

# STOCHASTIC DOMINANCE IN PORTFOLIO EFFICIENCY TESTING

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Decision making problem:

$$\max_{\lambda \in \Lambda} F(\lambda, \varrho)$$

where

- $\lambda$  is a decision vector
- $\Lambda$  is a set of possible decision vectors
- $\varrho \in \Omega$  is a random vector of outcomes
- $F : \Lambda \times \Omega \rightarrow \mathbb{R}$  is an objective criterion

# Portfolio optimization problem

$$\max_{\lambda \in \Lambda} Eu(\lambda' \varrho)$$

- $u$  is a utility function (non-decreasing function)
- $\lambda' \varrho$  is a final outcome of portfolio (decision vector)  $\lambda$
- maximizing expected utility criterion:  $F(\lambda, \varrho) = Eu(\lambda' \varrho)$
- choice of utility function - risk attitude

Two questions:

- How to find the true utility function that adequately describes the risk attitude of a decision maker
- How to find the true probability distribution of  $\varrho$

1952 - H. Markowitz - first portfolio optimization formulations:

- quadratic parametric program
- strong assumptions about the utility function (concavity) and the probability distribution (normality)
- nice results
- applied in finance
- Nobel Prize in Economics 1990

1952 - till now: portfolio optimization problems have become the most important issue of the decision making theory under risk

# Portfolio optimization problem

Utility function selection:

- consider some favorite type of a utility function (power, exponential,...) and estimate the parameters
- *consider a specified set of suitable utility functions - stochastic dominance approach*

Randomness:

- assume a particular probability distribution of random vector
- estimate the probability distribution from data
- *consider a set of suitable probability distributions - robustness, contamination, worst case analysis,...*

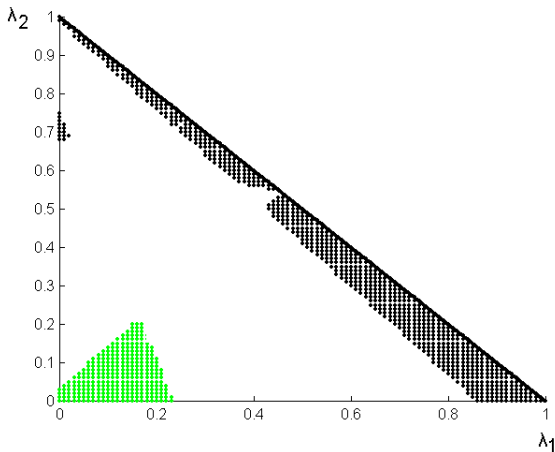
The goal of my research: to solve the portfolio selection problem when considering **a set of utility functions** and **a set of probability distributions**

The solution of the portfolio optimization problem: **efficient portfolios**.

- Crucial question of portfolio efficiency (in the sense of optimality):  
*Is a given portfolio a maximizer of expected utility for at least one considered utility function?*  
If yes, portfolio is call efficient.
- Crucial question of portfolio efficiency (in the sense of admissibility):  
*Does there exist a better portfolio (having higher expected utility) than a given portfolio for all considered utility functions*  
If no, portfolio called efficient.

# Example

All utility (non-decreasing) functions are considered (K. and Post, 2009):



Under Markowitz model assumption: all efficient portfolios form a line.

# First order stochastic dominance (FSD) - notation

Consider  $N$  alternatives and a random vector of their outcomes  $\varrho$  with distribution  $P$ . A decision maker may combine alternatives into portfolios and all portfolio possibilities are given by

$$\Lambda = \{\lambda \in \mathbb{R}^N | \mathbf{1}'\lambda = 1, \lambda_n \geq 0, n = 1, 2, \dots, N\}.$$

Let  $F_{\varrho'\lambda}(x)$  denote the cumulative probability distribution function of returns of portfolio  $\lambda$ .

## Definition

Portfolio  $\lambda \in \Lambda$  dominates portfolio  $\tau \in \Lambda$  by the first-order stochastic dominance ( $\varrho'\lambda \succ_{FSD} \varrho'\tau$ ) if

$$F_{\varrho'\lambda}(x) \leq F_{\varrho'\tau}(x) \quad \forall x \in \mathbb{R}$$

with strict inequality for at least one  $x \in \mathbb{R}$ .



# First order stochastic dominance (FSD) - interpretation

Necessary and sufficient conditions:  $\varrho'\lambda \succ_{FSD} \varrho'\tau$  if

- $Eu(\varrho'\lambda) \geq Eu(\varrho'\tau)$  for all utility functions and strict inequality holds for at least some utility function.
- $F_{\varrho'\lambda}^{-1}(y) \leq F_{\varrho'\tau}^{-1}(y) \quad \forall y \in [0, 1]$  with strict inequality for at least one  $y \in [0, 1]$ .
- $\text{VaR}_\alpha(-\varrho'\lambda) \leq \text{VaR}_\alpha(-\varrho'\tau) \quad \forall \alpha \in [0, 1]$  with strict inequality for at least one  $\alpha \in [0, 1]$ .

# Portfolio efficiency with respect to FSD

FSD efficiency: admissibility vs. optimality (K. and Post, 2009):

## Definition (Admissibility case)

A given portfolio  $\tau \in \Lambda$  is FSD inefficient if there exists portfolio  $\lambda \in \Lambda$  such that  $\varrho'\lambda \succ_{FSD} \varrho'\tau$ . Otherwise, portfolio  $\tau$  is FSD efficient.

## Definition (Optimality case)

Portfolio  $\tau \in \Lambda$  is FSD efficient if it is the optimal solution of

$$\max_{\lambda \in \Lambda} Eu(\varrho'\lambda)$$

for at least some utility function, i.e., there exists  $u$  such that

$$Eu(\varrho'\tau) - Eu(\varrho'\lambda) \geq 0 \quad \forall \lambda \in \Lambda.$$

Otherwise,  $\tau$  is FSD non-optimal.

