Stochastic Geometry

Viktor Beneš

December 11, 2018

Introduction

This text presents a supplementary material to the course Stochastic Geometry held by the author at Charles University, Faculty of Mathematics and Physics. It is mainly the list of definitions and theorems, with some remarks and corollaries. The proofs of the assertions are delivered in the lecture, some of them can be found in the literature.

1 Fell topology

Definition 1 *A topological space (S,σ) is a pair where $S \neq \emptyset$ and σ is a topology, system of open sets satisfying (i) $S \in \sigma$, $\emptyset \in \sigma$, (ii) S is closed with respect to finite intersections and arbitrary unions.*

Definition 2 *For $x \in S$, $A \in \sigma$ such that $x \in A$ we call A an (open) neighborhood of x. (S,σ) is a Hausdorff space if for each $x, y \in S$ there exist disjoint neighborhoods U_x, U_y of x, y, respectively.*

Definition 3 *A set $A \subset S$ is compact if each cover of A by open sets contains a finite subcover. A is locally compact if for each $x \in A$ there is a compact neighborhood K, i.e. $x \in K$ and there exists $G \in \sigma$ such that $x \in G \subset K$.*

Definition 4 *A system of sets $\tau \subset \sigma$ is a base of topology σ if any $B \in \sigma$ is a union of elements of τ. A system of sets $\tau' \subset \sigma$ is a subbase for σ and τ if each $B \in \tau$ is a finite intersection of elements of τ'.*

Let $d \in \mathbb{N}$ is arbitrary fixed. We will use the following notation: $\mathcal{F}$, $\mathcal{G}$, $\mathcal{C}$ is the system of closed, open, compact sets in $\mathbb{R}^d$, respectively, including $\emptyset$. Further we denote $\mathcal{F}' = \mathcal{F} \setminus \emptyset$, $\mathcal{G}' = \mathcal{G} \setminus \emptyset$, $\mathcal{C}' = \mathcal{C} \setminus \emptyset$.

Definition 5 *For $A \subset \mathbb{R}^d$ set*

$$\mathcal{F}_A = \{F \in \mathcal{F},\ F \cap A \neq \emptyset\}, \quad \mathcal{F}^A = \{F \in \mathcal{F},\ F \cap A = \emptyset\}.$$

For $A_1, \dots, A_k \subset \mathbb{R}^d$ we denote

$$\mathcal{F}^A_{A_1,\dots,A_k} = \mathcal{F}^A \cap \mathcal{F}_{A_1} \cap \dots \cap \mathcal{F}_{A_k},$$

$k \in \mathbb{N}$.

Remark 1 *We have* $(\mathcal{F}_A)^c = \mathcal{F}^A,\ (\mathcal{F}^A)^c = \mathcal{F}_A,\ \emptyset \in \mathcal{F}^A,\ \emptyset \notin \mathcal{F}_A,$

$$\mathcal{F}^{A_1} \cap \mathcal{F}^{A_2} = \mathcal{F}^{A_1 \cup A_2}.$$

Definition 6 *The Fell topology* σ *on* $\mathcal{F}$ *is given by the subbase*

$$\tau' = \{\mathcal{F}_G,\ G \in \mathcal{G}\} \cup \{\mathcal{F}^K,\ K \in \mathcal{C}\}.$$

Remark 2 *Finite intersections of sets from* τ' *have form* $\mathcal{F}^K_{G_1,\ldots,G_n}$*, thus the system*

$$\tau = \{\mathcal{F}^K_{G_1,\ldots,G_n},\ n \in \mathbb{N},\ G_1,\ldots,G_n \in \mathcal{G},\ K \in \mathcal{C}\}$$

forms a basis of the Fell topology.

Theorem 1 *The space* $(\mathcal{F},\sigma)$*, where* σ *is the Fell topology, is Hausdorff compact with a countable base of open sets. The space* $(\mathcal{F}',\sigma)$ *is only locally compact.*

Corollary 1 *The Borel* $\sigma-$*algebra* $\mathcal{B}(\mathcal{F})$ *on* $\mathcal{F}$ *is generated either by* $\{\mathcal{F}^K,\ K \in \mathcal{C}\}$ *or by* $\{\mathcal{F}_G,\ G \in \mathcal{G}\}$.

Definition 7 *Let* $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$*, we define* $\lim_{n\to\infty} F_n = F$ *if for each neighborhood* U *of* F *we have* $F_n \in U$ *a.a.* $n \in N$*, which means* $\exists_{n_0 \in N}\ \forall_{n \geq n_0}\ F_n \in U$.

Theorem 2 *Let* $F,\ F_n \in \mathcal{F},\ n \in \mathbb{N}$*. The conditions (a), (b) and (c) are equivalent, where (a)* $\lim_{n\to\infty} F_n = F$*,*
(b) it holds simultaneously (b1) and (b2), where

$$(b1) \quad \forall_{G\in\mathcal{G}}\ G \cap F \neq \emptyset \Longrightarrow G \cap F_n \neq \emptyset \text{ a.a. } n \in \mathbb{N},$$

$$(b2) \quad \forall_{K\in\mathcal{C}}\ K \cap F \neq \emptyset \Longrightarrow K \cap F_n \neq \emptyset \ a.a.\ n \in \mathbb{N}.$$

(c) it holds simultaneously (c1) and (c2), where
(c1) if $x \in F$ *then for a.a.* $n \in \mathbb{N}\, \exists_{x_n \in F_n}\, x_n \to x$ *in* $\mathbb{R}^d$*,*
(c2) for any subsequence $\{F_{n_j}\}_{j\in\mathbb{N}}$ *and* $x_{n_j} \in F_{n_j}$ *such that* $x_{n_j} \to x$ *it holds* $x \in F$.

