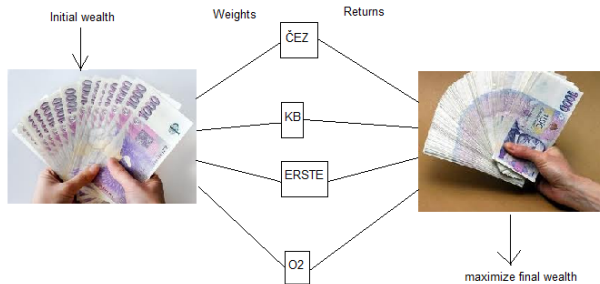


Decision problems with stochastic dominance constraints

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Motivation



- Mean–risk models

$$\max_{\lambda \in \Lambda} \quad m(\lambda' \mathbf{r}) - \nu r(\lambda' \mathbf{r})$$

or

$$\begin{aligned} \min_{\lambda \in \Lambda} \quad & r(\lambda' \mathbf{r}) \\ \text{s.t.} \quad & m(\lambda' \mathbf{r}) \geq \mu \end{aligned}$$

- \mathbf{r} is a random vector of assets returns
- maximizing mean $m(\lambda' \mathbf{r})$ & minimizing risk $r(\lambda' \mathbf{r})$
- risk measures (variance, semi variance,..., VaR, CVaR)
- risk or return parameter (ν, μ)

Alternative portfolio selection models

- probabilistic portfolio selection models - maximal reliability, minimal probability of default,...
- multiobjective portfolio selection models - more than two criteria: reward, risk, liquidity,...
- maximizing expected utility models - utility functions as a tool for risk modelling; non-decreasing, concave utility functions; absolute risk aversion,...
- **stochastic dominance constraint models** - first and second order stochastic dominance relations, allowing random benchmark,...
- other models - DEA based, robustified models, dynamic models (multiperiod, multistage),...

- Notation
- First (second) order stochastic dominance: FSD (SSD)
- Model formulations
- Empirical application

We consider a random vector $\mathbf{r} = (r_1, r_2, \dots, r_N)$ of returns of N assets with a discrete probability distribution described by T equiprobable scenarios. The returns of the assets for the various scenarios are given by

$$X = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^T \end{pmatrix}$$

where $\mathbf{x}^t = (x_1^t, x_2^t, \dots, x_N^t)$ is the t -th row of matrix X representing the assets returns along t -th scenario. We assume that the decision maker may also combine the alternatives into a portfolio. We will use $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)^T$ for a vector of portfolio weights and $X\boldsymbol{\lambda}$ represents returns of portfolio $\boldsymbol{\lambda}$. The portfolio possibilities are given by a simplex

$$\Lambda = \{\boldsymbol{\lambda} \in R^N | \mathbf{1}'\boldsymbol{\lambda} = 1, \lambda_j \geq 0, j = 1, 2, \dots, N\}.$$

In all considered models, we compare the performance of a portfolio with the performance of a benchmark. In the simplest case, the comparison of mean returns is considered. Alternative, FSD and SSD relation is used for comparisons.

The benchmark portfolio is denoted by τ . It may be a current portfolio, a market portfolio (index), random goal,...

The feasible set consists of portfolios which outperforms the benchmark, no matter what kind of comparison is applied.

First order stochastic dominance (FSD) - notation

Let $F_{\mathbf{r}'\lambda}(x)$ denote the cumulative probability distribution function of returns of portfolio λ .

Definition

Portfolio $\lambda \in \Lambda$ dominates portfolio $\tau \in \Lambda$ by the first-order stochastic dominance ($\mathbf{r}'\lambda \succeq_{FSD} \mathbf{r}'\tau$) if

$$F_{\mathbf{r}'\lambda}(x) \leq F_{\mathbf{r}'\tau}(x) \quad \forall x \in \mathbb{R}.$$

In general, FSD relation is expressed by infinitely many inequalities. However, under assumption of equiprobable scenarios, the number of inequalities is equal to the number of scenarios.

First order stochastic dominance (FSD) - interpretation

Other equivalent definitions: $\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$ if

- $Eu(\mathbf{r}'\boldsymbol{\lambda}) \geq Eu(\mathbf{r}'\boldsymbol{\tau})$ for all utility functions.
- No non-satiable decision maker prefers portfolio $\boldsymbol{\tau}$ to portfolio $\boldsymbol{\lambda}$.
- $F_{\mathbf{r}'\boldsymbol{\lambda}}^{-1}(y) \leq F_{\mathbf{r}'\boldsymbol{\tau}}^{-1}(y) \quad \forall y \in [0, 1]$.
- $\text{VaR}_{\alpha}(-\mathbf{r}'\boldsymbol{\lambda}) \leq \text{VaR}_{\alpha}(-\mathbf{r}'\boldsymbol{\tau}) \quad \forall \alpha \in [0, 1]$.
- $X\boldsymbol{\lambda} \geq PX\boldsymbol{\tau}$ for at least one permutation matrix P , that is, binary matrix with all row sums and all column sums equal 1, under assumption of equiprobable scenarios.