We focus on the admissibility case - easier case.

# Portfolio efficiency test with respect to FSD

Idea: To identify a FSD dominating portfolio.

In order to find a FSD dominating portfolio  $\lambda$ , we solve the following problem (Dupačová and K., 2014):

$$\begin{aligned}\xi_P(\tau) &= \max_{\lambda} d(\lambda, \tau) \\ \text{s.t.} \quad & H_P(\lambda, \tau) \leq 0 \\ & \lambda \in \Lambda.\end{aligned}$$

where

- $d(\lambda, \tau)$  is an arbitrary distance between portfolios  $\lambda$  and  $\tau$ , for example,  $d(\lambda, \tau) = (\lambda - \tau)'(\lambda - \tau)$ .
- $H_P(\lambda, \tau) := \max_{y \in \mathbb{R}} (F_{\varrho' \lambda}(y) - F_{\varrho' \tau}(y))$

where  $\xi_P(\tau)$  is called inefficiency measure. Or equivalently:

$$\begin{aligned}\xi_P(\tau) &= \max_{\lambda} d(\lambda, \tau) \\ \text{s.t.} \quad & F_{\varrho' \lambda}(y) - F_{\varrho' \tau}(y) \leq 0 \quad \forall y \in \mathbb{R} \\ & \lambda \in \Lambda.\end{aligned}$$

# Portfolio efficiency test with respect to FSD

Necessary and sufficient condition (Dupačová and K., 2014):

## Theorem

*A given portfolio  $\tau$  is FSD efficient if and only if  $\xi_P(\tau) = 0$ . If  $\xi_P(\tau) > 0$  then the optimal portfolio  $\lambda^*$  is FSD efficient and it dominates portfolio  $\tau$  by FSD.*

But how to compute inefficiency measure  $\xi_P(\tau)$ ? In general, we need to solve the optimization problem:

$$\begin{aligned}\xi_P(\tau) &= \max_{\lambda} d(\lambda, \tau) \\ \text{s.t.} \quad & F_{\varrho' \lambda}(y) - F_{\varrho' \tau}(y) \leq 0 \quad \forall y \in \mathbb{R} \\ & \lambda \in \Lambda.\end{aligned}$$

which:

- is non-convex
- is non-smooth
- has infinitely many constraints

# Possible approaches

Approximations of a true distribution:

- To assume a specific distribution (normal) and rewrite the problem in a few constraints - how good is such approximation...
- To assume a scenario approach, i.e.  $P$  is a discrete probability distribution (with non-equiprobable scenarios), and rewrite the problem in finitely many constraints, but non-convex, non-smooth...difficult to solve the problem even for a small number of scenarios (Dupačová and K., 2014)
- To assume that scenarios are equiprobable, and rewrite the problem as MIP... solvable for a small number of scenarios ( $< 100$ ) (K. and Post, 2009)

What is the quality of the solution under these approximations?

Quality analysis: stress testing

More robust FSD efficiency criteria

Consider a contamination of the original distribution of returns by additional scenario  $\mathbf{s}$ :  $P(t) = (1 - t)P + t\delta_{\{\mathbf{s}\}}$ ,  $t \in [0, 1]$ . Let  $\varrho(t)$  be a random variable with distribution  $P(t)$ . We now consider:

$$\begin{aligned} \xi_{P(t)}(\boldsymbol{\tau}) &= \max_{\boldsymbol{\lambda}} d(\boldsymbol{\lambda}, \boldsymbol{\tau}) \\ \text{s.t.} \quad &H_{P(t)}(\boldsymbol{\lambda}, \boldsymbol{\tau}) \leq 0 \\ &\boldsymbol{\lambda} \in \Lambda \end{aligned} \tag{1}$$

where  $H_{P(t)}(\boldsymbol{\lambda}, \boldsymbol{\tau}) = \max_{y \in \mathbb{R}} (F_{\varrho(t)'\boldsymbol{\lambda}}(y) - F_{\varrho(t)'\boldsymbol{\tau}}(y))$ .

Can be easily modified for any contaminating probability distribution.

# Directional FSD portfolio efficiency with respect to an additional scenario

A robust version of FSD efficiency (Dupačová and K., 2014) :

## Definition

A given portfolio  $\tau \in \Lambda$  is directionally FSD inefficient with respect to additional scenario  $\mathbf{s}$  if for each  $t$  exists  $\lambda(t)$  such that  $\varrho(t)' \lambda(t) \succ_{FSD} \varrho(t)' \tau$ . Moreover, a given portfolio  $\tau \in \Lambda$  is directionally FSD efficient with respect to additional scenario  $\mathbf{s}$  if there is no  $(t, \lambda(t))$  such that  $\varrho(t)' \lambda(t) \succ_{FSD} \varrho(t)' \tau$ .

The definition classifies portfolio  $\tau$  as directionally FSD efficient (inefficient) with respect to additional scenario  $\mathbf{s}$  if  $\tau$  is FSD efficient (inefficient) when using the original distribution  $P$  as well as in any contaminated case  $P(t)$ .

# Directional FSD portfolio efficiency with respect to an additional scenario

Necessary and sufficient conditions (Dupačová and K., 2014) :

## Theorem

*A given portfolio  $\tau \in \Lambda$  is directionally FSD efficient with respect to additional scenario  $\mathbf{s}$  if and only if*

$$\max_{t \in [0,1]} \xi_{P(t)}(\tau) = 0.$$

## Theorem

*A given portfolio  $\tau \in \Lambda$  is directionally FSD inefficient with respect to additional scenario  $\mathbf{s}$  if and only if*

$$\min_{t \in [0,1]} \xi_{P(t)}(\tau) > 0.$$

It leads to minimax....very difficult to solve even in the case when the probability distribution is approximated by a few equiprobable scenarios.