Definition 8 *Let* S *be a topological space, the map* $\phi : S \to \mathcal{F}$ *is upper, lower semicontinuous if* $\phi^{-1}(\mathcal{F}^K),\ \phi^{-1}(\mathcal{F}_G)$ *is open for all* $K \in \mathcal{C},\ G \in \mathcal{G}$*, respectively.*

Theorem 3 *The map* $\phi : S \to \mathcal{F}$ *is upper, lower semicontinuous if and only if for all* $t, t_i \in S,\ i \in \mathbb{N}$ *such that* $t_i \to t$ *we have*

$$\limsup_i \phi(t_i) \subset \phi(t), \quad \liminf_i \phi(t_i) \supset \phi(t),$$

respectively.

On a product space we consider the Tichonov topology, which is the coarsest topology such that all projections are continuous.

Theorem 4 *Consider the maps from* $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$*: (i)* $(F,F') \mapsto F \cup F'$ *is continuous; (ii)* $(F,F') \mapsto F \cap F'$ *is upper semicontinuous. For the maps from* $\mathcal{F} \to \mathcal{F}$*: (iii)* $F \mapsto cl\, F^c$ *is lower semicontinuous, (iv)*$F \mapsto bd\, F$ *is lower semicontinuous.*

Remark 3 *Intersection is not continuous.*

2 Random closed sets

Definition 9 *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space. A map*

$$Z : (\Omega, \mathcal{A}, P) \to (\mathcal{F}, \mathcal{B}(\mathcal{F}))$$

is called a random closed set (RACS). The measure $P_Z = PZ^{-1}$ *is the distribution of* Z.

Remark 4 *Let* Z, Y *be RACS's, then* $Z \cup Y$, $Z \cap Y$, clZ, bdZ *are RACS's.*

Definition 10 *Capacity functional* $T_Z : \mathcal{C} \to [0,1]$ *of a RACS* Z *is*

$$T_Z(K) = P_Z(\mathcal{F}_K),\ K \in \mathcal{C}.$$

Theorem 5 *The distribution of a RACS is uniquely determined by the capacity functional.*

Definition 11 *For a RACS* Z *and* $K \in \mathcal{C}$ *let* $S_0(K) = 1 - T_Z(K)$. *By induction*

$$S_n(K_0; K_1, \ldots, K_n) = S_{n-1}(K_0; K_1, \ldots, K_{n-1}) - S_{n-1}(K_0 \cup K_n; K_1, \ldots, K_{n-1}),$$

$n \in \mathbb{N}$, $K_0, \ldots, K_n \in \mathcal{C}$.

Remark 5 *It holds* $S_n(K_0; K_1, \ldots, K_n) = P_Z\left(\mathcal{F}^{K_0}_{K_1,\ldots,K_n}\right)$.

Theorem 6 *Let* Z *be a RACS, then (i)*

$$0 \le T_Z \le 1,\ T_Z(\emptyset) = 0,$$

(ii) if $K_j, K \in \mathcal{C},\ j \in \mathbb{N},\ K_j \downarrow K$, *then*

$$T_Z(K_j) \to T_Z(K).$$

(iii) $S_n(K_0; K_1, \ldots, K_n) \ge 0,\ n \in \mathbb{N},\ K_0, \ldots, K_n \in \mathcal{C}$.

Definition 12 *Function* $T : \mathcal{C} \to [0,1]$ *satisfying (i),(ii) from Theorem 6 is called Choquet capacity. Choquet capacity satisfying (iii) from Theorem 6 is called alternating capacity of infinite order.*

Theorem 7 *For an alternating capacity* T *of infinite order there exists a unique probability measure* Q *on* $\mathcal{F}$ *such that*

$$T(K) = Q(\mathcal{F}_K),\ K \in \mathcal{C}.$$

Remark 6 *Each alternating capacity of infinite order is a capacity functional of a RACS.*

Definition 13 *Let (Z, Z') be a pair of RACS's. We say that Z, Z' are independent if their joint distribution*

$$P_{(Z,Z')} = P_Z \otimes P_{Z'}.$$

Definition 13 can be extended to countably many RACS's.

Definition 14 *For a RACS Z the function $m(x) = P(x \in Z)$ is called inclusion probability of $x \in \mathbb{R}^d$ into Z. Denote*

$$k(x, y) = \mathrm{cov}(I_Z(x), I_Z(y)).$$

Remark 7 *We have*

$$m(x) = \int_{\mathcal{F}} I_F(x) \mathrm{d}P_Z(F),$$

$$k(x, y) + m(x)m(y) = P(\{x, y\} \subset Z).$$

Definition 15 *A RACS Z is stationary, isotropic if its distribution is invariant with respect to shifts, rotations, respectively.*

Theorem 8 *For a stationary RACS Z it holds $m(x) = p = \mathbb{E}[Leb([0, 1]^d \cap Z)]$ (volume fraction),*

$$k(x, 0) = k(x) = \mathbb{E}[Leb([0, 1]^d \cap Z \cap Z - x)] - p^2,\ x \in \mathbb{R}^d.$$

For $A, B \subset \mathbb{R}^d$ denote $\hat{A} = \{-x, x \in A\},\ A + B = \{x + y,\ x \in A, y \in B\}$ (Minkowski addition of sets). Let $\mathcal{K} \subset \mathcal{C}$ be the system of convex compact sets in $\mathbb{R}^d$, $\mathcal{K}' = \mathcal{K} \setminus \{\emptyset\}$.