Second order stochastic dominance – definitions

Let $F_{\mathbf{r}'\boldsymbol{\lambda}}(x)$ denote the cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$. The twice cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$ is defined as

$$F_{\mathbf{r}'\boldsymbol{\lambda}}^{(2)}(y) = \int_{-\infty}^y F_{\mathbf{r}'\boldsymbol{\lambda}}(x) dx. \quad (1)$$

Definition

Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by the second-order stochastic dominance ($\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$) if and only if

$$F_{\mathbf{r}'\boldsymbol{\lambda}}^{(2)}(y) \leq F_{\mathbf{r}'\boldsymbol{\tau}}^{(2)}(y) \quad \forall y \in \mathbb{R}.$$

In general, also SSD relation is expressed by infinitely many inequalities. However, under assumption of equiprobable scenarios, the number of inequalities is equal to the number of scenarios.

Second order stochastic dominance – interpretation

Other equivalent definitions of SSD relation: $\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$ if

- $Eu(\mathbf{r}'\boldsymbol{\lambda}) \geq Eu(\mathbf{r}'\boldsymbol{\tau})$ for all concave utility functions.
- No non-satiable and risk averse decision maker prefers portfolio $\boldsymbol{\tau}$ to portfolio $\boldsymbol{\lambda}$.
- $F_{\mathbf{r}'\boldsymbol{\lambda}}^{-2}(y) \leq F_{\mathbf{r}'\boldsymbol{\tau}}^{-2}(y) \quad \forall y \in [0, 1]$, where $F_{\mathbf{r}'\boldsymbol{\lambda}}^{-2}$ is a cumulated quantile function.
- $\text{CVaR}_{\alpha}(-\mathbf{r}'\boldsymbol{\lambda}) \leq \text{CVaR}_{\alpha}(-\mathbf{r}'\boldsymbol{\tau}) \quad \forall \alpha \in [0, 1]$, where

$$\begin{aligned} \text{CVaR}_{\alpha}(-\mathbf{r}'\boldsymbol{\lambda}) = \min_{v \in \mathbb{R}, z_t \in \mathbb{R}^+} \quad & v + \frac{1}{(1-\alpha)T} \sum_{t=1}^T z_t \\ \text{s.t.} \quad & z_t \geq -\mathbf{x}^t \boldsymbol{\lambda} - v, \quad t = 1, 2, \dots, T \end{aligned}$$

- $X\boldsymbol{\lambda} \geq WX\boldsymbol{\tau}$ for at least one double stochastic matrix W , that is, non-negative matrix with all row sums and all column sums equal 1, under assumption of equiprobable scenarios.

Risk measures

We assume discrete distribution - equiprobable scenarios

- variance:

$$\sigma^2(\mathbf{r}'\boldsymbol{\lambda}) = \frac{1}{T} \sum_{t=1}^T (\mathbf{x}^t \boldsymbol{\lambda} - \frac{1}{T} \sum_{s=1}^T (\mathbf{x}^s \boldsymbol{\lambda}))^2$$

- Value at Risk:

$$\begin{aligned} \text{VaR}_{\alpha}(-\mathbf{r}'\boldsymbol{\lambda}) &= \min_{\gamma, \delta_t} \gamma \\ \text{s.t. } &\gamma + M\delta_t \geq -\mathbf{x}^t \boldsymbol{\lambda}, \quad t = 1, \dots, T \\ &\sum_{t=1}^T \delta_t = \lfloor (1 - \alpha) T \rfloor \\ &\delta_t \in \{0, 1\}, \quad t = 1, \dots, T \end{aligned}$$

- Conditional Value at Risk:

$$\begin{aligned} \text{CVaR}_{\alpha}(-\mathbf{r}'\boldsymbol{\lambda}) &= \min_{v \in \mathbb{R}, z_t \in \mathbb{R}^+} v + \frac{1}{(1 - \alpha)T} \sum_{t=1}^T z_t \\ \text{s.t. } &z_t \geq -\mathbf{x}^t \boldsymbol{\lambda} - v, \quad t = 1, 2, \dots, T \end{aligned}$$

Model formulations I

Mean-variance model (quadratic programming):

$$\begin{aligned} \min_{\lambda \in \Lambda} \quad & \frac{1}{T} \sum_{t=1}^T (\mathbf{x}^t \lambda - \frac{1}{T} \sum_{s=1}^T (\mathbf{x}^s \lambda))^2 \\ \text{s.t.} \quad & \sum_{t=1}^T (\mathbf{x}^t \lambda) \geq \sum_{t=1}^T (\mathbf{x}^t \tau) \end{aligned}$$