# Directional FSD portfolio efficiency with respect to an additional scenario - sufficient condition

Applying contamination bounds [Dupačová and K. \(2014\)](#) proved: If  $H_{P(t)}(\lambda, \tau)$  is concave in  $t$  then  $\xi_{P(t)}(\tau)$  is quasiconcave in  $t$  and  $\xi_{P(t)}(\tau) \geq \min\{\xi_{P(0)}(\tau), \xi_{P(1)}(\tau)\}$ . As a consequence we can derive the following sufficient condition for directional FSD efficiency with respect to additional scenario  $\mathbf{s}$ .

## Theorem

*If*

- $H_{P(t)}(\lambda, \tau)$  is concave in  $t$ .
- $\tau$  is FSD efficient when using original probability distribution  $P$
- $\tau \in \operatorname{argmax}_{\lambda \in \Lambda} \mathbf{s}'\lambda$

*then  $\tau$  is directionally FSD efficient with respect to  $\mathbf{s}$ .*

# Directional FSD portfolio inefficiency with respect to an additional scenario - sufficient condition

Since  $F_{\varrho(t)'\lambda}(y)$  is linear in  $t$  for all  $\lambda \in \Lambda$  and  $y \in \mathbb{R}$  we may derive the following sufficient condition (Dupačová and K., 2014) :

## Theorem

*If there exists  $\lambda \in \Lambda$  such that  $\varrho'\lambda \succ_{FSD} \varrho'\tau$  and  $\mathbf{s}'\lambda \geq \mathbf{s}'\tau$  then  $\tau$  is directionally FSD inefficient with respect to  $\mathbf{s}$ .*

The proof makes use an contamination upper bound to show that  $\lambda \in \Lambda$  satisfying  $\varrho'\lambda \succ_{FSD} \varrho'\tau$  and  $\mathbf{s}'\lambda \geq \mathbf{s}'\tau$  FSD dominates  $\tau$  in any contaminated case, i.e.

$$\varrho(t)'\lambda \succ_{FSD} \varrho(t)'\tau.$$

# FSD efficiency - neighborhood robustness

Another approach to robustness (Dupačová and K., 2014) :  $\epsilon$ -FSD efficiency test

Let  $d(\bar{P}, P)$  be a distance between  $P$  and some alternative probability distribution  $\bar{P}$ .

## Definition

A given portfolio  $\tau \in \Lambda$  is  $\epsilon$ -FSD inefficient if there exists portfolio  $\lambda \in \Lambda$  and  $\bar{P}$  such that  $d(\bar{P}, P) \leq \epsilon$  with  $\bar{q}'\lambda \succ_{FSD} \bar{q}'\tau$ . Otherwise, portfolio  $\tau$  is  $\epsilon$ -FSD efficient.

The introduced  $\epsilon$ -FSD efficiency guarantees stability of the FSD efficiency classification with respect to small changes (prescribed by parameter  $\epsilon$ ) in probability distribution  $P$ . A given portfolio  $\tau$  is  $\epsilon$ -FSD efficient if and only if no portfolio  $\lambda$  FSD dominates  $\tau$  neither for the original distribution  $P$  nor for arbitrary distribution  $\bar{P}$  from  $\epsilon$ -neighborhood of  $P$ .

# $\epsilon$ -FSD efficiency test

For testing  $\epsilon$ -FSD efficiency of a given portfolio  $\tau$  we introduce a new measure of  $\epsilon$ -FSD efficiency:

$$\begin{aligned} \xi_{\epsilon}(\tau, R, \mathbf{p}) &= \min_{a_s, b_s, \lambda, \bar{\mathbf{p}}, \bar{\mathbf{q}}} \sum_{s=1}^S (a_s + b_s) & (2) \\ \text{s.t.} \quad & \text{VaR}_{\bar{\mathbf{q}}_s^{\lambda}}(-\bar{\boldsymbol{\rho}}' \boldsymbol{\lambda}) - \text{VaR}_{\bar{\mathbf{q}}_s^{\lambda}}(-\bar{\boldsymbol{\rho}}' \boldsymbol{\tau}) \leq a_s, \quad s = 1, \dots, S \\ & \text{VaR}_{\bar{\mathbf{q}}_s^{\tau}}(-\bar{\boldsymbol{\rho}}' \boldsymbol{\lambda}) - \text{VaR}_{\bar{\mathbf{q}}_s^{\tau}}(-\bar{\boldsymbol{\rho}}' \boldsymbol{\tau}) \leq b_s, \quad s = 1, \dots, S \\ & \bar{\mathbf{q}}_s^{\lambda} = \sum_{i=1}^S \bar{p}_i^{\lambda}, \quad \bar{\mathbf{q}}_s^{\tau} = \sum_{i=1}^S \bar{p}_i^{\tau}, \quad s = 1, \dots, S \\ & \sum_{i=1}^S \bar{p}_i = 1, \quad -\epsilon \leq \bar{p}_i - p_i \leq \epsilon, \quad \bar{p}_i \geq 0, \quad i = 1, 2, \dots, S \\ & \lambda \in \Lambda \quad a_s, b_s \leq 0, \quad s = 1, \dots, S \end{aligned}$$

## Theorem

Portfolio  $\tau \in \Lambda$  is  $\epsilon$ -FSD efficient if and only if  $\xi_{\epsilon}(\tau, R, \mathbf{p})$  given by (2) is equal to zero.