Definition 16 *For $K \in \mathcal{K}'$ with origin $0 \in K$, for $F \in \mathcal{F}$, $x \in \mathbb{R}^d$*

$$d_K(x, F) = \min\{r \geq 0;\ (x + rK) \cap F \neq \emptyset\}$$

is K-distance of x from F.

Remark 8 *It holds $d_K(x, F) \leq r$ if and only if $x \in F + r\hat{K}$.*

Definition 17 *Let Z be a stationary RACS with $0 < p < 1$. Contact distribution function*

$$H_K(r) = P(d_K(0, Z) \leq r | 0 \notin Z).$$

Covariance of Z is $C(x) = k(x) + p^2$.

Remark 9 *It holds*

$$H_K(r) = \frac{p_{Z+r\hat{K}} - p}{1 - p},$$

where $p_{Z+r\hat{K}}$ is the volume fraction of $Z + r\hat{K}$.

3 Boolean model

Definition 18 *Let* Ψ *be a stationary Poisson process in* $\mathbb{R}^d$ *with intensity* $\rho > 0$, $\mathrm{supp}\Psi = \{x_n\}_{n\in\mathbb{N}}$.*Let* Z_j, $j \in \mathbb{N}$, *be iid compact RACS's independent of* Ψ. *Then*

$$Z = \cup_{n\in\mathbb{N}}(Z_n + x_n)$$

is called a Boolean model, Z_n *are called grains.*

Remark 10 *The Boolean model is a stationary RACS.*

Theorem 9 *The capacity functional of the Boolean model* Z *is*

$$T_Z(K) = 1 - \exp(-\rho\mathbb{E}[Leb(K + \hat{Z}_1)]),\ K \in \mathcal{C}.$$

Remark 11 *The distribution of the Boolean model depends only on the constant* ρ *and the distribution of* Z_1.

Theorem 10 *For the Boolean model we have* $p = 1 - e^{-\rho\bar{Z}_1}$, *where* $\bar{Z}_1 = \mathbb{E}[Leb(Z_1)]$. *The covariance*

$$C(x) = 2p - 1 + (1-p)^2 e^{\rho\gamma_{Z_1}(x)},$$

where $\gamma_{Z_1}(x) = \mathbb{E}[Leb(Z_1 \cap (Z_1 - x))]$. *The contact distribution function*

$$H_K(r) = 1 - \exp(-\rho(\mathbb{E}[Leb(rK + \hat{Z}_1)] - \bar{Z}_1)).$$

Denote $Z' = \cup_{n\geq 2}(Z_n + x_n)$ given that $x_1 = 0$. Then from the Slivnyak theorem Z' has the same distribution as Z. Also given $x_1 = 0$ we call Z_1 a typical grain.

Theorem 11 *Let* $p_G = P(Z' \cap Z_1 \neq \emptyset)$, *then*

$$p_G = 1 - \mathbb{E}[\exp(-\rho\mathbb{E}[Leb(Z_1 + \hat{Z}_2)|Z_1])].$$

Example: Consider the Boolean model in $\mathbb{R}^2$ with deterministic circular grains of fixed radii $r > 0$. The probability that a typical grain is isolated is $1 - p_G$. We have

$$p = 1 - e^{-\pi\rho r^2},\ p_G = 1 - e^{-4\pi\rho r^2},\ (1-p)^4 = 1 - p_G$$

and the probability of an isolated typical grain is in the Table.

p	0.1	0.3	0.5	0.7
$1 - p_G$	0.66	0.24	0.063	0.008

Definition 19 *Hausdorff measure* $\mathcal{H}^k$ *of subsets* $\mathbb{R}^d$ *of order* $k \leq d$ *is*

$$\mathcal{H}^k(A) = \lim_{\delta\to 0_+} \inf_{\cup_{i\in\mathbb{N}}G_i} \sum_i \omega_k \left(\frac{diam G_i}{2}\right)^k,$$

where infimum is taken over all coverings $\cup_{i\in\mathbb{N}}G_i$ *of the set* A *by open sets* G_i *such that* $diam G_i \leq \delta$, *where*

$$diam G = \sup\{||x - y||;\ x, y \in G\}.$$

$\omega_k = \frac{\pi^{k/2}}{\Gamma(1+\frac{k}{2})} = \mathcal{H}^k(b(0,1))$ *(volume of unit ball in* $\mathbb{R}^k$*).*

Theorem 12 *Let*

$$p_s = \mathbb{E}\left[\frac{\mathcal{H}^{d-1}(bdZ_1 \cap Z')}{\mathcal{H}^{d-1}(bdZ_1)}\right],$$

then $p_s = p$.

