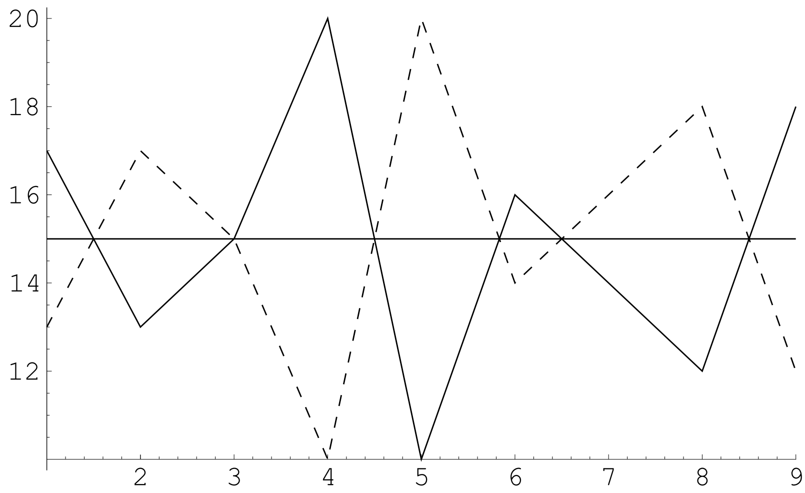
Advanced Topics of Financial Management

Interactive Demonstrations

■ Portfolio Theory

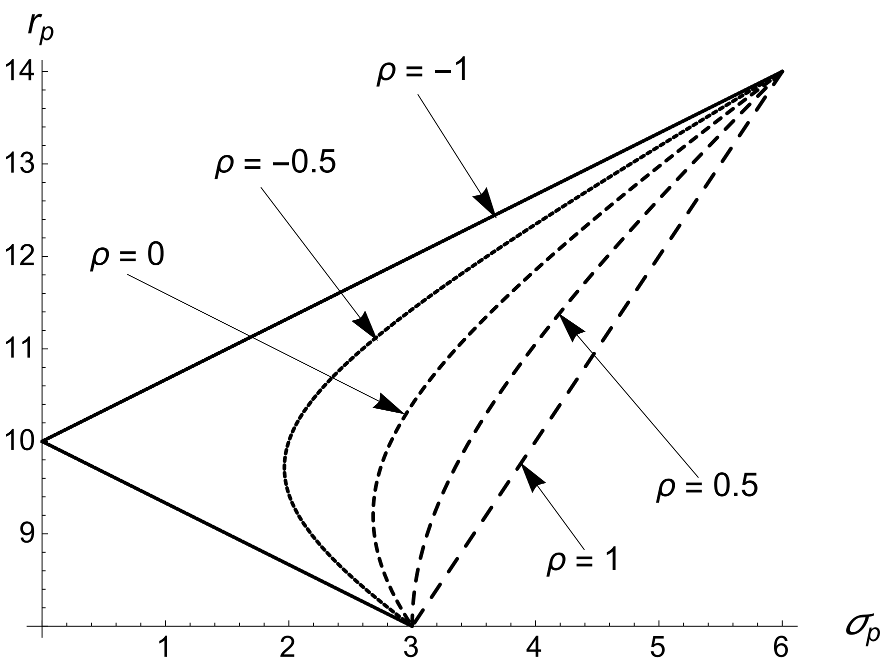
Returns of two perfectly negatively correlated assets (Výnosy dvou perfektně negativně korelovaných aktiv):

```
Import["fig3.eps"]
```



Efficient Frontier of the two assets portfolio (Eficientní hranice portfolia dvou aktiv):

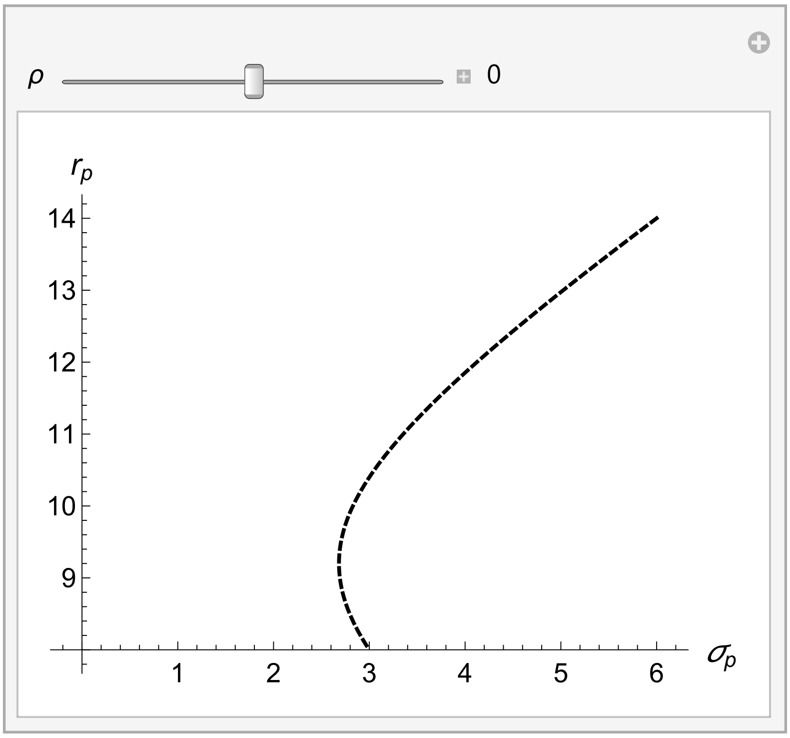
```
Import["EfficientFrontier.eps"]
```



Example 9.2.1 from [1], p. 82

Use slider to change the correlation coefficient:

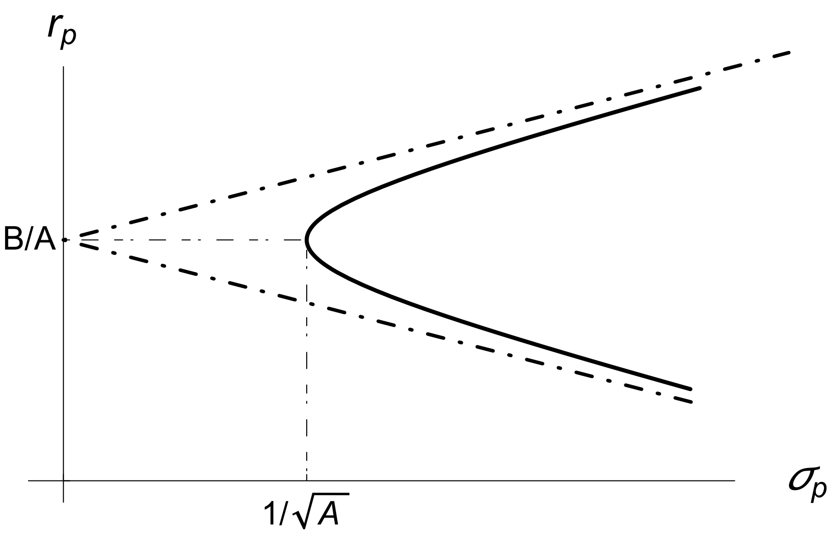
```
Manipulate[ParametricPlot[{Sqrt[36 - 72 x1 + 45 x1^2 + 36 x1 ρ - 36 x1^2 ρ], 14 - 6 x1},
  {x1, 0, 1}, AxesOrigin → {0, 8}, AspectRatio → 3/4,
  AxesLabel → {Style["σp", Italic, 16], Style["rp", Italic, 16]},
  AxesStyle → Directive[14], PlotPoints → 50,
  PlotStyle → {Black, Thick, Dashing[0.01 + 0.01 ρ]}],
 {{ρ, 0}, -1, 1, Appearance → "Labeled"}, SaveDefinitions → True]
```



Minimum Variance Portfolios (Portfolia s minimálním rizikem):

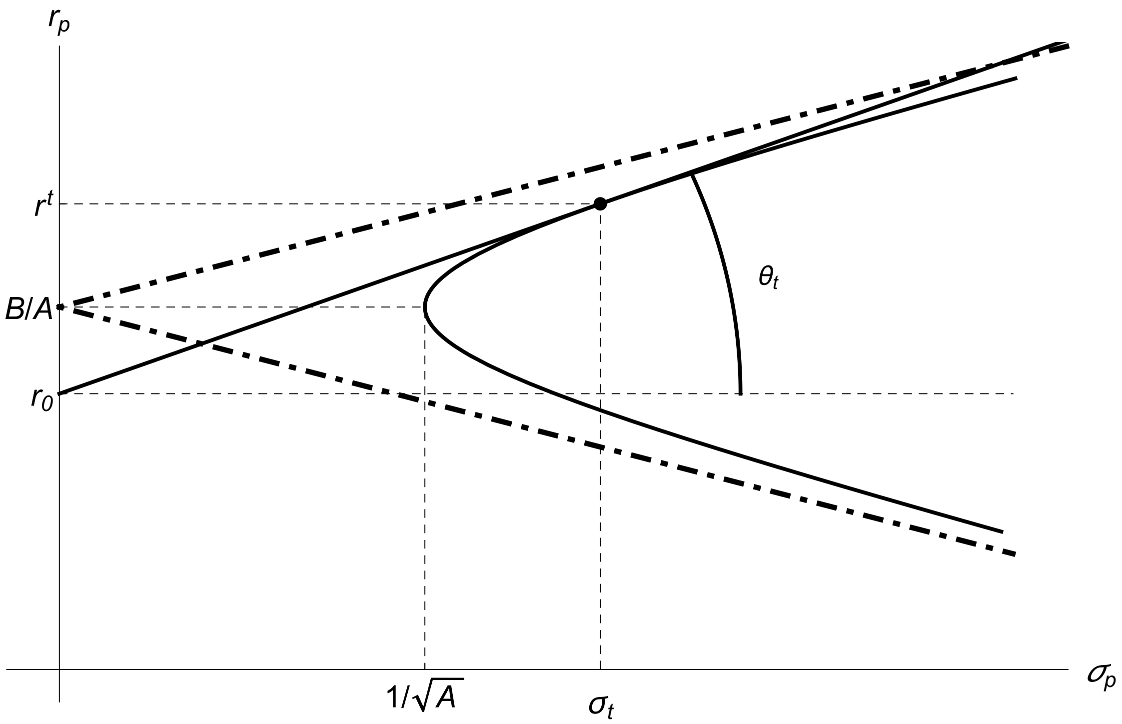
Risky assets only:

```
Import["portfoliorisky2.eps"]
```



Minimum Variance Portfolios with Risky and Riskless Assets (Portfolia s minimálním rizikem při existenci bezrizikového aktiva):

```
Import["portfolior2riskyplusriskless.eps"]
```



Exercise (In Czech)

[3] Cipra, T.: Matematika cenných papírů. Kamil Mařík Professional Publishing. Praha 2013.

Kapitola 2. 3. Analýza akcie z hlediska výnosnosti a rizika, str. 145-175 (bez odstavce 2. 3. 5.)

Úkol 1: přeformulovat v jazyce a symbolice užívané v přednášce a v [1].