# Second order stochastic dominance – definitions

Let  $F_{\mathbf{r}'\lambda}(x)$  denote the cumulative probability distribution function of returns of portfolio  $\lambda$ . The twice cumulative probability distribution function of returns of portfolio  $\lambda$  is defined as

$$F_{\mathbf{r}'\lambda}^{(2)}(y) = \int_{-\infty}^y F_{\mathbf{r}'\lambda}(x) dx. \quad (25)$$

## Definition

Portfolio  $\lambda \in \Lambda$  dominates portfolio  $\tau \in \Lambda$  by the second-order stochastic dominance ( $\mathbf{r}'\lambda \succ_{SSD} \mathbf{r}'\tau$ ) if and only if

$$F_{\mathbf{r}'\lambda}^{(2)}(y) \leq F_{\mathbf{r}'\tau}^{(2)}(y) \quad \forall y \in \mathbb{R}$$

with strict inequality for at least one  $y \in \mathbb{R}$ .

## Definition

A given portfolio  $\tau \in \Lambda$  is SSD inefficient if there exists portfolio  $\lambda \in \Lambda$  such that  $\mathbf{r}'\lambda \succ_{SSD} \mathbf{r}'\tau$ . Otherwise, portfolio  $\tau$  is SSD efficient.

# Second order stochastic dominance – interpretation

Other equivalent definitions of SSD relation:  $\mathbf{r}'\boldsymbol{\lambda} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$  if

- $Eu(\mathbf{r}'\boldsymbol{\lambda}) \geq Eu(\mathbf{r}'\boldsymbol{\tau})$  for all concave utility functions and strict inequality holds for at least some concave utility function.
- No non-satiable and risk averse decision maker prefers portfolio  $\boldsymbol{\tau}$  to portfolio  $\boldsymbol{\lambda}$  and at least one prefers  $\boldsymbol{\lambda}$  to  $\boldsymbol{\tau}$ .
- $F_{\mathbf{r}'\boldsymbol{\lambda}}^{-2}(y) \leq F_{\mathbf{r}'\boldsymbol{\tau}}^{-2}(y) \quad \forall y \in [0, 1]$  with strict inequality for at least one  $y \in [0, 1]$ , where  $F_{\mathbf{r}'\boldsymbol{\lambda}}^{-2}$  is a cumulated quantile function.
- $CVaR_{\alpha}(-\mathbf{r}'\boldsymbol{\lambda}) \leq CVaR_{\alpha}(-\mathbf{r}'\boldsymbol{\tau}) \quad \forall \alpha \in [0, 1]$  with strict inequality for at least one  $\alpha \in [0, 1]$ , where

$$\begin{aligned} CVaR_{\alpha}(-\mathbf{r}'\boldsymbol{\lambda}) = \min_{\mathbf{v} \in \mathbb{R}, \mathbf{z}_t \in \mathbb{R}^+} \quad & \mathbf{v} + \frac{1}{1-\alpha} \sum_{t=1}^S p_t z_t \\ \text{s.t.} \quad & z_t \geq -\mathbf{x}^t \boldsymbol{\lambda} - \mathbf{v}, \quad t = 1, 2, \dots, S \end{aligned} \quad (26)$$

Let

$$\begin{aligned} \xi(\tau, X, \mathbf{p}) &= \min_{a_s, \lambda} \sum_{s=0}^{S-1} a_s \\ \text{s.t.} \quad &\text{CVaR}_{q_s^{\tau}}(-\mathbf{r}'\lambda) - \text{CVaR}_{q_s^{\tau}}(-\mathbf{r}'\tau) \leq a_s, \quad s = 0, 1, \dots, S-1 \\ &a_s \leq 0, \quad s = 0, 1, \dots, S-1 \\ &\lambda \in \Lambda. \end{aligned} \tag{27}$$

## Theorem

*A given portfolio  $\tau$  is SSD efficient if and only if  $\xi(\tau, X, \mathbf{p}) = 0$ . If  $\xi(\tau, X, \mathbf{p}) < 0$  then the optimal portfolio  $\lambda^*$  in (27) is SSD efficient and it dominates portfolio  $\tau$  by SSD.*

# Linear SSD efficiency test

Expressions for CVaR:

$$\begin{aligned} \text{CVaR}_\alpha(-\mathbf{r}'\boldsymbol{\lambda}) = \min_{v \in \mathbb{R}, \mathbf{z}_t \in \mathbb{R}^+} \quad & v + \frac{1}{1-\alpha} \sum_{t=1}^S p_t z_t \\ \text{s.t.} \quad & z_t \geq -\mathbf{x}^t \boldsymbol{\lambda} - v, \quad t = 1, 2, \dots, S \end{aligned} \quad (28)$$

and the similar expression can be considered for portfolio  $\boldsymbol{\tau}$ . However, for portfolio  $\boldsymbol{\tau}$ , we will rather use the dual formulation:

$$\begin{aligned} \text{CVaR}_\alpha(-\mathbf{r}'\boldsymbol{\tau}) = \max_{\kappa_t \in \mathbb{R}^+} \quad & \frac{1}{1-\alpha} \sum_{t=1}^S \kappa_t (-\mathbf{x}^t \boldsymbol{\tau}) \\ \text{s.t.} \quad & \kappa_t \leq p_t, \quad t = 1, 2, \dots, S \\ & \sum_{t=1}^S \kappa_t = 1 - \alpha \end{aligned} \quad (29)$$

See Rockafellar & Uryasev (2000, 2002) for more details.



# Robustness in SSD portfolio efficiency testing - notation

Following Dupačová & Kopa (2012), we consider  $\epsilon$ -SSD efficiency approach as a robustification of the classical SSD portfolio efficiency. It guarantees stability of the SSD efficiency classification with respect to small changes (prescribed by parameter  $\epsilon > 0$ ) in probability vector  $\mathbf{p}$ . Assume that the probability distribution  $\bar{P}$  of random returns  $\bar{\mathbf{r}}$  takes again values  $\mathbf{x}^s$ ,  $s = 1, 2, \dots, S$  but with other probabilities  $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_S)$ . We define the distance between  $P$  and  $\bar{P}$  as  $d(\bar{P}, P) = \max_i |\bar{p}_i - p_i|$ .