Remark 12 $1 - p_s$ *is the ratio of exposed surface area of the Boolean model.*

Corollary 2 *Define a surface measure* S *on* $\mathcal{B}(\mathbb{R}^d)$ *as*

$$S(A) = \mathcal{H}^{d-1}(A \cap bdZ).$$

Let S_d *be the surface density of* Z*, then*

$$S_d = \rho\bar{S}(1-p), \quad \bar{S} = \mathbb{E}[\mathcal{H}^{d-1}(Z_1)].$$

4 Modeling of RACS's

Let E be a locally compact space with countable base of open sets and Borel σ-algebra $\mathcal{B} = \mathcal{B}(E)$. Denote $\mathcal{M} = \mathcal{M}(E)$ system of locally finite measures on $\mathcal{B}$ equipped with the smallest σ-algebra $\mathfrak{M}$ such that maps $\eta \mapsto \eta(A)$ on $\mathcal{M}$ are continuous for each $A \in \mathcal{B}$. Let $\mathcal{N} \subset \mathcal{M}$ be the system of counting measures, $\mathfrak{N}$ the trace of $\mathfrak{M}$. A random measure

$$X : (\Omega, \mathcal{A}, P) \to (\mathcal{M}, \mathfrak{M})$$

such that $P(X \in \mathcal{N}) = 1$ is called a point process. The point process X in E is simple if $X(\{x\}) \leq 1$ for all $x \in E$. The intensity measure θ of the point process is a locally finite measure on $\mathcal{B}$ such that

$$\theta(A) = \mathbb{E}X(A), \ A \in \mathcal{B}.$$

A Poisson process in E is a simple point process in E such that
(i) $X(A)$ has Poisson distribution with parameter $\theta(A)$ $\forall A \in \mathcal{B}$ with $\theta(A) < \infty$.
(ii) $X(A_1), \ldots, X(A_m)$ are independent random variables for each $A_1, \ldots, A_m \in \mathcal{B}$ pairwise disjoint, $m \in \mathbb{N}$.

Remark 13 *There exists a unique Poisson process with given intensity measure* θ.

Next we deal with the locally compact space $\mathcal{F}'$ of nonempty closed subsets of $\mathbb{R}^d$.

Definition 20 *A particle process in* $\mathbb{R}^d$ *is a simple point process in* $\mathcal{F}'$ *concentrated on* $\mathcal{C}'$*, i.e.* $\theta(\mathcal{F}' \setminus \mathcal{C}') = 0$*. A stationary particle process is such that its distribution is invariant with respect to shifts in* $\mathbb{R}^d$.

The local finiteness of a particle process X can be expressed by $X(\mathcal{F}_C) < \infty$, $C \in \mathcal{C}$.

Theorem 13 *Let X be a point process on $\mathcal{F}'$, then $Z_X = \cup_{\mathrm{supp}X} F$ is RACS. The Boolean model is precisely the union set of a stationary Poisson particle process.*

A particle process can be interpreted as a marked point process Φ in $\mathbb{R}^d$ with a mark in $\mathcal{C}'$. The points are determined by means of a center function.

Definition 21 *For $K \in \mathcal{C}'$ the circumball $B(K)$ with circumcenter $c(K)$ and circumradius $r(K)$ is the smallest closed ball containing K. We have $c : \mathcal{C}' \to \mathbb{R}^d$, let*

$$\mathcal{C}_0 = \{K \in \mathcal{C}';\ c(K) = 0\},$$
$$\alpha : \mathcal{C}' \to \mathbb{R}^d \times \mathcal{C}_0,\ K \mapsto (c(K), K - c(K)).$$

Theorem 14 *For a stationary particle process X there is a decomposition of intensity measure*

$$\theta(A) = \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} I_A(x + K) \mathrm{d}x \mathrm{d}Q(K), \tag{1}$$

where $A \in \mathcal{B}(\mathcal{C}')$, $\gamma > 0$ and Q is a probability measure on $\mathcal{C}_0$. θ is invariant w.r.t. translations and if X is isotropic, θ is invariant w.r.t. rotations.

Corollary 3 *Under the assumptions of Theorem 14 we have*

$$\gamma = \lim_{r \to +\infty} \frac{\theta(\mathcal{F}_{b(0,r)})}{r^d \omega_d}.$$

Definition 22 *Let*

$$\mathcal{R}' = \{B \subset \mathbb{R}^d,\ B = \cup_{i=1}^n K_i,\ K_i \in \mathcal{K}', n \in \mathbb{N}\},$$

then $\mathcal{R} = \mathcal{R}' \cup \{\emptyset\}$ is called the convex ring. A map $\varphi : \mathcal{R} \to \mathbb{R}$ is an additive functional if

$$\varphi(A \cap B) + \varphi(A \cup B) = \varphi(A) + \varphi(B)$$

for all $A, B \in \mathcal{R}$ such that $A \cap B \in \mathcal{R}$.

Consider a particle process X with intensity measure θ concentrated on $\mathcal{B}(\mathcal{K}')$,

$$Z = \cup_{suppX} Z_j$$

the union set. Let $K_0 \in \mathcal{K}'$ be a window, we are interested in the random variable $\varphi(Z \cap K_0)$.

Lemma 1 *It holds*

$$\varphi(Z \cap K_0) = \sum_{k=1}^{M} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le M} \varphi(K_0 \cap Z_{i_1} \cap \cdots \cap Z_{i_k}),$$

where

$$M = card\{j :\ Z_j \cap K_0 \neq \emptyset\}.$$

Definition 23 *A map* $\varphi : \mathcal{R} \to \mathbb{R}$ *is conditionally bounded if it is bounded on* $\{K \in \mathcal{K},\ K \subset K_0\}$ *for each* $K_0 \in \mathcal{K}'$.