Interest Rates and Yield Curves

Decomposition of the Interest Rates (Determinants of the Interest Rate, Dekompozice úrokové míry)

Determinants of the IR

$$1 + r = (1 + r_0)(1 + r_{infl})(1 + r_{default})(1 + r_{liquid})(1 + r_{mat})$$

r ... quoted or nominal interest rate (IR) in a general sense (nominální, kótovaná úroková míra v zobecněném slova smyslu)

Here r_0 denotes the risk-free, riskless IR if we do not consider inflation, (dependend on domestic and international economical conditions, time preference of consumption), r_{infl} inflation premium (prémie za inflaci) expected rate of inflation, r_{default} default risk premium, credit risk premium (prémie za riziko nesplacení, kreditní riziko) is the premium charged for the default risk, that is the risk that the debtor will not pay either principal or interest or both, completely or partly, deliberately or unwittingly. The term r_{liquid}, liquidity premium (prémie za likviditu) stands for the risk that an asset in question is not readily convertible into cash without considerable cost, Finally r_{mat} maturity risk premium (prémie za riziko v okamžiku splatnosti, riziko z vypršení) is the premium for the risk produced by possible changes of interest rates during the life of an asset. There are two types of the maturity risk. Consider bonds, e.g. For long-term bonds, it is the *interest rate risk*; if the market interest rate rises, the prices of bonds go down. This kind of premium rises when the interest rates are more volatile. For short-term bonds, it is the *reinvestment rate risk*; if these bills become due and the actual interest rates are low, the reinvestment will result in interest income loss. It expresses the investor’s impotency what to do with the gained cash. This is also the reason why banks charge extra fees for the early repayment of a loan. At maturity the investment opportunities may substantially differ from those existing at the beginning.

Additive form

Sometimes the decomposition is given in the additive form

$$\mathrm{r} = \mathrm{r}_0 + \mathrm{r}_{\text{default}} + \mathrm{r}_{\text{infl}} + \mathrm{r}_{\text{liquid}} + \mathrm{r}_{\text{mat}}$$

which is a good approximation if the components of r are sufficiently small since the cross-factors of type $r_0\, r_{\text{infl}}$ are small of twice higher order than the original components.

Riskless IR with expected inflation

Riskless IR $r_{0\,\text{infl}}$ with the expected inflation:

$$(1 + r_{0\,\text{infl}}) = (1 + r_0)\,(1 + r_{\text{infl}})$$

Example (calculation r_{default})

Companies rated AAA, AA, A in years 1984 and 1991

```
(* in per cent *)
Clear[r, r0infl, rdefault];
Solve[1 + r/100 == (1 + r0infl/100) (1 + rdefault/100), rdefault]
```

$$\left\{\left\{\text{rdefault} \to \frac{100\,(\text{r} - \text{r0infl})}{100 + \text{r0infl}}\right\}\right\}$$

```
Grid[{{" ", "r1991", "rdefault 1991", "r1984", "rdefault 1984"},
  {"T-bonds", 8, 0, 12.34, 0},
  {"AAA", 8.9, 100 (r - r0infl)/(100 + r0infl) /. {r → 8.9, r0infl → 8},
   12.50, 100 (r - r0infl)/(100 + r0infl) /. {r → 12.50, r0infl → 12.34}},
  {"AA", 9.1, 100 (r - r0infl)/(100 + r0infl) /. {r → 9.1, r0infl → 8}, 12.70,
   100 (r - r0infl)/(100 + r0infl) /. {r → 12.70, r0infl → 12.34}},
  {"A", 9.4, 100 (r - r0infl)/(100 + r0infl) /. {r → 9.4, r0infl → 8}, 13.41,
   100 (r - r0infl)/(100 + r0infl) /. {r → 13.41, r0infl → 12.34}}},
 Dividers → {{True, True}, {True, True}}, Frame → True]
```

	r_{1991}	$r_{default\ 1991}$	r_{1984}	$r_{default\ 1984}$
T-bonds	8	0	12.34	0
AAA	8.9	0.833333	12.5	0.142425
AA	9.1	1.01852	12.7	0.320456
A	9.4	1.2963	13.41	0.952466

Rating

Example (Matrix of transition probabilities, transition matrix, matice pravděpodobností přechodu)

Credit Rating Transition Probabilities

Rating categories, credit classes (in Markov chains’ terminology states of the chain) Standard & Poor’s: AAA, AA, A, BBB, BB, B, C, D. (In practice, we meet more detailed subdivision: AA+, AA-, &c.). AAA means the best credit quality (extremely reliable with regard to financial obligations), AA very good credit quality (very reliable), ..., C close to or already bankrupt, D payment default on some financial obligation has actually occurred (bankrupt firms).

Source: Bluhm & al.: Credit Risk Modeling

Let X_t be a random variable taking values AAA, AA, A, BBB, BB, B, CCC, D at time t. Transition matrix (matice pravděpodobností přechodu) among credit classes is a square matrix $\boldsymbol{P} = (p_{ij})$, $i, j =$ AAA, AA, A, BBB, BB, B, CCC, D, where $p_{i,j} = P(X_{t+1} = j \mid X_t = i)$. Time is usually in years. In literature the last row is often missing because it is always (0, 0, ..., 0, 1). State (class) D is so called absorbing state (pohlcující stav). If a firm falls in D, it cannot quit.

If the transition probabilities remain constant in time (a homogenous Markov chain, e. g.) then the transition matrix from time t to time $t + n$ is simply the n-th power $\boldsymbol{P}^{(n)} = \boldsymbol{P}^n$.

Dimitrios Kavvathas
Goldman Sachs Group, Inc.
Estimating Credit Rating Transition Probabilities for Corporate Bonds,

AFA 2001 New Orleans Meetings
http://web.sakarya.edu.tr/~adurmus/basel_II/SSRN-id252517 %20 rating.pdf

$t \longrightarrow t+1$	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.9446	0.0477	0.0060	0.0010	0.0007	0.0001	0.0000	0.0000
AA	0.0049	0.9179	0.0694	0.0059	0.0007	0.0009	0.0002	0.0001
A	0.0010	0.0187	0.9199	0.0522	0.0055	0.0025	0.0001	0.0001
BBB	0.0004	0.0032	0.0547	0.8819	0.0496	0.0083	0.0007	0.0011
BB	0.0004	0.0014	0.0084	0.0675	0.8388	0.0713	0.0072	0.0050
B	0.0001	0.0007	0.0027	0.0060	0.0562	0.8479	0.0481	0.0384
CCC	0.0012	0.0001	0.0051	0.0046	0.0170	0.0891	0.5401	0.3428
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

The corresponding transition matrix:

```
(transitionmatrix = {{0.9446`, 0.0477`, 0.006`, 0.001`, 0.0007, 0.0001`, 0.`, 0.`},
   {0.0049`, 0.9179`, 0.0694`, 0.0059`, 0.0007, 0.0009`, 0.0002`, 0.0001`},
   {0.001`, 0.0187`, 0.9198999999999999`, 0.052199999999999996`,
    0.005500000000000005`, 0.0025`, 0.0001`, 0.0001`},
   {0.0004`, 0.0032`, 0.0547`, 0.8819`, 0.0496`, 0.0083`, 0.0007, 0.0011`},
   {0.0004`, 0.0014, 0.0084`, 0.0675`, 0.8388`, 0.0713`, 0.0072`, 0.005`},
   {0.0001`, 0.0007, 0.0027`, 0.006`, 0.0562`, 0.8479, 0.0481`, 0.0384`},
   {0.0012`, 0.0001`, 0.0051`, 0.0046`, 0.017`, 0.0891`, 0.5401`, 0.3428`},
   {0.`, 0.`, 0.`, 0.`, 0.`, 0.`, 0.`, 1.`}}) // MatrixForm
```

```
0.9446 0.0477 0.006  0.001  0.0007 0.0001  0.     0.
0.0049 0.9179 0.0694 0.0059 0.0007 0.0009 0.0002 0.0001
 0.001 0.0187 0.9199 0.0522 0.0055 0.0025 0.0001 0.0001
0.0004 0.0032 0.0547 0.8819 0.0496 0.0083 0.0007 0.0011
0.0004 0.0014 0.0084 0.0675 0.8388 0.0713 0.0072  0.005
0.0001 0.0007 0.0027  0.006 0.0562 0.8479 0.0481 0.0384
0.0012 0.0001 0.0051 0.0046  0.017 0.0891 0.5401 0.3428
  0.     0.     0.     0.     0.     0.     0.     1.
```

The transition matrix after 3 years

```
Map[PaddedForm[#, {4, 4}] &, MatrixPower[transitionmatrix, 3], {2}] // MatrixForm
```

```
0.8435  0.1245  0.0251  0.0044  0.0021  0.0006  0.0001  0.0001
0.0130  0.7777  0.1771  0.0244  0.0038  0.0030  0.0005  0.0006
0.0029  0.0481  0.7900  0.1289  0.0201  0.0083  0.0007  0.0010
0.0013  0.0108  0.1354  0.7026  0.1127  0.0284  0.0030  0.0055
0.0011  0.0045  0.0303  0.1530  0.6094  0.1559  0.0183  0.0274
0.0005  0.0021  0.0094  0.0243  0.1232  0.6296  0.0720  0.1392
0.0021  0.0008  0.0100  0.0120  0.0366  0.1345  0.1662  0.6379
0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  1.0000
```

The transition matrix after 5 years

```
Map[PaddedForm[#, {4, 4}] &, MatrixPower[transitionmatrix, 5], {2}] // MatrixForm
```

```
0.7540  0.1810  0.0500  0.0100  0.0036  0.0014  0.0002  0.0002
0.0191  0.6639  0.2528  0.0475  0.0088  0.0055  0.0008  0.0016
0.0047  0.0691  0.6913  0.1789  0.0361  0.0152  0.0015  0.0031
0.0022  0.0192  0.1886  0.5774  0.1452  0.0482  0.0056  0.0133
0.0018  0.0080  0.0540  0.1967  0.4624  0.1927  0.0243  0.0601
0.0009  0.0036  0.0173  0.0441  0.1523  0.4824  0.0653  0.2345
0.0023  0.0016  0.0127  0.0180  0.0449  0.1233  0.0587  0.7386
0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  1.0000
```

```
MatrixPower[transitionmatrix, 5] // MatrixForm
```

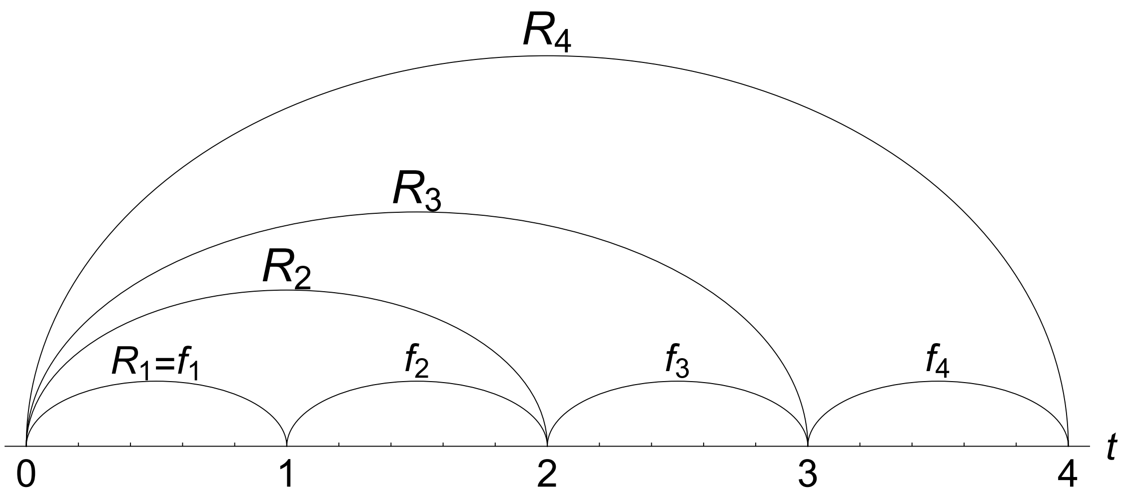
0.754047	0.181039	0.0500082	0.00997099	0.003613	0.00137168	0.000160563	0.0002
0.0190958	0.66389	0.252783	0.0475338	0.00882278	0.00551528	0.000770907	0.001
0.0047493	0.0691222	0.69126	0.178899	0.0361265	0.0151726	0.00154961	0.003
0.00217193	0.0191538	0.188601	0.577437	0.145191	0.0481752	0.00559326	0.013
0.00177282	0.00799809	0.0540338	0.196662	0.462431	0.192664	0.0243407	0.066
0.000865104	0.00356906	0.0172748	0.0441493	0.152305	0.482371	0.0653018	0.23
0.0022732	0.0016416	0.0126774	0.0179962	0.0448539	0.123263	0.0587495	0.73
0.	0.	0.	0.	0.	0.	0.	1

Term Structure of Interest Rates (Časová struktura úrokových měr)

Time structure of interest rates explains the relationships between the yields of **comparable** types of investment for different maturities. Usually it is applied to bonds but generally it may be used for an arbitrary investment. By comparability we mean a set of corporate bonds rated AAA, say. The term structure is changing in time so we must clearly state the respective date τ to which the term structure relates. Sometimes we speak on interest rates implied in the term structure at the particular time τ. Here, for the sake of simplified notation we set $\tau = 0$.