VaR-FSD model (mixed integer programming)

$$\begin{aligned} \min_{\gamma, \delta_t} \quad & \gamma \\ \text{s.t.} \quad & \gamma + M\delta_t \geq -\mathbf{x}^t \lambda, \quad t = 1, \dots, T \\ & \sum_{t=1}^T \delta_t = \lfloor (1 - \alpha) T \rfloor \\ & X\lambda \geq PX\tau \\ & 1'P = 1', \quad P1 = 1 \\ & P, \delta_t \in \{0, 1\}, \quad t = 1, \dots, T \end{aligned}$$

CVaR-SSD model (linear programming)

$$\begin{aligned} \min_{v \in \mathbb{R}, z_t, W \in \mathbb{R}^+} \quad & v + \frac{1}{(1 - \alpha)T} \sum_{t=1}^T z_t \\ \text{s.t.} \quad & z_t \geq -\mathbf{x}^t \boldsymbol{\lambda} - v, \quad t = 1, 2, \dots, T \\ & X\boldsymbol{\lambda} \geq WX\boldsymbol{\tau} \\ & \mathbf{1}'W = \mathbf{1}', \quad W\mathbf{1} = 1 \end{aligned}$$

Other combinations - 9 models - 9 optimal portfolios.

- We take US stock market data from the Kenneth French library. We consider a standard set of 10 active benchmark stock portfolios as the base assets. They are formed, and annually rebalanced, based on individual stocks market capitalization of equity, each representing a decile of the cross-section of stocks in a given year. The first decile stocks (the smallest size) are called "small" and the last decile stocks are called "large".
- Furthermore, we include CRISP proxy of the market portfolio as the benchmark and US Treasury bill as a riskless asset.
- We use data on annual excess returns from 1977 to 2006 (30 observations).
- Out-of-sample analysis: 2007-2011

Empirical application - results

Portfolio compositions:

Portfolio	Variance			VaR			CVaR		
	Mean return	FSD	SSD	Mean return	FSD	SSD	Mean return	FSD	SSD
Riskless	0.27	0.03	0.27	0.30	0.00	0.25	0.28	0.04	0.27
Small	0.00	0.00	0.00	0.32	0.00	0.33	0.06	0.14	0.06
2nd decile	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3rd decile	0.00	0.08	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4th decile	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5th decile	0.00	0.02	0.00	0.24	0.00	0.20	0.00	0.00	0.00
6th decile	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.00
7th decile	0.73	0.42	0.73	0.00	0.67	0.07	0.66	0.71	0.67
8th decile	0.00	0.28	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9th decile	0.00	0.02	0.00	0.14	0.01	0.15	0.00	0.00	0.00
Large	0.00	0.15	0.00	0.00	0.12	0.00	0.00	0.07	0.00
Market	0.00	0.00	0.00	0.00	0.19	0.00	0.00	0.00	0.00

Empirical application - results

Portfolio performance:

In-sample descriptive statistics

	Variance			VaR			CVaR		
	Mean return	FSD	SSD	Mean return	FSD	SSD	Mean return	FSD	SSD
mean	7.16	8.67	7.17	7.16	8.90	7.67	7.16	9.39	7.26
st. deviation	11.04	13.98	11.04	12.82	14.61	13.50	11.16	14.96	11.30
min	-12.87	-19.07	-12.87	-18.26	-19.38	-18.88	-12.00	-16.85	-12.16
max	31.87	38.46	31.87	44.50	38.86	46.88	34.16	47.10	34.61
skewness	0.07	-0.08	0.07	0.46	-0.08	0.45	0.16	0.20	0.16
kurtosis	-0.47	-0.42	-0.47	1.32	-0.62	1.28	-0.16	0.09	-0.16

Out-of-sample descriptive statistics

	Variance			VaR			CVaR		
	Mean return	FSD	SSD	Mean return	FSD	SSD	Mean return	FSD	SSD
mean	3.16	3.94	3.16	2.04	3.52	2.17	2.88	3.54	2.92
st. deviation	23.22	29.03	23.23	22.83	29.63	24.55	23.15	30.16	23.45
min	-30.99	-39.22	-31.00	-29.27	-41.15	-31.70	-30.76	-40.15	-31.15
max	30.40	36.66	30.41	29.02	37.49	31.27	30.24	39.06	30.63
skewness	-0.55	-0.64	-0.55	-0.24	-0.71	-0.26	-0.48	-0.47	-0.48
kurtosis	0.42	0.39	0.42	-0.73	0.81	-0.62	0.24	0.15	0.24

End of the presentation

Thank you for your attention.
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