Theorem 15 *Let* X *be a Poisson particle process,* $\varphi : \mathcal{R} \to \mathbb{R}$ *additive, conditionally bounded,* $\varphi(\emptyset) = 0$. *Then for each* $K_0 \in \mathcal{K}'$ *we have* $\varphi(Z \cap K_0) \in L_1(P)$ *and*

$$\mathbb{E}\varphi(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \int_{\mathcal{K}'} \dots \int_{\mathcal{K}'} \varphi(K_0 \cap K_1 \cap \dots \cap K_k) \mathrm{d}\theta(K_1) \dots \mathrm{d}\theta(K_k).$$

Corollary 4 *In Theorem 15 let* X *be stationary,* $\mathcal{K}_0 = \{K \in \mathcal{K}',\, c(K) = 0\}$. *Then*

$$\mathbb{E}\varphi(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\gamma^k}{k!} \int_{\mathcal{K}_0} \dots \int_{\mathcal{K}_0} \Phi(K_0,\, K_1, \dots, K_k) \mathrm{d}Q(K_1) \dots \mathrm{d}Q(K_k)$$

for a constant $\gamma > 0$ *and probability measure* Q *on* $\mathcal{K}_0$. *Here*

$$\Phi(K_0,\, K_1, \dots, K_k) = \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \varphi(K_0 \cap (K_1 + x_1) \cap \dots \cap (K_k + x_k)) \mathrm{d}x_1 \dots \mathrm{d}x_k.$$

Lemma 2 *Let* α *be a* σ*-finite measure on* $\mathbb{R}^d$, $K_0, \dots, K_k \in \mathcal{B}^d$, *then*

$$\int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \alpha(K_0 \cap (K_1 + x_1) \cap \dots \cap (K_k + x_k)) \mathrm{d}x_1 \dots \mathrm{d}x_k = \alpha(K_0) \prod_{j=1}^{k} Leb(K_j).$$

Corollary 5 *Under the assumptions of Corollary 4*

$$\frac{\mathbb{E}Leb(Z \cap K_0)}{Leb(K_0)} = p$$

(volume fraction).

Further consider a RACS

$$Z = \cup_{(x,C) \in \tilde{\Psi}} (x + C),$$

where $\tilde{\Psi}$ is independently marked point process Ψ with mark space $\mathcal{C}'$.

Theorem 16 *For the capacity functional* T_Z *it holds*

$$T_Z(K) = 1 - \mathbb{E}\left[\prod_{x \in supp\Psi} (1 - T_{C_0}(K - x)) \right],\ K \in \mathcal{C}',$$

where C_0 *is a typical mark.*

5 Processes of k-flats

Definition 24 *Let $G(d,k)$, $1 \leq k < d$, be the system of k-dimensional linear subspaces in $\mathbb{R}^d$. The system of k-flats is a set*

$$A(d,k) = \{F = L + x;\ x \in L^\perp,\ L \in G(d,k)\}.$$

For $L \in G(d,k)$, $F \in A(d,k)$ we denote Leb_L, Leb_F the k-dimensional Lebesgue measure on L, F, respectively.

Remark 14 *It holds $G(d,k) \subset A(d,k)$ is a closed set in $\mathcal{F}$.*

Theorem 17 *Let θ be a locally finite measure concentrated on $A(d,k)$, invariant w.r.t. shifts. Then there is a unique finite measure θ_0 on $G(d,k)$ such that for all $A \in \mathcal{B}(A(d,k))$ we have*

$$\theta(A) = \int_{G(d,k)} \int_{L^\perp} I_A(L+x) \mathrm{d}Leb_{L^\perp}(x) \mathrm{d}\theta_0(L).$$

Definition 25 *Process of k-flats in $\mathbb{R}^d$ is a point process on $\mathcal{F}'$ with intensity measure concentrated on $A(d,k)$.*

Theorem 18 *Let X be a stationary process of k-flats with intensity measure $\theta \neq 0$. Then there are unique constant $\gamma > 0$ and probability measure Q on $G(d,k)$ such that for all $f : A(d,k) \to \mathbb{R}_+$ measurable we have*

$$\int_{A(d,k)} f \mathrm{d}\theta = \gamma \int_{G(d,k)} \int_{L^\perp} f(L+x) \mathrm{d}Leb_{L^\perp}(x) \mathrm{d}Q(L).$$

Denote π_0 the projection from

$$\cup_{k=1}^{d-1} A(d,k)$$

onto

$$\cup_{k=1}^{d-1} G(d,k).$$

Corollary 6 *For $A \in \mathcal{B}(A(d,k))$ we have in Theorem 18*

$$\gamma = \frac{\mathbb{E}[X(\mathcal{F}_{b(0,1)})]}{\omega_{d-k}}$$

(intensity constant) and

$$Q(A) = \frac{\mathbb{E}[X(\mathcal{F}_{b(0,1)} \cap \pi_0^{-1}(A))]}{\mathbb{E}[X(\mathcal{F}_{b(0,1)})]}$$

(directional distribution).

Definition 26 *Stationary Poisson process X of k-flats is given by the condition*

$$X(\mathcal{F}_{b(0,1)}) \sim Poisson(\gamma \omega_{d-k}).$$

Let $D \in \mathcal{B}^d,\ F \in A(d,k)$, we denote $Leb_F(D) = Leb_F(D \cap F)$.