R_t spot (okamžité), f_t forward (forwardové) interest rates

$$(1 + R_t)^t = \prod_{j=1}^{t} (1 + f_j), \quad t = 1, \ \ldots, \ T \qquad (1)$$



Yield Curves (YC, Výnosové křivky) I

YC ... dependence of the interest rates on maturity (závislost úrokové míry na době splatnosti): $r = r(t)$, plot of this function (graf této závislosti)

Generally, a yield curve (výnosová křivka), YC plots interest rates paid on interest bearing securities against the time to maturity. Sometimes we also speak on the term structure of the interest rates. Usually it is applied to zero-coupon bonds but similarly it may be used for an arbitrary investment.

We must emphasize that we have to consider only comparable investments. Yield curves differ both in time and with the type of investment. Thus at the same time we may plot yield curves for government zero coupon bonds for the maturities 1, 3, 6, 9, 12 months getting a completely different picture for AA rated firm's bonds for the same maturities, symbolically $YC_{T-\text{bills},2016} \neq YC_{\text{AA},2016}$. We should also take into account the risk factors (cf. decomposition of interest rate) and also comparable taxation conditions. Even for the same type of securities (like T-bills), the shape of the yield curve differs in time, i.e., the shape is different in years 2015 and 2016, say, ceteris paribus. Symbolically, for an AAA rated company $YC_{\text{AA},2015} \neq YC_{\text{AA},2016}$. This feature may be explained by many factors, like the change in spot riskless rate, inflation, and other exogenous factors. Another important feature is the internal need of the issuer for short, medium, or long financial funds.

$R_t - R_1$ ... *yield spread (výnosové rozpětí)*

Example (Comparison of the yield curves at different reference dates, Porovnání výnosových křivek v různých referenčních časech, Treasury YC USA)

December 15, 2014:

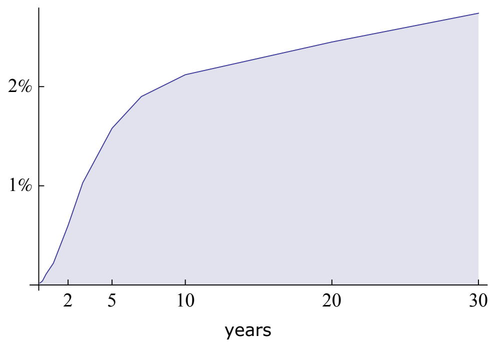
yield curves

Input interpretation:

United States	treasury yield curve

Results:

More



3-month treasury bill	0.04%
1-year treasury bill	0.22%
2-year treasury note	0.6%
5-year treasury note	1.58%
10-year treasury note	2.12%
30-year treasury bond	2.74%

(December 15, 2014)

WolframAlpha

January 5, 2015:

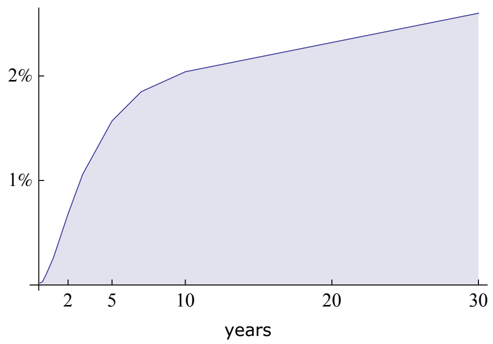
yield curves

Input interpretation:

United States	treasury yield curve

Results:

More



3-month treasury bill	0.03%
1-year treasury bill	0.26%
2-year treasury note	0.68%
5-year treasury note	1.57%
10-year treasury note	2.04%
30-year treasury bond	2.6%

(January 5, 2015)

WolframAlpha

Dec 10, 2015:

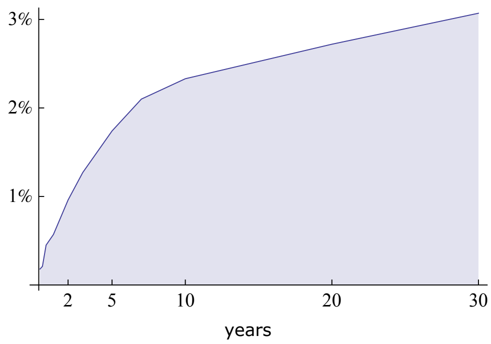
yield curves

Input interpretation:

United States | treasury yield curve

Results:

More



3-month treasury bill	0.21%
1-year treasury bill	0.57%
2-year treasury note	0.96%
5-year treasury note	1.74%
10-year treasury note	2.33%
30-year treasury bond	3.07%

(December 3, 2015)

WolframAlpha

Dec 6, 2016:

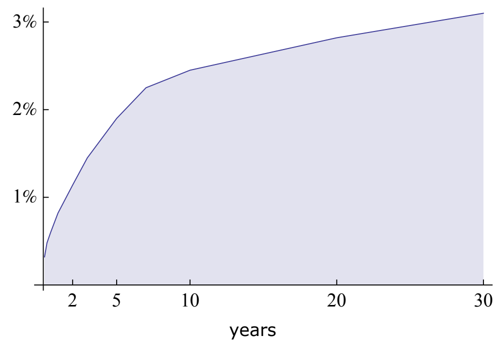
yield curves

Input interpretation:

United States	treasury yield curve

Results:

More



3-month treasury bill	0.48%
1-year treasury bill	0.82%
2-year treasury note	1.14%
5-year treasury note	1.9%
10-year treasury note	2.45%
30-year treasury bond	3.1%

(December 1, 2016)

WolframAlpha

Notes:

Example (Oldřich Alfons Vasicek, EWGFM Oct 2010 Prague)

Here we show that on an efficient market the yield curve cannot be constant (flat). Suppose on the contrary that the (annual) yields for all maturities are constant and equal to $r(t) = r$, $0 \leq t \leq 10$ years. Let us consider three zero-coupon bonds B_1, B_5, and B_{10} with maturities 1, 5, and 10 years and yields $1 + r$, $(1 + r)^5$, and $(1 + r)^{10}$, respectively. Let W be an initial investment. We create two portfolios. Portfolio A consists of B_5 only, portfolio B consists of a combination of B_1 and B_{10} with weights w and $1 - w$ such that the average time to maturity (not the duration!) of such a portfolio is 5 years, the same as of B_5. Hence w must fulfill $1.\,w + 10\,(1 - w) = 5$ and thus $w = 5/9$. With the initial investment $W = 100$ and $r = 0.05$ the corresponding cashflows for the portfolios A and B are

```
cfA = Cashflow[{-100, 0, 0, 0, 0, 100 (1 + r)^5} /. r → 0.05]
cfB = Cashflow[
  {-100, w 100 (1 + r), 0, 0, 0, 0, 0, 0, 0, 0, (1 - w) 100 (1 + r)^10} /. {w → 5/9, r → 0.05}]
```

```
Cashflow[{-100, 0, 0, 0, 0, 127.628}]

Cashflow[{-100, 58.3333, 0, 0, 0, 0, 0, 0, 0, 0, 72.3953}]
```

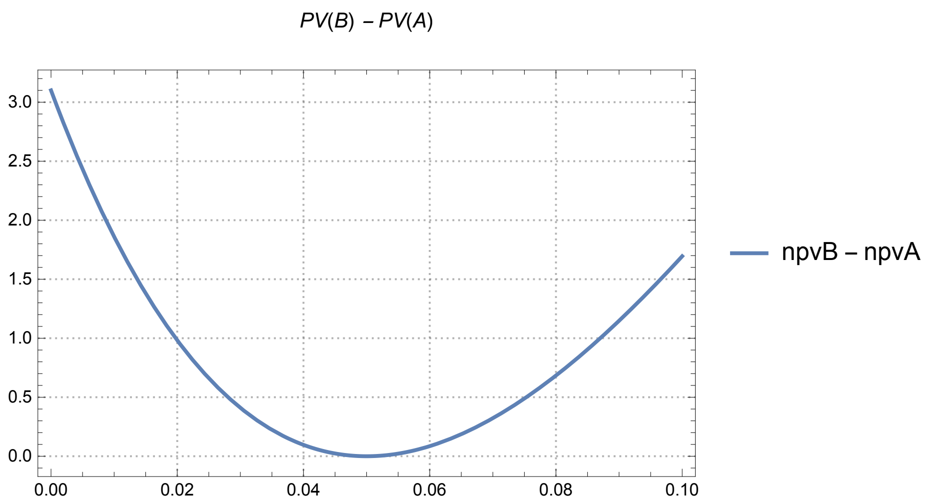
The present values at the valuation interest rate *i*:

```
{npvA = TimeValue[cfA, i], npvB = TimeValue[cfB, i]} // Simplify
```

$$\left\{-100 + \frac{127.628}{(1+i)^5},\ -100 + \frac{72.3953}{(1+i)^{10}} + \frac{58.3333}{1+i}\right\}$$

The dependence of the difference between the present values of cfB and cfA on *i*.

```
Plot[npvB - npvA, {i, 0, 0.1}, PlotStyle → Thick,
 PlotTheme → "Detailed", PlotLabel → "PV(B)  - PV(A)" <> "\n"]
```



Thus the investment *B* provides at least the same yield as the innvestment *B*. In an efficient market, supply and demand would drive the price of the five-year bond down and the prices of the one-year and ten-year bonds up.

```
iVasicek7 = {3, 4, 5, 6, 7} / 100.;
(tableVasicek7 = {{"i", Map[PaddedForm[#, {4, 2}] &, iVasicek7]} // Flatten,
     {"PV_A", Map[PaddedForm[#, {4, 2}] &,
        Table[npvA /. i → i1, {i1, iVasicek7}] // Map[Chop, #] &]} // Flatten,
     {"PV_B", Map[PaddedForm[#, {4, 2}] &, Table[npvB /. i → i1, {i1, iVasicek7}] //
         Map[Chop, #] &]} // Flatten,
     {"Difference PV_B- PV_A ", Map[PaddedForm[#, {4, 2}] &,
        (Table[npvB /. i → i1, {i1, iVasicek7}] - Table[npvA /. i → i1,
            {i1, iVasicek7}]) // Map[Chop, #] &]} // Flatten}) // TableForm;
tableVasicek7Grid = Text@Grid[tableVasicek7, Frame → All]
```

i	0.03	0.04	0.05	0.06	0.07
PV_A	10.09	4.90	0.00	–4.63	–9.00
PV_B	10.50	5.00	0.00	–4.54	–8.68
Difference $PV_B - PV_A$	0.41	0.10	0.00	0.09	0.32

Yield Curves (YC, Výnosové křivky) 2

Statistické a numerické metody konstrukce výnosových křivek

N srovnatelných aktiv (cenných papírů) 1, ..., N,
pozorované, kótované výnosy (úrokové míry) $y_1, \ldots, y_N$ v časech $T_1, \ldots, T_N$

Pro účely finančního rozhodování je zapotřebí znát výnosy také v časech různých od kótovaných minimálně v intervalu $[T_1, T_N]$.