## Definition

A given portfolio  $\tau \in \Lambda$  is  $\epsilon$ -SSD inefficient if there exists portfolio  $\lambda \in \Lambda$  and  $\bar{P}$  such that  $d(\bar{P}, P) \leq \epsilon$  with  $\bar{\mathbf{r}}' \lambda \succ_{SSD} \bar{\mathbf{r}}' \tau$ . Otherwise, portfolio  $\tau$  is  $\epsilon$ -SSD efficient.

A portfolio  $\tau$  is  $\epsilon$ -SSD efficient if and only if no portfolio  $\lambda$  SSD dominates  $\tau$  neither for the original probabilities  $\mathbf{p}$  nor for arbitrary probabilities  $\bar{\mathbf{p}}$  from  $\epsilon$ -neighborhood of the original vector  $\mathbf{p}$ .

# Robustness analysis via Contamination

Introduced for general stochastic programs by Dupačová et. al. (1996, 2000, 2006...) Assume that a problem was solved for original distribution  $P$ . Changes in probability distribution  $P$  are modeled using **contaminated distributions**

$$P(t) := (1 - t)P + tQ, t \in [0, 1]$$

with  $Q$  another *fixed* probability distribution such that optimal value function  $\varphi(Q)$  is finite.

Via contamination, robustness analysis wrt. changes in  $P$  gets reduced to much simpler analysis of **parametric program with scalar parameter  $t$** .

One can compute lower and upper bound for optimal value function  $\varphi(t)$ . We apply this notion in the easiest manner - the alternative distribution is just one scenario (can be seen as stress test scenario or worst case scenario)

# Robust portfolio efficiency with respect to the additional scenario

For a contamination parameter  $t \in [0, 1]$ , we assume that the random return  $\tilde{q}(t)$  takes values  $\mathbf{r}^1, \mathbf{r}^2, \dots, \mathbf{r}^{S+1}$  with probabilities  $\tilde{\mathbf{p}}(t) = ((1-t)p_1, (1-t)p_2, \dots, (1-t)p_S, t)$ . We denote the extended scenario matrix by  $\tilde{R}$ , that is,

$$\tilde{R} = \begin{pmatrix} R \\ \mathbf{r}^{S+1} \end{pmatrix}.$$

## Definition

A given portfolio  $\tau \in \Lambda$  is directionally SSD inefficient with respect to  $\mathbf{r}^{S+1}$  if it exists  $t_0 > 0$  such that for every  $t \in [0, t_0]$  there is a portfolio  $\lambda(t) \in \Lambda$  satisfying  $\tilde{q}(t)' \lambda(t) \succ_{SSD} \tilde{q}(t)' \tau$ .

## Definition

A given portfolio  $\tau \in \Lambda$  is directionally SSD efficient with respect to  $\mathbf{r}^{S+1}$  if there does not exist  $t_0 > 0$  such that for every  $t \in [0, t_0]$  there is a portfolio  $\lambda(t) \in \Lambda$  satisfying  $\tilde{q}(t)' \lambda(t) \succ_{SSD} \tilde{q}(t)' \tau$ .

# Robust portfolio efficiency – con't

Using contamination bounds derived in Dupačová & Kopa (2012) we can derive a sufficient condition for directional SSD efficiency and directional SSD inefficiency.

## Theorem

*Let  $\tau \in \Lambda$  be a SSD efficient portfolio for the original distribution  $P$ . Let*

$$\mathbf{r}^{S+1}\tau \geq \mathbf{r}^{S+1}\lambda \text{ for all } \lambda \in \Lambda. \quad (30)$$

*Then  $\tau \in \Lambda$  is directionally SSD efficient with respect to  $\mathbf{r}^{S+1}$ .*

## Theorem

*Let  $\tau \in \Lambda$  be a SSD inefficient portfolio for the original distribution  $P$ . If there exists a portfolio  $\lambda \in \Lambda$  such that*

$$\text{CVaR}_{q_s^\tau}(-\boldsymbol{\varrho}'\lambda) - \text{CVaR}_{q_s^\tau}(-\boldsymbol{\varrho}'\tau) < 0, s = 0, 1, \dots, S-1 \quad (31)$$

$$\mathbf{r}^{S+1}\lambda \geq \min((R\tau)^{[1]}, \mathbf{r}^{S+1}\tau) \quad (32)$$

*then  $\tau$  is directionally SSD inefficient with respect to  $\mathbf{r}^{S+1}$ .*

# Risk measures

We assume discrete distribution - equiprobable scenarios

- variance:

$$\sigma^2(\mathbf{r}'\boldsymbol{\lambda}) = \frac{1}{T} \sum_{t=1}^T (\mathbf{x}^t \boldsymbol{\lambda} - \frac{1}{T} \sum_{s=1}^T (\mathbf{x}^s \boldsymbol{\lambda}))^2$$

- Value at Risk:

$$\begin{aligned} \text{VaR}_{\alpha}(-\mathbf{r}'\boldsymbol{\lambda}) &= \min_{\gamma, \delta_t} \gamma \\ \text{s.t. } &\gamma + M\delta_t \geq -\mathbf{x}^t \boldsymbol{\lambda}, \quad t = 1, \dots, T \\ &\sum_{t=1}^T \delta_t = \lfloor (1 - \alpha) T \rfloor \\ &\delta_t \in \{0, 1\}, \quad t = 1, \dots, T \end{aligned}$$

- Conditional Value at Risk:

$$\begin{aligned} \text{CVaR}_{\alpha}(-\mathbf{r}'\boldsymbol{\lambda}) &= \min_{v \in \mathbb{R}, z_t \in \mathbb{R}^+} v + \frac{1}{(1 - \alpha)T} \sum_{t=1}^T z_t \\ \text{s.t. } &z_t \geq -\mathbf{x}^t \boldsymbol{\lambda} - v, \quad t = 1, 2, \dots, T \end{aligned}$$