Theorem 19 *Let X be a stationary process of k-flats with intensity constant $\gamma > 0$. Then*

$$\mathbb{E} \sum_{F \in X} Leb_F = \gamma Leb.$$

Further we investigate sections of flat processes.

Theorem 20 *Let X be a stationary k-flat process in $\mathbb{R}^d$, $1 \leq k \leq d-1$ and $S \in A(d, d-k+j)$, $0 \leq j \leq k-1$. Then $Y = X \cap S$ is a stationary j-flat process. If X is simple, then also Y is simple.*

Definition 27 *Determinant $[L_1, \ldots, L_m]$ of linear subspaces $L_1, \ldots, L_m$ in $\mathbb{R}^d$ for which*

$$\sum_{i=1}^{m} dim L_i = q \leq d$$

is equal to the q-dimensional Lebesgue measure of parallelogram generated by the union of orthonormal bases of all L_i, $i = 1, \ldots, m$. Further for $q \geq (m-1)d$ we set

$$[L_1, \ldots, L_m] = [L_1^{\perp}, \ldots, L_m^{\perp}].$$

Theorem 21 *Let X be a stationary k-flat process in $\mathbb{R}^d$ with intensity $\gamma > 0$ and directional distribution Q. Let $S \in G(d, d-k)$, $Y = X \cap S$. Then for the intensity γ_Y of the stationary point process Y it holds:*

$$\gamma_Y = \gamma \int_{G(d,k)} [S, L] \mathrm{d}Q(L).$$

Remark 15 *For $k = 1$ or $k = d-1$ the directional distribution of a stationary k-flat process can be defined as an even measure φ on the unit sphere S^{d-1} in $\mathbb{R}^d$. For a Borel subset B of S^{d-1} without antipodal points we set*

$$\varphi(B) = \tfrac{1}{2} Q(\{span(u),\, u \in B\}), \qquad k = 1$$
$$\varphi(B) = \tfrac{1}{2} Q(\{u^{\perp},\, u \in B\}), \qquad k = d-1$$

From the previous theorem it holds

$$\gamma_Y(u) = \gamma \int_{S^{d-1}} |\langle u, v \rangle| \mathrm{d}\varphi(v),\ u \in S^{d-1},$$

where $\gamma_Y(u) = \gamma_{Y \cap span(u)}$, $k = d-1$; $\gamma_Y(u) = \gamma_{Y \cap u^{\perp}}$, $k = 1$.

5.1 Processes of $k-$surfaces

Definition 28 *The Hausdorff dimension H_d of a set $A \in \mathcal{B}^d$ is*

$$H_d(A) = \inf\{s \geq 0;\ \mathcal{H}^s(A) = 0\}.$$

Let $\mathcal{K}^{(k)}$ be the system of $K \in \mathcal{K}$ with $H_d(K) = k$, $k \in \{0, 1, \ldots, d-1\}$. Denote $\mathcal{R}^{(k)}$ the system of finite unions of sets from $\mathcal{K}^{(k)}$. Elements of $\mathcal{R}^{(k)}$ are called k-surfaces.

Example: Polygonal surface is a $(d-1)-$surface.

Definition 29 *A process of k-surfaces is a particle process in $\mathbb{R}^d$ with intensity measure concentrated on $\mathcal{R}^{(k)}$.*

Definition 30 *For $B \in \mathcal{R}^{(k)}$ we call $V_k(B) = \mathcal{H}^k(B)$ the $k-$volume of B. For a stationary process X of $k-$surfaces with intensity $\gamma > 0$ and $k-$surface distribution Qon $\mathcal{B}(\mathcal{R}_0^{(k)})$,*

$$\mathcal{R}_0^{(k)} = \{K \in \mathcal{R}^{(k)};\, c(K) = 0\},$$

we call the specific k-volume of X the quantity

$$\bar{V}_k(X) = \gamma \int_{\mathcal{R}_0^{(k)}} V_k(B) \mathrm{d}Q(B).$$

Definition 31 *Let X be a process of k-surfaces, random measure $\eta = \sum_{B \in X} \mathcal{H}^k_{|B}$ on $\mathbb{R}^d$ is given by*

$$\eta(A) = \sum_{B \in X} \mathcal{H}^k(A \cap B),\ A \in \mathcal{B}^d.$$

Theorem 22 *Let X be a stationary process of k-surfaces with $\bar{V}_k(X) < \infty$. Then η is a stationary random measure and*

$$\bar{V}_k(X) = \frac{\mathbb{E}[\eta(A)]}{Leb(A)}$$

is its intensity, for any $A \in \mathcal{B}^d$, $0 < Leb(A) < \infty$.

For $B \in \mathcal{R}^{(k)}$, $B = \cup_{i=1}^m$, $B_i \in \mathcal{K}^{(k)}$ denote

$$\mathcal{O} = \{y \in B_i \cap B_j,\, i \neq j,\, aff B_i \neq aff B_j\},$$

$B_i \subset aff B_i \in A(d,k)$. For $y \in B \setminus \mathcal{O}$ define the tangent plane $T_y(B) \in G(d,k)$ as a subspace parallel to $aff B_i$ such that $y \in B_i$.