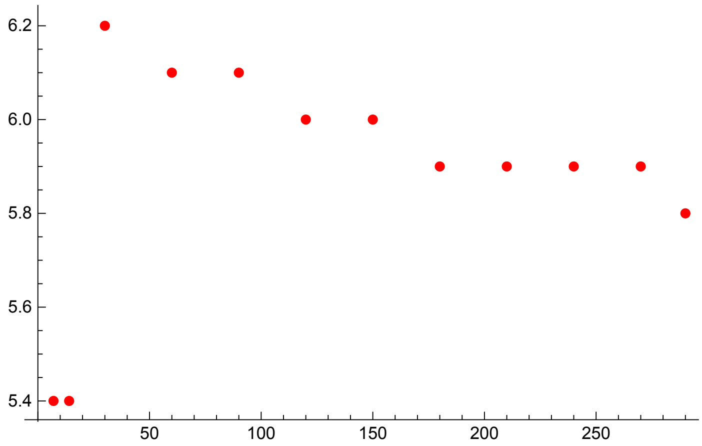
Numerický přístup (interpolace)

Intepolační funkce $y = f(t)$ musí splňovat $y_n = f(T_n)$, $n = 1, \ldots, N$.

Příklad (v grafech pozor na počátky souřadnic)

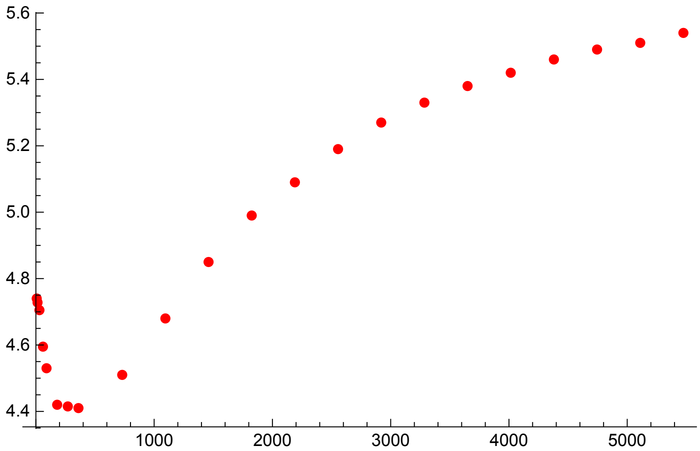
Data ir2 jsou reálná ve tvaru {splatnost (ve dnech), okamžitá (spotová) úroková míra (v procentech)}, $N = 12$. Jedná se o termínovaný vklad České spořitelny, sazby z února 1999.

```
plot2 =
 ListPlot[ir2 = {{7, 5.4`}, {14, 5.4`}, {30, 6.2`}, {60, 6.1`}, {90, 6.1`}, {120, 6.`},
    {150, 6.`}, {180, 5.9`}, {210, 5.9`}, {240, 5.9`}, {270, 5.9`}, {290, 5.8`}},
  PlotStyle → {Red, PointSize[0.015]}]
```



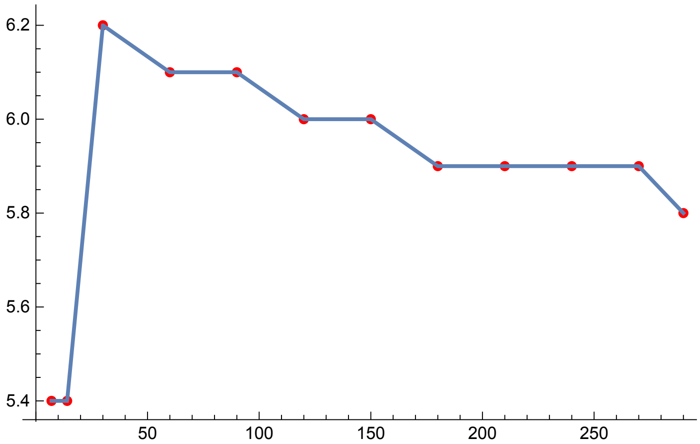
Data ir3 jsou reálná ve tvaru {splatnost (ve dnech), okamžitá (spotová) úroková míra (v procentech)}, $N = 22$.

```
plot3 = ListPlot[
  (ir3 = {{7, 4.74}, {14, 4.728}, {30, 4.705}, {60, 4.595}, {90, 4.53}, {180, 4.42},
     {270, 4.415}, {360, 4.41}, {730, 4.51}, {1095, 4.68}, {1460, 4.85}, {1825, 4.99},
     {2190, 5.09}, {2555, 5.19}, {2920, 5.27}, {3285, 5.33}, {3650, 5.38},
     {4015, 5.42}, {4380, 5.46}, {4745, 5.49}, {5110, 5.51}, {5475, 5.54}}),
  PlotStyle → {Red, PointSize[0.015]}]
```

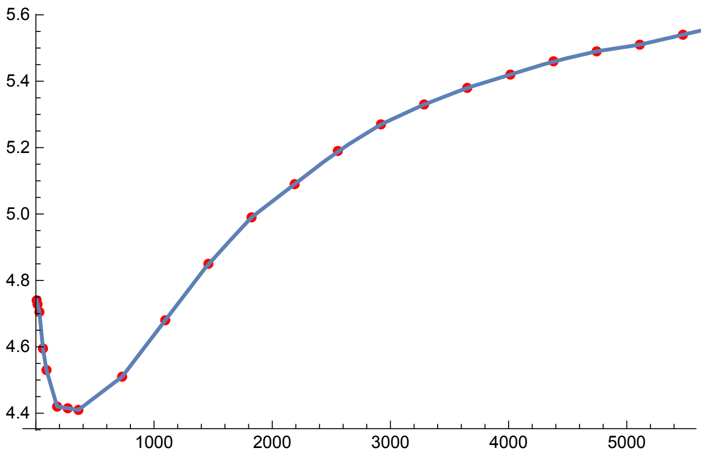


Nejjednodušší (v řadě případů nejlepší) je lineární interpolace, tj. pospojování bodů úsečkami.

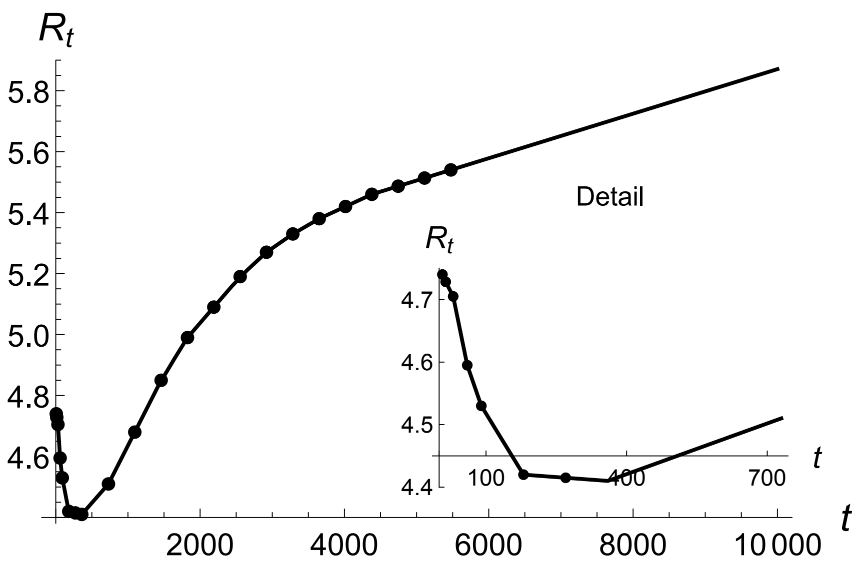
```
int21 = Interpolation[ir2, InterpolationOrder → 1];
plotint21 = Plot[int21[t], {t, 7, 290}, PlotRange → All, PlotStyle → Thick];
Show[plot2, plotint21]
```



```
int31 = Interpolation[ir3, InterpolationOrder → 1];
plotint31 = Plot[int31[t], {t, 7, 10000}, PlotRange → All, PlotStyle → Thick];
Show[plot3, plotint31]
```

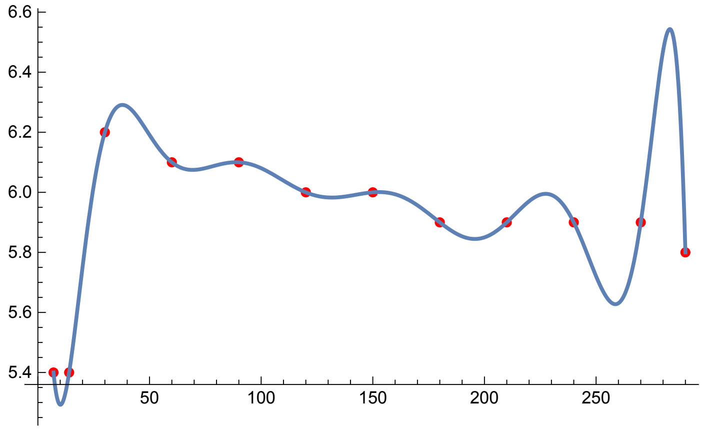


In detail:



The interpolation by polynomials results in a catastrophe:

```
poly2 = InterpolatingPolynomial[ir2, t];
poly2plot = Plot[poly2,
   {t, Min[First[ir2][[1]]], Max[Last[ir2][[1]]]}, PlotRange → All, PlotStyle → Thick];
Show[plot2, poly2plot, PlotRange → All]
```



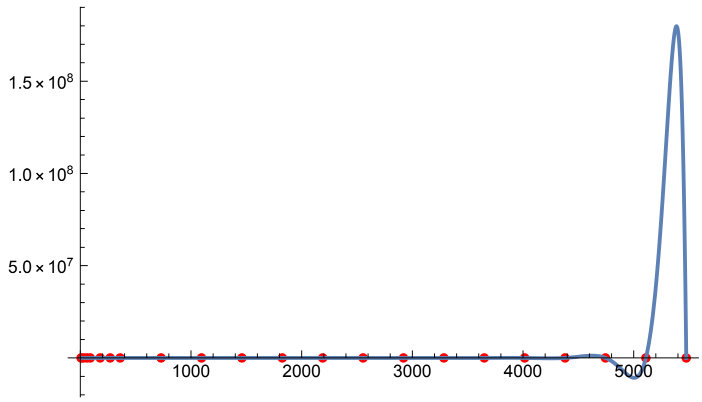
Interpolační polynom je stupně $N-1$, v našem případě 11:

```
poly2 // Expand // TraditionalForm
```

$$-2.70955\times10^{-22}\,t^{11}+4.33735\times10^{-19}\,t^{10}-3.02325\times10^{-16}\,t^{9}+1.20446\times10^{-13}\,t^{8}-3.02631\times10^{-11}\,t^{7}+4.99031\times10^{-9}\,t^{6}-5.44447\times10^{-7}\,t^{5}+0.000038621\,t^{4}-0.00169791\,t^{3}+0.0418424\,t^{2}-0.459345\,t+7.06338$$

Koeficienty polynomů nejsou malá čísla. Uvědomme si, že $t\in[7, 290]$, takže třeba $200^{11}=2.048\times10^{25}$.

```
poly3 = InterpolatingPolynomial[ir3, t];
poly3plot = Plot[poly3,
   {t, Min[First[ir3][[1]]], Max[Last[ir3][[1]]]}, PlotRange → All, PlotStyle → Thick];
Show[plot3, poly3plot, PlotRange → All]
```



