is also deviation measure, if $\lambda_k > 0$ and $\sum_{k=1}^K \lambda_k = 1$.

Rockafellar et al (2006 A, B): Duality representation using *risk envelopes*, subdifferentiability and optimality conditions.

We say that general deviation measure \mathcal{D} is

(LSC) **lower semicontinuous** (lsc) if all the subsets of $\mathcal{L}_2(\Omega)$ having the form $\{X : \mathcal{D}(X) \leq c\}$ for $c \in \mathbb{R}$ (level sets) are closed;

We say that general deviation measure \mathcal{D} is

- (LSC) **lower semicontinuous** (lsc) if all the subsets of $\mathcal{L}_2(\Omega)$ having the form $\{X : \mathcal{D}(X) \leq c\}$ for $c \in \mathbb{R}$ (level sets) are closed;
- (D5) **lower range dominated** if $\mathcal{D}(X) \leq EX - \inf_{\omega \in \Omega} X(\omega)$ for all X .

Theorem 2 in Rockafellar et al (2006 A):

Theorem

Deviation measures correspond one-to-one with strictly expectation bounded risk measures under the relations

- $\mathcal{D}(X) = \mathcal{R}(X - \mathbb{E}X)$
- $\mathcal{R}(X) = \mathbb{E}[-X] + \mathcal{D}(X)$

In this correspondence, \mathcal{R} is coherent if and only if \mathcal{D} is lower range dominated.

- Follmer, H., Schied, A.: **Stochastic Finance: An Introduction In Discrete Time**. Walter de Gruyter, Berlin, 2002.
- Pflug, G.Ch., Romisch, W.: **Modeling, measuring and managing risk**. World Scientific Publishing, Singapore, 2007.
- Rockafellar, R.T., Uryasev, S. (2002). **Conditional Value-at-Risk for General Loss Distributions**, *Journal of Banking and Finance* 26, 1443–1471.
- Rockafellar, R.T., Uryasev, S., Zabarankin M. (2006A). **Generalized Deviations in Risk Analysis**. *Finance and Stochastics* 10 , 51–74.
- Rockafellar, R.T., Uryasev, S., Zabarankin M. (2006B). **Optimality Conditions in Portfolio Analysis with General Deviation Measures**. *Mathematical Programming* 108, No. 2-3, 515–540.