Remark 16 *For $\mathcal{H}^k$-almost all $y \in B$ the tangent plane is uniquely determined.*

Theorem 23 *Let X be a stationary process of k-surfaces with $0 < \bar{V}_k(X) < \infty$. Then there exists a unique probability measure $\mathcal{T}$ on $G(d,k)$ such that for all $C \in \mathcal{B}(G(d,k))$ and $A \in \mathcal{B}^d$, $0 < Leb(A) < \infty$, we have*

$$\mathbb{E} \sum_{B \in X} \int_{B \cap A} I_C(T_y(B)) \mathrm{d}\mathcal{H}^k(y) = Leb(A) \bar{V}_k(X) \mathcal{T}(C).$$

Definition 32 *The probability measure $\mathcal{T}$ from the previous theorem is called the directional distribution of the process of $k-$surfaces.*

Remark 17 *For $k = 1$ or $k = d-1$ the directional distribution $\mathcal{T}$ of a stationary process of $k-$surfaces can be interpreted as an even measure φ on the unit sphere S^{d-1} in $\mathbb{R}^d$. For a Borel subset A of S^{d-1} without antipodal points we set*

$$\varphi(A) = \tfrac{1}{2}\mathcal{T}(\{span(u),\, u \in A\}), \qquad k = 1$$
$$\varphi(A) = \tfrac{1}{2}\mathcal{T}(\{u^\perp,\, u \in A\}), \qquad k = d-1.$$

Let X be a stationary process of k-surfaces in $\mathbb{R}^d$, $k = \{1, \dots, d-1\}$ and $S \in G(d, d-k)$. We consider the stationary point process $Y = X \cap S$ with intensity $\gamma_Y > 0$.

Theorem 24 *It holds*

$$\gamma_Y = \bar{V}_k(X) \int_{G(d,k)} [S, L] \mathrm{d}\mathcal{T}(L).$$

Remark 18 *For $k = 1$, $w \in S^{d-1}$ we have*

$$\gamma_{X \cap w^\perp} = \bar{V}_1(X) \int_{S^{d-1}} |\langle u, w \rangle| \mathrm{d}\varphi(u),$$

for $k = d-1$ we have

$$\gamma_{X \cap span\{w\}} = \bar{V}_{d-1}(X) \int_{S^{d-1}} |\langle u, w \rangle| \mathrm{d}\varphi(u).$$

6 Minkowski functionals

Definition 33 *For $0 \le k \le d$ and $K \in \mathcal{K}$ the $k-$th Minkowski functional*

$$W_k(K) = \frac{\omega_d}{\omega_{d-k}} \int_{G(d,k)} Leb_{L^\perp}(p_{L^\perp} K) U_k(\mathrm{d}L), \tag{2}$$

where $p_{L^\perp} K$ is the orthogonal projection of K onto $L^\perp$, U_k is the uniform probability distribution on $G(d,k)$.

Remark 19 *It holds for $K \in \mathcal{K}'$ (i) $W_0(K) = Leb(K)$, $W_d(K) = \omega_d$,*
(ii) $W_{d-1}(K) = \frac{\omega_d}{2}\bar{b}(K)$, where $\bar{b}(K)$ is the mean width of K,
(iii) $W_k(b(0,r)) = r^{d-k}\omega_d$, $W_k(rK) = r^{d-k}W_k(K)$.

Lemma 3 *Minkowski functionals are continuous, nondecreasing, additive and motion invariant.*

Theorem 25 *(Hadwiger) Each functional $f : \mathcal{K} \longrightarrow \mathbb{R}$ which is continuous, additive and motion invariant, is a linear combination of Minkowski functionals, i.e. $f = \sum_{k=0}^d a_k W_k$ for some $a_0, \dots, a_d \in \mathbb{R}$.*

Theorem 26 *(Steiner) Let $K \in \mathcal{K}'$, then for $0 \le i \le d$ we have*

$$W_i(K + b(0,r)) = \sum_{k=0}^{d-i} \binom{d-i}{k} W_{k+i}(K) r^k, \; r > 0. \tag{3}$$

Remark 20 *For $K \in \mathcal{K}'$ we have $\mathcal{H}^{d-1}(bd\,K) = dW_1(K)$.*

Further we will generalize Minkowski functionals to the convex ring $\mathcal{R}$.

Definition 34 *Euler-Poincaré characteristic is a functional $\chi : \mathcal{R} \to \mathbb{Z}$ such that $\chi(\emptyset) = 0$, $\chi(K) = 1$ for $K \in \mathcal{K}$ and it is additive.*

Definition 35 *Minkowski functionals for $B \in \mathcal{R}$ are defined as*

$$W_k(B) = \frac{\omega_d}{\omega_{d-k}} \int_{G(d,k)} \int_{L^\perp} \chi(B \cap (L+s)) Leb_{L^\perp}(\mathrm{d}s) U_k(\mathrm{d}L).$$

Remark 21 *In $\mathcal{R}$ we have $W_d(B) = \omega_d \chi(B)$.*

6.1 Boolean model with convex grains

Let $Z = \cup_{Z_i \in X} Z_i$ be a Boolean model, where X is a stationary Poisson particle process with intensity $\rho > 0$. We assume that X is concentrated on $\mathcal{K}'$, Z_1 has isotropic distribution and $\bar{W}_j = \mathbb{E}[W_j(Z_1)] < \infty$. Denote

$$\psi(K) = \rho \mathbb{E}[Leb(\hat{K} + Z_1)].$$

Theorem 27 *For $K \in \mathcal{K}$ the value of $T_Z(K)$ depends only on ρ and the means of Minkowski functionals of the typical grain. We have*

$$\psi(K) = \frac{\rho}{\omega_d} \sum_{k=0}^{d} \binom{d}{k} W_k(K) \bar{W}_{d-k}. \tag{4}$$