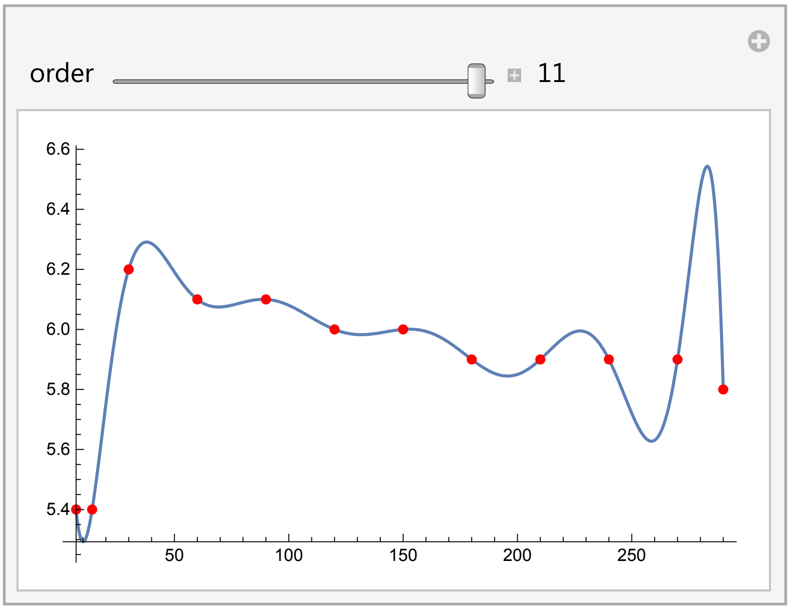
Interpolační polynom je stupně 21:

```
poly3 // Expand // TraditionalForm
```

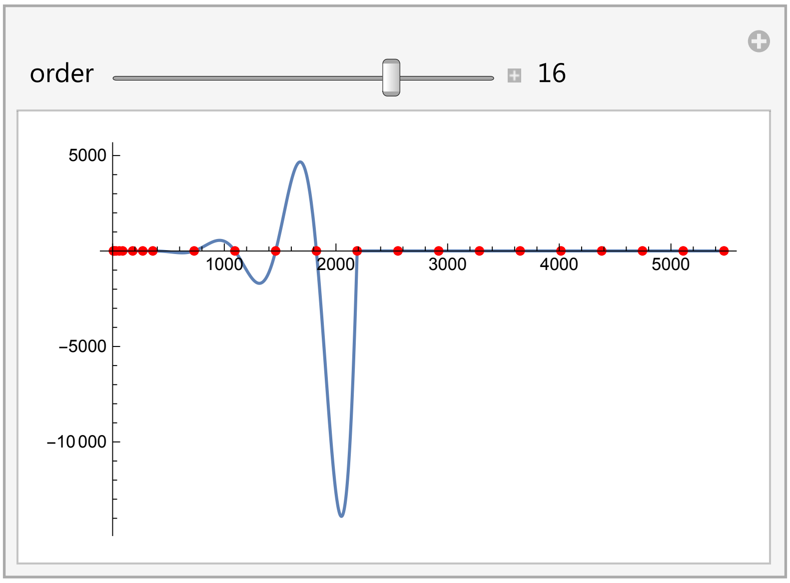
$$-3.33159\times10^{-63}\,t^{21}+1.47944\times10^{-58}\,t^{20}-3.00944\times10^{-54}\,t^{19}+3.71713\times10^{-50}\,t^{18}-3.1155\times10^{-46}\,t^{17}+1.87574\times10^{-42}\,t^{16}-8.37671\times10^{-39}\,t^{15}+2.82525\times10^{-35}\,t^{14}-7.25962\times10^{-32}\,t^{13}+1.42357\times10^{-28}\,t^{12}-2.12108\times10^{-25}\,t^{11}+2.37702\times10^{-22}\,t^{10}-1.97132\times10^{-19}\,t^{9}+1.1825\times10^{-16}\,t^{8}-4.97787\times10^{-14}\,t^{7}+1.41542\times10^{-11}\,t^{6}-2.59175\times10^{-9}\,t^{5}+2.86641\times10^{-7}\,t^{4}-0.00001739\,t^{3}+0.000496623\,t^{2}-0.00747474\,t+4.77331$$

Interpolation by the polynomials of lower order, given by the option InterpolationOrderInterpolace polynomy nižšího stupně, dáno volbou InterpolationOrder. Polynomy na sebe vzájemně navazují (viz pospojování bodů úsečkami, InterpolationOrder → 1):

```
Manipulate[intpol2 = Interpolation[ir2, InterpolationOrder → order];
 Show[plotpol2 = Plot[intpol2[t], {t, Min[First[ir2][[1]]], Max[Last[ir2][[1]]]},
    PlotRange → All], plot2, PlotRange → All],
 {{order, 4, "order"}, 1, 11, 1, Appearance → "Labeled"}, SaveDefinitions → True]
```




```
Manipulate[intpol3 = Interpolation[ir3, InterpolationOrder → order];
 Show[plotpol3 = Plot[intpol3[t], {t, Min[First[ir3][[1]]], Max[Last[ir3][[1]]]},
    PlotRange → All], plot3, PlotRange → All],
 {{order, 4, "order"}, 1, 21, 1, Appearance → "Labeled"}, SaveDefinitions → True]
```



Check it by fitting by the least squares method. If the degree of the polynomial is number of observed pairs minus one, the resulting function is the interpolating polynomial.

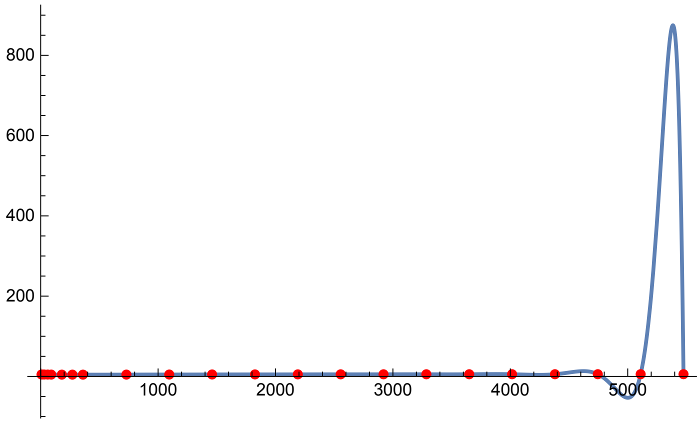
```
t^(Range[Length[ir3]]-1)
```

$\{1, t, t^2, t^3, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, t^{11}, t^{12}, t^{13}, t^{14}, t^{15}, t^{16}, t^{17}, t^{18}, t^{19}, t^{20}, t^{21}\}$

```
Fit[ir3, t^(Range[Length[ir3]]-1), t]
```

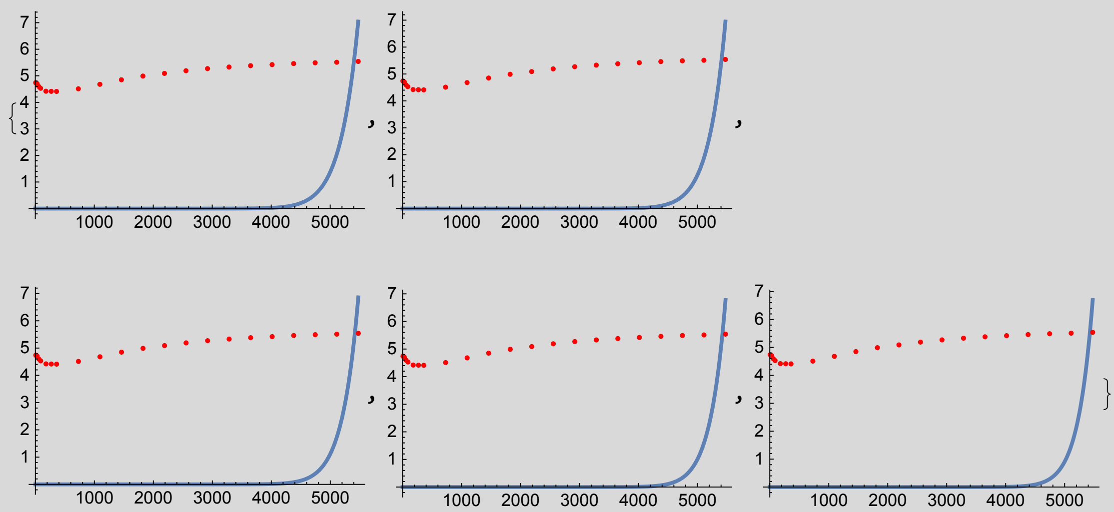
$$4.75025 - 0.000784626\,t - 0.0000525157\,t^2 + 5.63714\times10^{-7}\,t^3 - 2.66501\times10^{-9}\,t^4 + 7.24395\times10^{-12}\,t^5 - 1.25517\times10^{-14}\,t^6 + 1.46479\times10^{-17}\,t^7 - 1.18478\times10^{-20}\,t^8 + 6.68996\times10^{-24}\,t^9 - 2.58683\times10^{-27}\,t^{10} + 6.36526\times10^{-31}\,t^{11} - 7.44797\times10^{-35}\,t^{12} - 5.81929\times10^{-39}\,t^{13} + 2.91258\times10^{-42}\,t^{14} - 8.78535\times10^{-47}\,t^{15} - 8.47293\times10^{-50}\,t^{16} + 8.63044\times10^{-54}\,t^{17} + 2.17425\times10^{-57}\,t^{18} - 5.94336\times10^{-61}\,t^{19} + 5.51152\times10^{-65}\,t^{20} - 1.89735\times10^{-69}\,t^{21}$$

```
Show[Plot[Fit[ir3, t^(Range[Length[ir3]]-1), t] // Evaluate,
  {t, 0, 5475}, PlotRange → All, PlotStyle → Thick], plot3]
```

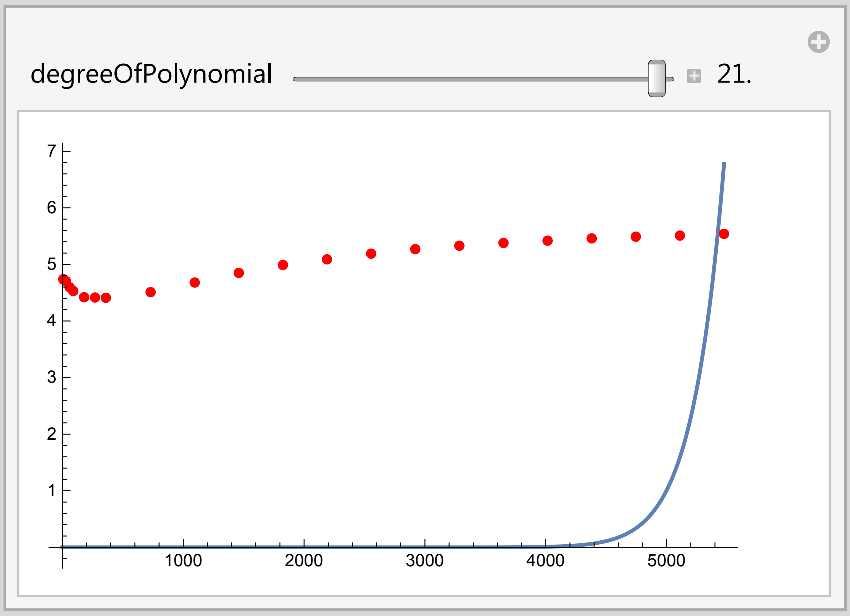


***Skip it, not yet finished.

```
Table[Show[Plot[Fit[ir3, t^k, t] // Evaluate, {t, 0, 5475},
   PlotRange → All, PlotStyle → Thick], plot3], {k, 18, Length[ir3]}]
```



```
Manipulate[Show[Plot[Fit[ir3, t^degreeOfPolynomial, t] // Evaluate,
    {t, 0, 5475}, PlotRange → All, PlotStyle → Thick], plot3],
  {{degreeOfPolynomial, 1}, 1, Length[ir3] - 1, Appearance -> "Labeled"}]
```



Statistický přístup (vyrovnání, fitting): parametrický

In the simplest case we are trying to find a function $y = g(t)$ such that $g(T_n)$ does not differ from y_n too much. V nejjednodušším případě hledáme funkci $y = g(t)$ takovou, že $g(T_n)$ se od y_n "příliš neliší", např. ve smyslu součtu čtverců odchylek.