# Model formulations I

Mean-variance model (quadratic programming):

$$\begin{aligned} \min_{\lambda \in \Lambda} \quad & \frac{1}{T} \sum_{t=1}^T (\mathbf{x}^t \lambda - \frac{1}{T} \sum_{s=1}^T (\mathbf{x}^s \lambda))^2 \\ \text{s.t.} \quad & \sum_{t=1}^T (\mathbf{x}^t \lambda) \geq \sum_{t=1}^T (\mathbf{x}^t \tau) \end{aligned}$$

VaR-FSD model (mixed integer programming)

$$\begin{aligned} \min_{\gamma, \delta_t} \quad & \gamma \\ \text{s.t.} \quad & \gamma + M\delta_t \geq -\mathbf{x}^t \lambda, \quad t = 1, \dots, T \\ & \sum_{t=1}^T \delta_t = \lfloor (1 - \alpha) T \rfloor \\ & X\lambda \geq PX\tau \\ & 1'P = 1', \quad P1 = 1 \\ & P, \delta_t \in \{0, 1\}, \quad t = 1, \dots, T \end{aligned}$$

CVaR-SSD model (linear programming)

$$\begin{aligned} \min_{v \in \mathbb{R}, z_t, W \in \mathbb{R}^+} \quad & v + \frac{1}{(1 - \alpha)T} \sum_{t=1}^T z_t \\ \text{s.t.} \quad & z_t \geq -\mathbf{x}^t \boldsymbol{\lambda} - v, \quad t = 1, 2, \dots, T \\ & X\boldsymbol{\lambda} \geq W X \boldsymbol{\tau} \\ & \mathbf{1}' W = \mathbf{1}', \quad W \mathbf{1} = 1 \end{aligned}$$

Other combinations - 9 models - 9 optimal portfolios.

- We take US stock market data from the Kenneth French library. We consider a standard set of 10 active benchmark stock portfolios as the base assets. They are formed, and annually rebalanced, based on individual stocks market capitalization of equity, each representing a decile of the cross-section of stocks in a given year. The first decile stocks (the smallest size) are called "small" and the last decile stocks are called "large".
- Furthermore, we include CRISP proxy of the market portfolio as the benchmark and US Treasury bill as a riskless asset.
- We use data on annual excess returns from 1977 to 2006 (30 observations).
- Out-of-sample analysis: 2007-2011



# Empirical application - results

Portfolio compositions:

Portfolio	Variance			VaR			CVaR		
	Mean return	FSD	SSD	Mean return	FSD	SSD	Mean return	FSD	SSD
Riskless	0.27	0.03	0.27	0.30	0.00	0.25	0.28	0.04	0.27
Small	0.00	0.00	0.00	0.32	0.00	0.33	0.06	0.14	0.06
2nd decile	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3rd decile	0.00	0.08	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4th decile	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5th decile	0.00	0.02	0.00	0.24	0.00	0.20	0.00	0.00	0.00
6th decile	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.00
7th decile	0.73	0.42	0.73	0.00	0.67	0.07	0.66	0.71	0.67
8th decile	0.00	0.28	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9th decile	0.00	0.02	0.00	0.14	0.01	0.15	0.00	0.00	0.00
Large	0.00	0.15	0.00	0.00	0.12	0.00	0.00	0.07	0.00
Market	0.00	0.00	0.00	0.00	0.19	0.00	0.00	0.00	0.00

# Empirical application - results

## Portfolio performance:

### In-sample descriptive statistics

	Variance			VaR			CVaR		
	Mean return	FSD	SSD	Mean return	FSD	SSD	Mean return	FSD	SSD
mean	7.16	8.67	7.17	7.16	8.90	7.67	7.16	9.39	7.26
st. deviation	11.04	13.98	11.04	12.82	14.61	13.50	11.16	14.96	11.30
min	-12.87	-19.07	-12.87	-18.26	-19.38	-18.88	-12.00	-16.85	-12.16
max	31.87	38.46	31.87	44.50	38.86	46.88	34.16	47.10	34.61
skewness	0.07	-0.08	0.07	0.46	-0.08	0.45	0.16	0.20	0.16
kurtosis	-0.47	-0.42	-0.47	1.32	-0.62	1.28	-0.16	0.09	-0.16

### Out-of-sample descriptive statistics

	Variance			VaR			CVaR		
	Mean return	FSD	SSD	Mean return	FSD	SSD	Mean return	FSD	SSD
mean	3.16	3.94	3.16	2.04	3.52	2.17	2.88	3.54	2.92
st. deviation	23.22	29.03	23.23	22.83	29.63	24.55	23.15	30.16	23.45
min	-30.99	-39.22	-31.00	-29.27	-41.15	-31.70	-30.76	-40.15	-31.15
max	30.40	36.66	30.41	29.02	37.49	31.27	30.24	39.06	30.63
skewness	-0.55	-0.64	-0.55	-0.24	-0.71	-0.26	-0.48	-0.47	-0.48
kurtosis	0.42	0.39	0.42	-0.73	0.81	-0.62	0.24	0.15	0.24

K. and Post (2009), Post and K. (2013), Post, Fang and K. (2015) applied the theory of portfolio efficiency testing (for various types of stochastic dominance criterion) to US market portfolio efficiency testing. While US market portfolio is generally considered to be efficient they found:

- US market portfolio is FSD inefficient in both meanings (optimality & admissibility)
- US market portfolio is inefficient also when considering other types of stochastic dominance (SSD, NSD, DARA SD, IRRA SD)
- A dominating portfolio was identified
- Consequences on the US market behavior were analyzed