Example: For $d = 3$ using $S(.)$, $\bar{S}$ for the surface area we have

$$\psi(K) = \bar{W}_0 + \frac{1}{2}\bar{b}(K)\bar{S} + \frac{1}{2}\bar{\bar{b}}S(K) + W_0(K).$$

Corollary 7 *Under the above assumptions the contact distribution function of the Boolean model is*

$$H_K(r) = 1 - \exp\left(-\frac{\rho}{\omega_d} \sum_{k=1}^{d} \binom{d}{k} r^k \bar{W}_k W_{d-k}(K) \right), \tag{5}$$

$\log(1 - H_K(r))$ *is polynomial in r.*

Example: (i) For $d = 3$ the contact distribution function (CDF) is

$$H_K(r) = 1 - \exp\left(-\rho \left(\frac{r}{2}\bar{b}(K)\bar{S} + \frac{r^2}{2}\bar{\bar{b}}S(K) + r^3 W_0(K) \right) \right).$$

(ii) Spherical CDF, $K = b(0,1)$,

$$H_S(r) = 1 - \exp\left(-\rho \sum_{k=1}^{d} \binom{d}{k} \bar{W}_k r^k \right).$$

(iii) linear CDF: $K = [0, u]$ unit segment in $\mathbb{R}^2$, $u \in S^1$. $U(.)$ denotes perimeter of a set. We have

$$H_l(r) = 1 - \exp\left(-\rho\,\bar{U}\frac{r}{\pi}\right).$$

We apply this formula to quantify the range of vision in a forest modelled by Boolean model. Let the mean radius of a trunk be $\mathbb{E}R = 0.2$ m, intensity of trees $\rho = 0.01m^{-2}$, $H_l(r) = 1 - \exp(-0.004r)$, $H_l(500) = 0.865$. Standing in a random location in the forest in arbitrary direction an observer would have probability 0.135 of being able to see more than 500m far in a uniformly random direction.

7 Stereology

Stereology makes inference on geometrical parameters of particle processes observed on lower-dimensional probes like sections or projections. General formulas in $\mathbb{R}^d$ are derived from Crofton theorem from integral geometry. Here we restrict to basic problems in $\mathbb{R}^3$ and $\mathbb{R}^2$. Here we use a traditional notation: V volume, S surface area, $\bar{b}$ mean width, A area in $\mathbb{R}^2$, U perimeter. Let X be a stationary isotropic particle process in $\mathbb{R}^3$ with intensity ρ_X.

7.1 Planar section

Consider $L \in A(3, 2)$ and $Y = X \cap L$. Then Y is a stationary isotropic particle process in $\mathbb{R}^2$ with some intensity ρ_Y. We denote by bar expected values of particle characteristics, e.g. $\bar{\bar{b}}$ is the expected mean width of a particle.

Theorem 28 *Under the above assumptions it holds*

$$\rho_X\bar{V} = \rho_Y\bar{A}, \quad \rho_X\bar{S} = \frac{4}{\pi}\rho_Y\bar{U}, \quad \rho_X\bar{\bar{b}} = \rho_Y.$$

In practice one can estimate $\bar{A},\ \bar{U},\ \rho_Y$ from observation in planar section, but three equations in Theorem 28 do not suffice to obtain four spatial characteristics $\bar{V},\ \bar{S},\ \bar{\bar{b}},\ \rho_X$. In order to be able to do it we shall add an assumption on particle shape.

Let X have a.s. disjoint spherical particles X_j with distribution function D_X of diameters,

$$d_X = \int_0^\infty (1 - D_X(x))\mathrm{d}x$$

is the average particle diameter. Let $L \in A(3, 2)$, $Y = X \cap L$. Particle sections $Y_j = X_j \cap L$ are circular with a distribution function D_Y of diameters. From Theorem 28 we have $\rho_X d_X = \rho_Y$.

Theorem 29 *It holds*

$$D_Y(r) = 1 - \frac{1}{d_X}\int_0^\infty (1 - D_X(\sqrt{r^2 + x^2}))\mathrm{d}x \tag{6}$$

Corollary 8 *Let f, g be densities of D_X, D_Y, respectively, then*

$$g(y) = \frac{y}{d_X} \int_0^\infty \frac{f(x)}{\sqrt{x^2 - y^2}} \mathrm{d}x.$$

7.2 Projection of a thin slab

Let $t > 0$ and $X_t = \{(x, y) \in \mathbb{R}^2;\ \exists_{z\in[0,t]}\ (x, y, z) \in X\}$ be a projection of the process in a slab of thickness t. We have to assume that a.s. the projections of disjoint particles do not overlap in X_t, which is a stationary particle process in $\mathbb{R}^2$ with an intensity ρ_t.

Theorem 30 *Expected values of characteristics of X and X_t are related as*

$$\rho_X \left(\bar{V} + \frac{\bar{S}t}{4} \right) = \rho_t \bar{A},$$

$$\rho_X \pi \left(\frac{\bar{S}}{4} + \bar{\bar{b}}t \right) = \rho_t \bar{U},$$

$$\rho_X (\bar{\bar{b}} + t) = \rho_t.$$

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