Příklad (vyrovnání kubickým polynomem)

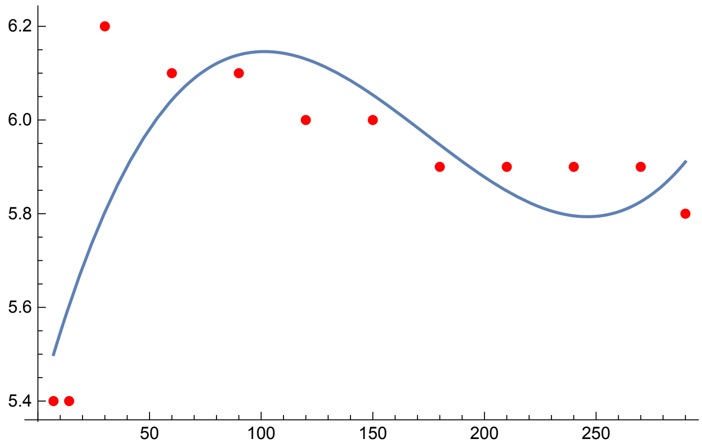
$$g(t;\ \beta_0,\ \beta_1,\ \beta_2,\ \beta_3) = \beta_0 + \beta_1\, t + \beta_2\, t^2 + \beta_3\, t^3$$

β_i neznámé parametry

```
{β0, β1, β2, β3}
```

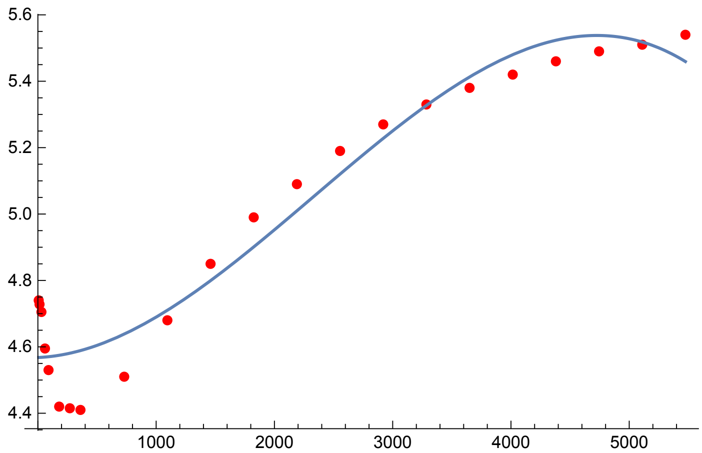
```
(cubic2fit = LinearModelFit[ir2, {t, t^2, t^3}, t]) // Normal
Show[plot2, Plot[cubic2fit[t], {t, Min[First[ir2][[1]]], Max[Last[ir2][[1]]]}]]
```

$5.3832 + 0.017437\,t - 0.000121375\,t^2 + 2.32824\times 10^{-7}\,t^3$




```
(cubic3fit = LinearModelFit[ir3, {t, t^2, t^3}, t]) // Normal
Show[plot3, Plot[cubic3fit[t], {t, Min[First[ir3][[1]]], Max[Last[ir3][[1]]]}]]
```

$4.56783 + 0.000015768\,t + 1.23613\times 10^{-7}\,t^2 - 1.76721\times 10^{-11}\,t^3$



**Příklad (Bradley-Crane)

$g_{BC}(t;\,\alpha,\,\beta,\,\gamma) = \alpha t^{\beta}\,e^{\gamma t}$

$\alpha,\ \beta,\ \gamma$ neznámé parametry

Transformace, která převede funkci nelineární v parametrech na funkci lineární v parametrech; pak je možné použít metod lineární regrese. V našem případě logaritmická transformace

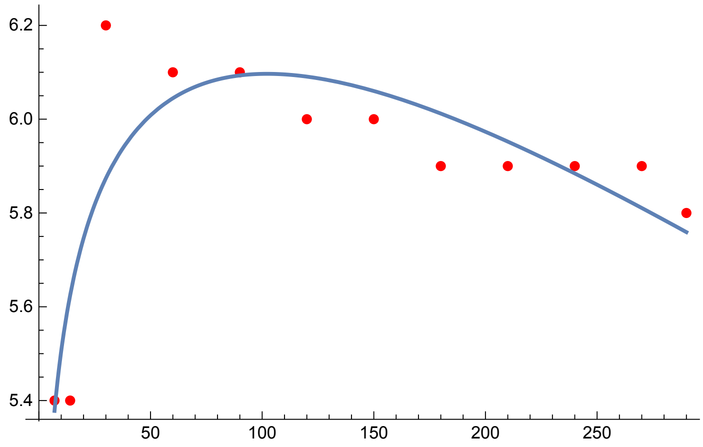
$y^{\star} = \log g(t;\,\alpha,\,\beta,\,\gamma) = \log\alpha + \beta\log t + \gamma t$, po reparamerizaci $\alpha^{\star} := \log\alpha$ je

$y^{\star} = \alpha^{\star} + \beta\log t + \gamma t$

```
(* Logaritmy kótovaných výnosů y* *)
(ir2log = Map[{# // First, Log[# // Last]} &, ir2]) // Transpose // TableForm
(BC2fit =
  NonlinearModelFit[ir2log, αstar + β Log[t] + γ t, {αstar, β, γ}, t] // Normal // Exp //
   Simplify)
Show[plot2, Plot[BC2fit, {t, 7, 290}, PlotStyle → Thick]]
```

7	14	30	60	90	120	150	180	21
1.6864	1.6864	1.82455	1.80829	1.80829	1.79176	1.79176	1.77495	1.

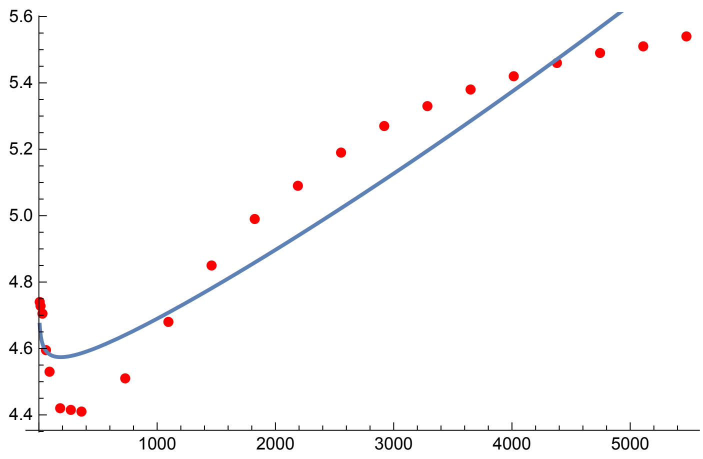
$4.70078\, e^{-0.000701438\, t}\, t^{0.0716908}$




```
(* Logaritmy kótovaných výnosů y* *)
(ir3log = Map[{# // First, Log[# // Last]} &, ir3]) // Transpose // TableForm
(BC3fit =
  NonlinearModelFit[ir3log, αstar + β Log[t] + γ t, {αstar, β, γ}, t] // Normal // Exp //
   Simplify)
Show[plot3, Plot[BC3fit, {t, Min[First[ir3][[1]]], Max[Last[ir3][[1]]]}, PlotStyle → Thick]]
```

7	14	30	60	90	180	270	360	7
1.55604	1.5535	1.54863	1.52497	1.51072	1.48614	1.48501	1.48387	1

$\frac{4.75351\, e^{0.0000495824\, t}}{t^{0.00913463}}$



!Příklad (Nelson-Siegel)

$g_{NS}(t;\, \beta_0,\, \beta_1,\, \beta_2,\, c) =$ ...

```
Clear[nelsonsiegelspot]; (* Spot curve or zero-coupon*)
```

nelsonsiegelspot[b0_, b1_, b2_, c_, t_] := $b0 + \frac{(b1 + b2)\left(1 - e^{-\frac{t}{c}}\right)}{\frac{t}{c}} - b2\, e^{-\frac{t}{c}}$

The parameters have the following meaning: the function starts at level *b*0 + *b*1 at time 0 and has asymptote at level *b*0; a local extreme is determined by parameter *c*.

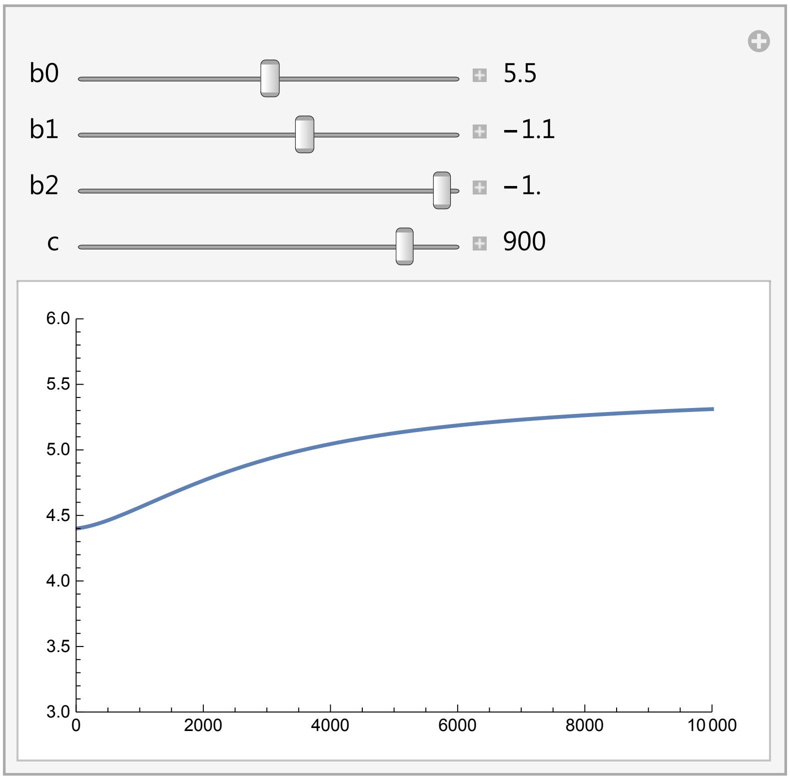
```
Limit[nelsonsiegelspot[b0, b1, b2, c, t], t → 0]
Limit[nelsonsiegelspot[b0, b1, b2, c, t],
 t → ∞, Assumptions → {b0 > 0 ∧ b1 > 0 ∧ b2 > 0 ∧ c > 0}]
```

b0 + b1

b0

Manipulating spot:

```
Manipulate[Plot[nelsonsiegelspot[b0, b1, b2, c, t], {t, 0, 10000},
   PlotRange → {{0, 10000}, {3, 6}}, AxesOrigin → {0, 3}, PlotStyle → {Thick}],
 {{b0, 5.8, "b0"}, 5, 6, 0.1, Appearance → "Labeled"},
 {{b1, -1, "b1"}, -2, -0.5, 0.1, Appearance → "Labeled"},
 {{b2, -3, "b2"}, -5, -1, 0.2, Appearance → "Labeled"},
 {{c, 400, "c"}, 100, 1000, 10, Appearance → "Labeled"}, SaveDefinitions → True
]
```



Remember: *r* ... spot rate, *f* ...forward rate, then

$$R(t) = \frac{1}{t}\int_0^t f(\tau)\, d\tau \qquad (2)$$

(the integral mean, integrální průměr)

Motivation:

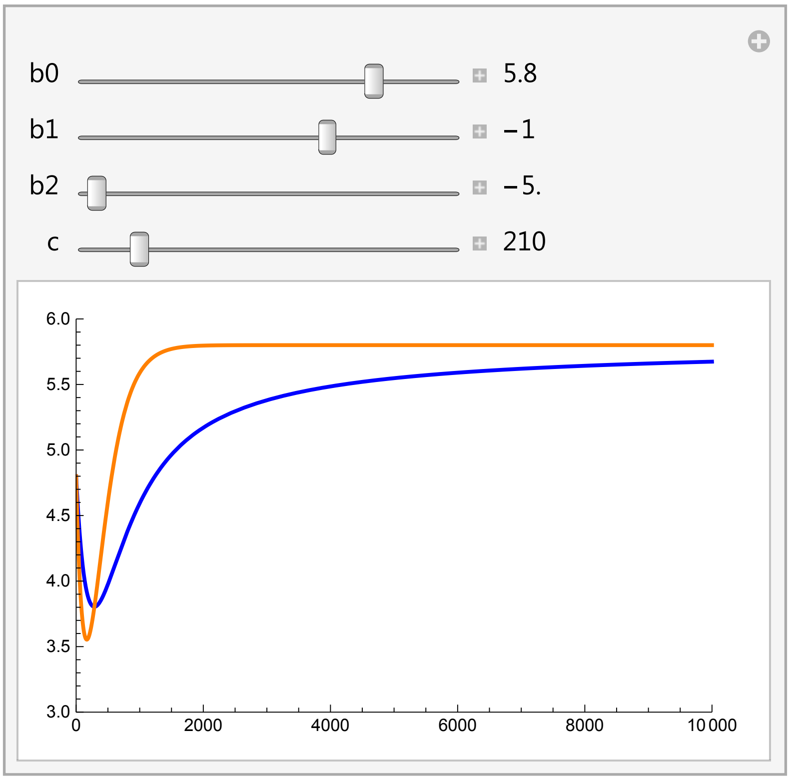
$\log(1 + R_t)$ ... ..

```
(* Forward curve *)
Clear[nelsonsiegelforward];
```

$$\texttt{nelsonsiegelforward[b0\_, b1\_, b2\_, c\_, t\_] := } b0 + b1\, e^{-t/c} + b2\, \frac{t}{c}\, e^{-t/c}$$

Both spot and forward curves:

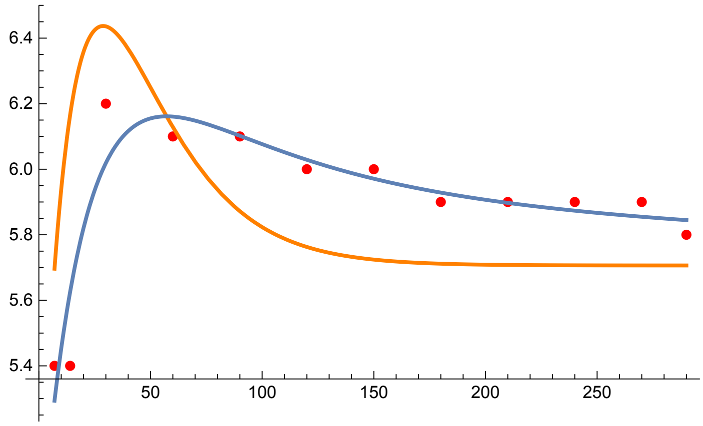
```
Manipulate[
 Plot[{nelsonsiegelspot[b0, b1, b2, c, t], nelsonsiegelforward[b0, b1, b2, c, t]},
  {t, 0, 10000}, PlotRange → {{0, 10000}, {3, 6}},
  AxesOrigin → {0, 3}, PlotStyle → {{Blue, Thick}, {Orange, Thick}}],
 {{b0, 5.8, "b0"}, 5, 6, 0.1, Appearance → "Labeled"},
 {{b1, -1, "b1"}, -2, -0.5, 0.1, Appearance → "Labeled"},
 {{b2, -3, "b2"}, -5, -1, 0.2, Appearance → "Labeled"},
 {{c, 400, "c"}, 100, 1000, 10, Appearance → "Labeled"}, SaveDefinitions → True
]
```



Fitting Nelson-Siegel: on the graph, estimated spot is blue and estimated forward is orange.

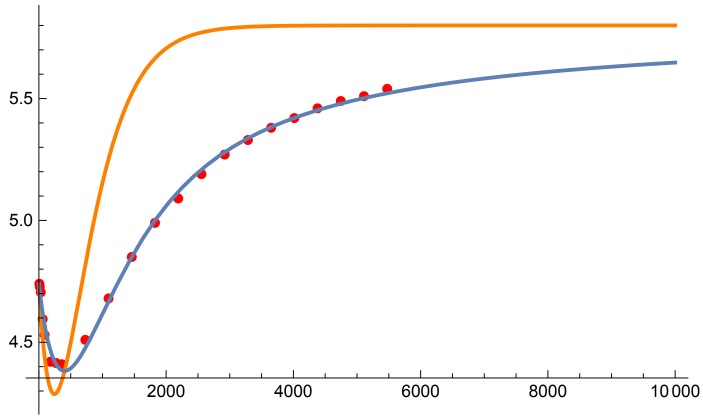
```
fitnelsonsiegel2 = FindFit[ir2, nelsonsiegelspot[b0, b1, b2, c, t], {b0, b1, b2, c}, t]
curvenel2 = nelsonsiegelspot[b0, b1, b2, c, t] /. fitnelsonsiegel2;
curvenelfc2[t_] := nelsonsiegelspot[b0, b1, b2, c, t] /. fitnelsonsiegel2;
curvenel2forward = nelsonsiegelforward[b0, b1, b2, c, t] /. fitnelsonsiegel2;
Show[plot2,
 Plot[curvenel2forward, {t, 7, 290}, PlotRange → All, PlotStyle → {Orange, Thick}],
 Plot[curvenel2, {t, 7, 290}, PlotRange → All, PlotStyle → Thick], PlotRange → All]
```

{b0 → 5.70628, b1 → -0.904452, b2 → 2.75749, c → 21.6675}




```
fitnelsonsiegel3 = FindFit[ir3, nelsonsiegelspot[b0, b1, b2, c, t], {b0, b1, b2, c}, t]
curvenel3 = nelsonsiegelspot[b0, b1, b2, c, t] /. fitnelsonsiegel3;
curvenelfc3[t_] := nelsonsiegelspot[b0, b1, b2, c, t] /. fitnelsonsiegel3;
curvenel3forward = nelsonsiegelforward[b0, b1, b2, c, t] /. fitnelsonsiegel3;
Show[plot3,
 Plot[curvenel3forward, {t, 0, 10000}, PlotRange → All, PlotStyle → {Orange, Thick}],
 Plot[curvenel3, {t, 0, 10000}, PlotRange → All, PlotStyle → Thick], PlotRange → All]
```

{b0 → 5.80001, b1 → -1.06274, b2 → -2.82135, c → 392.246}



!Příklad (Swensson)

Zobecňuje předchozí, obsahuje navíc dva parametry.

$g_{Sw}(t; \beta_0, \beta_1, \beta_2, \beta_3, c_1, c_2) =$ ...

Začneme s jednošším tvarem forwardové křivky:

```
Clear[swenssonforward];
```

swenssonforward[b0_, b1_, b2_, b3_, c1_, c2_, t_] := $b0 + b1\, e^{-t/c1} + b2 \frac{t}{c1} e^{-t/c1} + b3 \frac{t}{c2} e^{-t/c2}$

***Editing notes

```
b0 + b1 e^(-t/c1) + b2 t/c1 e^(-t/c1) + b3 t/c2 e^(-t/c2) /.
 {b0 → β0, b1 → β1, b2 → β2, c1 → γ1, b3 → β3, c2 → γ2}
% // TeXForm
```

$$\beta_0 + e^{-\frac{t}{\gamma_1}} \beta_1 + \frac{e^{-\frac{t}{\gamma_1}} t \beta_2}{\gamma_1} + \frac{e^{-\frac{t}{\gamma_2}} t \beta_3}{\gamma_2}$$

```
\beta _0+\beta _1 e^{-\frac{t}{\gamma _1}}+\frac{\beta _2 t
   e^{-\frac{t}{\gamma _1}}}{\gamma _1}+\frac{\beta _3 t
   e^{-\frac{t}{\gamma _2}}}{\gamma _2}
```

***End of Editing notes

Swensson spot rate obtained as the integral mean:

```
Clear[swenssonspot];
(swenssonspot[b0_, b1_, b2_, b3_, c1_, c2_, t_] =
   1/t ∫_0^t swenssonforward[b0, b1, b2, b3, c1, c2, τ] dτ // Simplify) // TraditionalForm
```

$$\frac{1}{t}\left(\mathrm{b0}\, t + \mathrm{b1}\left(\mathrm{c1} - \mathrm{c1}\, e^{-\frac{t}{\mathrm{c1}}}\right) + \mathrm{b2}\left(\mathrm{c1} - e^{-\frac{t}{\mathrm{c1}}}(\mathrm{c1}+t)\right) + \mathrm{b3}\left(\mathrm{c2} - e^{-\frac{t}{\mathrm{c2}}}(\mathrm{c2}+t)\right)\right)$$

The parameters have the following meaning: the function starts at level $b0 + b1$ at time 0 and has asymptote at level $b0$; a local extreme is determined by parameter c.

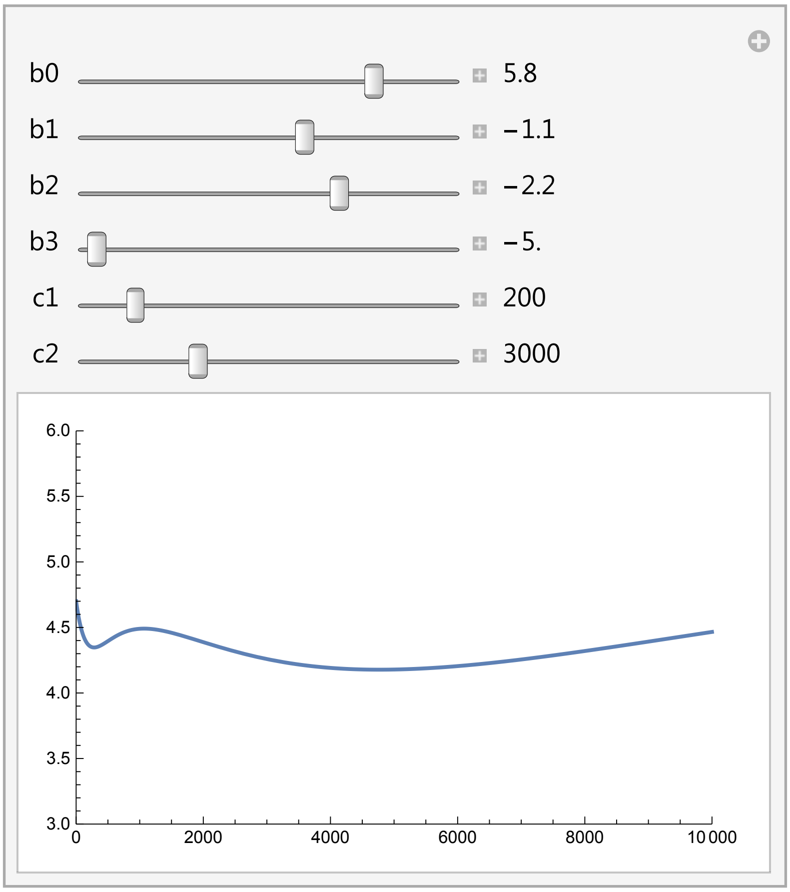
```
Limit[swenssonspot[b0, b1, b2, b3, c1, c2, t], t → 0,
 Assumptions → {b0 > 0 && b1 > 0 && b2 > 0 && b3 > 0 && c1 > 0 && c2 > 0}]
Limit[swenssonspot[b0, b1, b2, b3, c1, c2, t], t → ∞,
 Assumptions → {b0 > 0 && b1 > 0 && b2 > 0 && b3 > 0 && c1 > 0 && c2 > 0}]
```

b0 + b1

b0

Manipulating spot:

```
Manipulate[Plot[swenssonspot[b0, b1, b2, b3, c1, c2, t], {t, 0, 10000},
  PlotRange → {{0, 10000}, {3, 6}}, AxesOrigin → {0, 3}, PlotStyle → {Thick}],
 {{b0, 5.8, "b0"}, 5, 6, 0.1, Appearance → "Labeled"},
 {{b1, -1, "b1"}, -2, -0.5, 0.1, Appearance → "Labeled"},
 {{b2, -3, "b2"}, -5, -1, 0.2, Appearance → "Labeled"},
 {{b3, -3, "b3"}, -5, -1, 0.2, Appearance → "Labeled"},
 {{c1, 200, "c1"}, 100, 1000, 10, Appearance → "Labeled"},
 {{c2, 5000, "c2"}, 100, 10000, 10, Appearance → "Labeled"}, SaveDefinitions → True
]
```