# Conclusions

- Stochastic dominance is an useful tool in portfolio optimization because it allows to consider a set of utility functions
- The tests for portfolio efficiency with respect to stochastic dominance criteria are very complicated mathematical programming problems, very sensitive to underlying probability distribution. Therefore, all kinds of robustness, worst case or stress testing analysis are very useful.
- The tests are computationally manageable only under some assumptions, the complexity is very high.
- Portfolio efficiency with respect to stochastic dominance criteria is a relatively new field of research (from 2003 - Post (2003), Kuosmanen (2004)...) however, strongly motivated by the classical portfolio optimization problems
- Efficiency in the sense of admissibility is also related to work of Dentcheva, Fabian, Henrion, Ruszczyński, Schultz,... (2003 - now)
- A lot of open (difficult) problems for future research in mathematics, computer science, finance...
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# Main references

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- ④ J. Dupačová, M. Kopa: Robustness in stochastic programs with risk constraints, *Annals of Operations Research* 200, 1, (2012), 55 – 74. IF= 1.029
- ⑤ M. Kopa: Measuring of second-order stochastic dominance portfolio efficiency, *Kybernetika* 46, 3, (2010) 488 – 500, IF= 0.619
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# $N$ -th order stochastic dominance

Let  $U_N$  be the set of  $N$  times differentiable utility functions such that:  $(-1)^k u^{(k)} \leq 0$  for all  $k = 1, 2, \dots, N$ .

## Definition

Portfolio  $\lambda$  dominates portfolio  $\tau$  with respect to  $N$ -th order stochastic dominance ( $\lambda \succ_{NSD} \tau$ ) if  $Eu(\mathbf{r}'\lambda) \geq Eu(\mathbf{r}'\tau)$  for all utility functions  $u \in U_N$  with strict inequality for at least one such utility function.

The general definition of NSD efficiency for  $N \geq 2$  can be seen as an extension of SSD efficiency and, following Post and K. (2013), we formulate it in the “NSD optimality” form. We allow for non-equal probabilities of scenarios  $(p_1, \dots, p_T)$ .

## Definition

A given portfolio  $\tau$  is NSD efficient ( $N \geq 2$ ), if there exists at least one utility function  $u \in U_N$  such that  $Eu(\mathbf{r}'\tau) - Eu(\mathbf{r}'\lambda) \geq 0$  for all  $\lambda \in \Lambda$  with strict inequality for at least one  $\lambda \in \Lambda$ .

# Necessary and sufficient condition for NSD efficiency

Using KKT condition in problem  $\max_{\lambda \in \Lambda} \sum_{t=1}^T p_t u(\mathbf{x}^t \lambda)$ , Post and K. (2013) derived the following NSD efficiency test:

Assume that scenarios are ordered in the ascending order according to returns of portfolio  $\tau$ , i.e.  $\mathbf{x}^t \tau \leq \mathbf{x}^{t+1} \tau, t = 1, 2, \dots, T-1$ . Let

$$\theta^*(\tau) = \min_{\beta_n, \gamma_k, \theta} \theta \quad (4)$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{t=1}^T (\mathbf{x}^t \tau - \mathbf{x}_j^t) p_t \left( \sum_{n=1}^{N-2} n \beta_n (\mathbf{x}^t \tau - \mathbf{x}^T \tau)^{n-1} + \right. \\ & \left. (N-1) \sum_{k=t}^T \gamma_k (\mathbf{x}^t \tau - \mathbf{x}^k \tau)^{N-2} \right) + \theta \geq 0, \quad j = 1, \dots, M \end{aligned}$$

$$(-1)^n \beta_n \leq 0, \quad n = 1, \dots, N-2$$

$$(-1)^{N-1} \gamma_k \leq 0, \quad k = 1, 2, \dots, T$$

$$\sum_{t=1}^T \left( \sum_{n=1}^{N-2} n \beta_n (\mathbf{x}^t \tau - \mathbf{x}^T \tau)^{n-1} + (N-1) \sum_{k=t}^T \gamma_k (\mathbf{x}^t \tau - \mathbf{x}^k \tau)^{N-2} \right) p_t = 1.$$

A portfolio  $\tau$  is NSD efficient  $\Leftrightarrow \theta^*(\tau)$  given by (4) is equal to zero.

# Ordering of returns

To be able to use the necessary and sufficient condition for NSD efficiency one needs to order the returns of any portfolio. We may do it, for example, using so-called permutations matrix  $P = \{p_{i,k}\}_{i,k=1}^T$ , that is, a 0-1 matrix that satisfies:

$$\sum_{i=1}^T p_{i,k} = \sum_{k=1}^T p_{i,k} = 1, \quad p_{i,k} \in \{0, 1\}, \quad i, k = 1, \dots, T.$$

Then for any portfolio returns  $\mathbf{x}^t \boldsymbol{\tau}$ ,  $t = 1, 2, \dots, T$ , a permutation matrix  $P$  exists such that:

$$(\mathbf{X}\boldsymbol{\tau})^{[t]} = \sum_{k=1}^T p_{t,k} \mathbf{x}^k \boldsymbol{\tau}$$

that is,  $P\mathbf{X}\boldsymbol{\tau}$  is a vector of ordered returns of portfolio  $\boldsymbol{\tau}$  from the smallest one.



# Final model - risk minimization under NSD efficiency

$$\min_{\tau \in \Lambda} \sigma^2(\mathbf{r}\tau) = \frac{1}{T} \sum_{t=1}^T \left( \mathbf{x}^t \tau - \frac{1}{T} \sum_{s=1}^T \mathbf{x}^s \tau \right)^2$$

s.t.

$$\frac{1}{T} \sum_{t=1}^T \mathbf{x}^t \tau \geq m$$

$$y^t = \sum_{k=1}^T p_{t,k} \mathbf{x}^k \tau, \quad t = 1, 2, \dots, T$$

$$\sum_{i=1}^T p_{i,k} = \sum_{k=1}^T p_{i,k} = 1, \quad p_{i,k} \in \{0, 1\}, \quad i, k = 1, \dots, T$$

$$y^{t+1} \geq y^t, \quad t = 1, 2, \dots, T-1$$

$$\begin{aligned} & \sum_{t=1}^T (y^t - \sum_{k=1}^T p_{t,k} x_j^k) p_t \left( \sum_{n=1}^{N-2} n \beta_n (y^t - y^T)^{n-1} + \right. \\ & \left. (N-1) \sum_{k=t}^T \gamma_k (y^t - y^k)^{N-2} \right) \geq 0, \quad j = 1, \dots, M \\ & (-1)^n \beta_n \leq 0, \quad n = 1, \dots, N-2 \\ & (-1)^{N-1} \gamma_k \leq 0, \quad k = 1, 2, \dots, T \\ & \sum_{t=1}^T \left( \sum_{n=1}^{N-2} n \beta_n (y^t - y^T)^{n-1} + (N-1) \sum_{k=t}^T \gamma_k (y^t - y^k)^{N-2} \right) p_t = 1. \end{aligned}$$

# Alternative models

One can easily use another measure of risk instead of variance, for example for equiprobable scenarios:

- semivariance:

$$\sigma^2(\mathbf{r}'\boldsymbol{\tau}) = \frac{1}{T} \sum_{t=1}^T \left( (\mathbf{x}^t \boldsymbol{\tau} - \frac{1}{T} \sum_{s=1}^T (\mathbf{x}^s \boldsymbol{\tau}))^- \right)^2$$

- Value at Risk:

$$\begin{aligned} \text{VaR}_\alpha(-\mathbf{r}'\boldsymbol{\tau}) &= \min_{\gamma, \delta_t} \gamma \\ \text{s.t. } &\gamma + M\delta_t \geq -\mathbf{x}^t \boldsymbol{\tau}, \quad t = 1, \dots, T \\ &\sum_{t=1}^T \delta_t = \lfloor (1 - \alpha)T \rfloor, \quad \delta_t \in \{0, 1\}, \quad t = 1, \dots, T \end{aligned}$$

- Conditional Value at Risk:

$$\begin{aligned} \text{CVaR}_\alpha(-\mathbf{r}'\boldsymbol{\tau}) &= \min_{v \in \mathbb{R}, z_t \in \mathbb{R}^+} v + \frac{1}{(1 - \alpha)T} \sum_{t=1}^T z_t \\ \text{s.t. } &z_t \geq -\mathbf{x}^t \boldsymbol{\tau} - v, \quad t = 1, 2, \dots, T \end{aligned}$$

- US stock market data from the Kenneth French library.
- We consider a standard set of 10 active benchmark stock portfolios as the base assets. They are formed, and annually rebalanced, based on individual stocks market capitalization of equity, each representing a decile of the cross-section of stocks in a given year. The first decile stocks (the smallest size) are called "small" and the last decile stocks are called "large".
- We include US Treasury bill as a riskless asset.
- We use data on annual excess returns (in %) from 1982 to 2011 (30 observations).
- Hence we have  $n=11$  base assets and  $T=30$  scenarios.

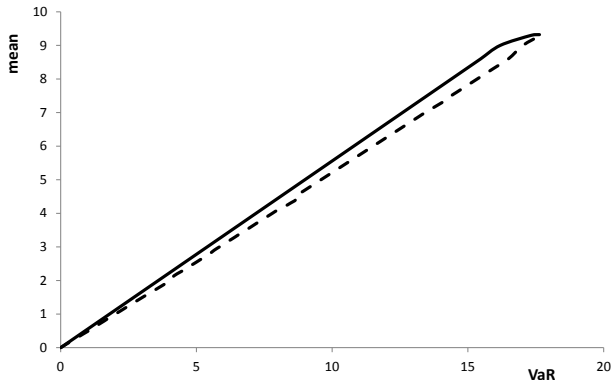
# Empirical study - descriptive statistics

	mean	st. deviation	min	max	skewness	c. kurtosis
Small	8.43	26.66	-44.67	90.27	0.65	1.88
2nd decile	8.06	22.44	-37.92	60.56	0.04	-0.07
3rd decile	8.58	19.56	-34.80	49.95	-0.20	-0.21
4th decile	7.83	18.64	-30.75	47.68	-0.17	-0.15
5th decile	8.97	19.50	-36.86	45.49	-0.19	0.00
6th decile	8.75	17.01	-29.90	40.97	-0.20	-0.15
7th decile	9.32	18.44	-42.48	43.68	-0.49	0.95
8th decile	8.69	17.77	-40.89	39.67	-0.57	0.92
9th decile	8.62	17.12	-43.38	37.90	-0.82	1.67
Large	7.18	16.57	-36.56	32.47	-0.74	0.48

**Table 1:** Base assets 1982-2011 descriptive statistics

# Empirical study - results

Fig. 1: Mean-VaR efficiency frontiers with additional SSD efficiency constraints (dashed line) and without SSD efficiency constraints (solid line)



- We formulated a new type of optimization problems which “combines” two most common approaches to portfolio efficiency
- The new problem can be seen as a generalization of mean-risk models
- The new idea is to add constraints which reduce the feasibility set to the NSD efficient portfolios
- One can use several different risk measures and orders of stochastic dominance, including DARA SD (Post, Fang & K. 2014)
- The disadvantage: computational complexity
- Another disadvantage: the optimal portfolio is very sensitive to changes in probability distribution of returns → some stability analysis is needed, for example stress testing using contamination techniques as proposed in Dupačová & K. (2012, 2014)

# Main related recent references

- J. Dupačová, M. Kopa: Robustness of optimal portfolios under risk and stochastic dominance constraints, *European Journal of Operational Research*, **234**, 2 (2014), 434–441.
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