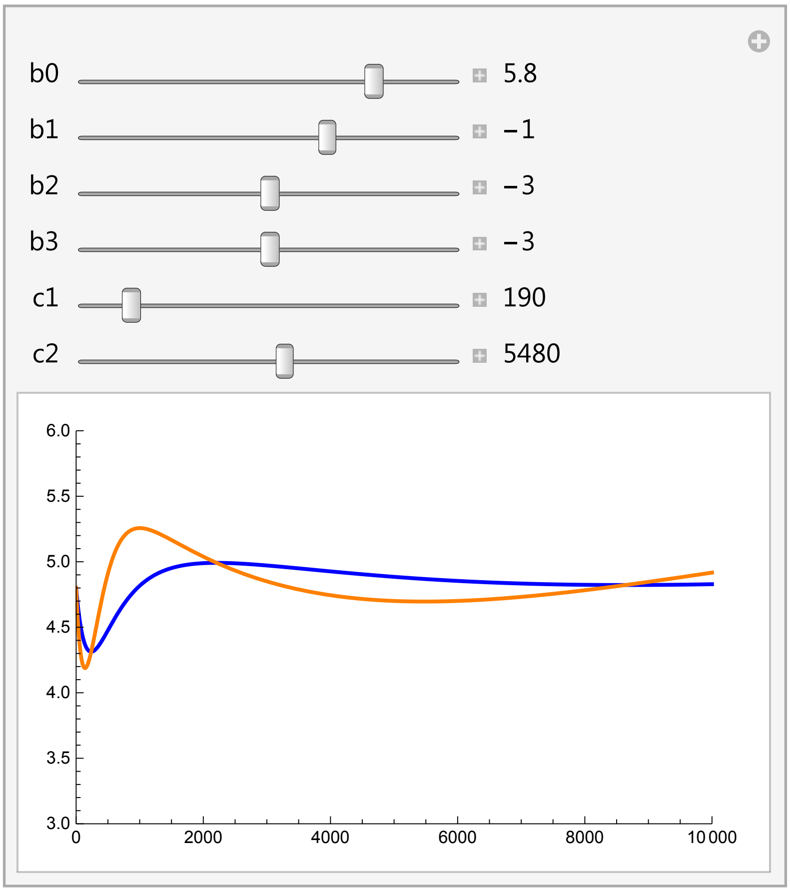


Remeber: *r* ... spot rate, *f* ...forward rate, then

$$R(t) = \frac{1}{t} \int_0^t f(\tau)\, d\tau \qquad (3)$$

Both spot and forward curves

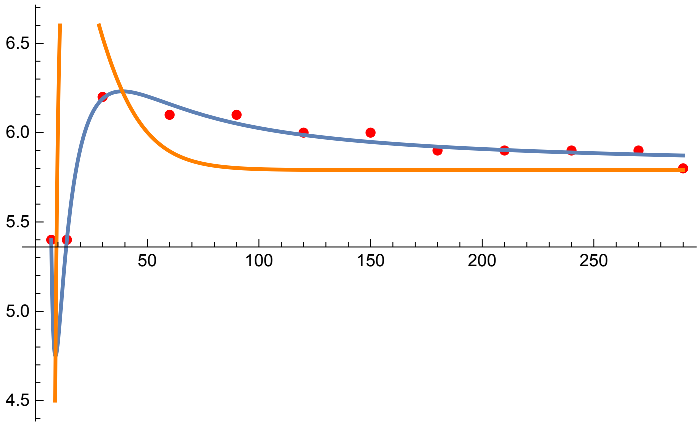
```
Manipulate[Plot[
   {swenssonspot[b0, b1, b2, b3, c1, c2, t], swenssonforward[b0, b1, b2, b3, c1, c2, t]},
   {t, 0, 10000}, PlotRange → {{0, 10000}, {3, 6}},
   AxesOrigin → {0, 3}, PlotStyle → {{Blue, Thick}, {Orange, Thick}}],
  {{b0, 5.8, "b0"}, 5, 6, 0.1, Appearance → "Labeled"},
  {{b1, -1, "b1"}, -2, -0.5, 0.1, Appearance → "Labeled"},
  {{b2, -3, "b2"}, -5, -1, 0.2, Appearance → "Labeled"},
  {{b3, -3, "b3"}, -5, -1, 0.2, Appearance → "Labeled"},
  {{c1, 200, "c1"}, 100, 1000, 10, Appearance → "Labeled"},
  {{c2, 5000, "c2"}, 100, 10000, 10, Appearance → "Labeled"}, SaveDefinitions → True
]
```



Fitting Swensson: on the graph, estimated spot is blue and estimated forward is orange.

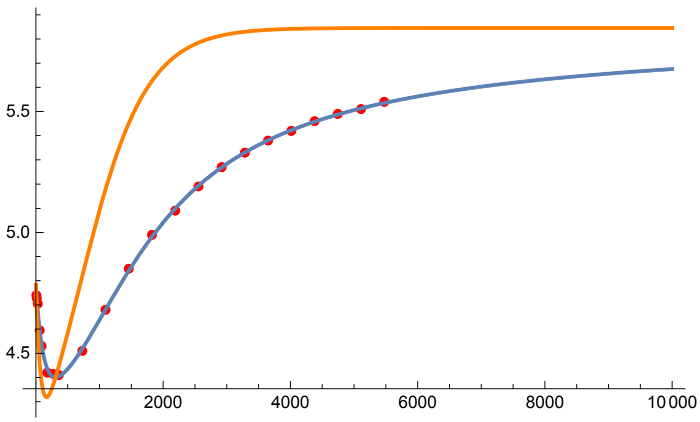
```
fitswensson2 =
 FindFit[ir2, swenssonspot[b0, b1, b2, b3, c1, c2, t], {b0, b1, b2, b3, c1, c2}, t]
(curveswe2 = swenssonspot[b0, b1, b2, b3, c1, c2, t] /. fitswensson2) // Simplify;
curveswefc2[t_] := swenssonspot[b0, b1, b2, b3, c1, c2, t] /. fitswensson2
(curveswe2forward = swenssonforward[b0, b1, b2, b3, c1, c2, t] /. fitswensson2) //
  Simplify;
Show[plot2, Plot[curveswe2, {t, 7, 290}, PlotRange → All, PlotStyle → Thick],
 Plot[curveswe2forward, {t, 7, 290}, PlotRange → {4.5, 6.6},
  (* ,ClippingStyle→Directive[Red,Thick] *)
  PlotStyle → {Orange, Thick}], PlotRange → All]
```

{b0 → 5.79075, b1 → 422.416, b2 → -439.066, b3 → 3.9501, c1 → 1.27503, c2 → 11.338}




```
fitswensson3 =
 FindFit[ir3, swenssonspot[b0, b1, b2, b3, c1, c2, t], {b0, b1, b2, b3, c1, c2}, t]
(curveswe3 = swenssonspot[b0, b1, b2, b3, c1, c2, t] /. fitswensson3) // Simplify;
curveswefc3[t_] := swenssonspot[b0, b1, b2, b3, c1, c2, t] /. fitswensson3
(curveswe3forward = swenssonforward[b0, b1, b2, b3, c1, c2, t] /. fitswensson3) //
  Simplify;
Show[plot3, Plot[curveswe3, {t, 0, 10000}, PlotRange → All, PlotStyle → Thick],
 Plot[curveswe3forward, {t, 0, 10000}, PlotRange → All,
  PlotStyle → {Orange, Thick}], PlotRange → All]
```

{b0 → 5.84541, b1 → -1.06241, b2 → -1.22259, b3 → -3.10279, c1 → 131.083, c2 → 450.74}



!Statistical approach (vyrovnání, fitting): spliny

Ilustrace je na datech ir3

```
Hyperlink[Button[" Fitting by cubic splines "],
 "http://www.karlin.mff.cuni.cz/~hurt/Fitting_by_cubic_splines.pdf"]
```

Fitting by cubic splines

***Statistical approach (vyrovnání, fitting): neparametrický: jádrové odhady (kernel estimators)

http://demonstrations.wolfram.com/EstimatingTheLocalMeanFunction/

***Example

Splines

The principle

```
SetDirectory[NotebookDirectory[]]
Import["20161214_FM1e_Cubic_Splines.png"]
```

2. Základy finanční mat

2.3.6 Fitting by cubic splines (Vyrovnání kubickými spliny)

One of the successful and recently frequently used models is the model of *cubic s* Assuming $T_1 < T_2 < \cdots < T_N$, we consider functions g such that (i) g is a piecewise function, i.e., g equals

$$g_n(t) := \alpha_n + \beta_n t + \gamma_n t^2 + \delta_n t^3 \quad \text{for } t \in [T_{n-1}, T_n], \quad n = 2, \dots, N, \tag{2.81}$$

(ii) g is twice continuously differentiable everywhere; this is (together with (i)) equi to

$$g_n(T_n) = g_{n+1}(T_n), \quad g_n'(T_n) = g_{n+1}'(T_n), \quad g_n''(T_n) = g_{n+1}''(T_n), \quad n = 2, \dots, N -$$

We then choose the function $\widehat{g}$ from this class that minimizes a combination of the re sum of squares and the integrated squared 2nd derivative of g:

$$\widehat{g} = \underset{g}{\text{argmin}} \left\{ \sum_{n=1}^{N} (y_n - g(T_n))^2 + \lambda \int_{T_1}^{T_N} (g''(t))^2 dt \right\}$$

with a smoothing constant $\lambda > 0$. The resulting g represents a compromise be
fit of data and smoothness of the fitting curve. Values of the smoothing const
cover ordinary least squares fitting by a straight line ($\lambda \to \infty$) as one extreme, an
numerical interpolation by a piecewise cubic functions ($\lambda = 0$) as the other one. I
of the method together with an algorithm can be found in [150].

$$\hat{g} = \underset{g}{\operatorname{argmin}} \{ \sum_{n=1}^{N} w_n (y_n - g(T_n))^2 + \int_{T_1}^{T_N} \lambda(t)(g''(t))^2 dt \} \tag{2.82}$$

w_i may take into account the market capitalization, e.g. (Risk J. 2(1), p. 25), $\lambda(\tau)$
$0 \le \tau < 1$, $\lambda(\tau) = 100$, $1 \le \tau < 10$, $\lambda(\tau) = 100000$, $\tau \ge 10$,

2.3.18 Example. We saw that the wildest behavior of the interest rates from R
2.3.16 is in the region of the shortest maturites. Hence here we illustrate the fittin
for the first 10 quotations corresponding to $T_{10} = 1095$ days. The method is very se
to the choice of the smoothing constant λ. On Figure 2.14 we show the fitted cur
$\lambda = 10^7$ and $\lambda = 10000$.

60 [14.12.2016,

Procedure for fitting data by cubic splines

SplinesJanHurt.nb
Reference 1: Späth, Helmuth: Algorithmen für elementare Ausgleichsmodelle. Oldenbourg Verlag, München Wien 1973 (pp. 59-61). Beware of a lot of errors in the formulas!!!

Reference 2 (added in proofs) : Fisher, M. and Zervos, D.: YieldCurve. In: Varian, H. R. (ed.) Computational Economics and Finance. Springer. New York 1996. (pp. 269-302).

Notice that with size n of the input data the system of 3n linear equations must be solved. Generally, this is simple task for Mathematica since the matrix of the system is five-diagonal, but rendering graphics of the spline functions usually takes a longer time.

It is also useful to compare this method with the method of running medians (Tukey).

splinesJH::usage = "splinesJH[x, u, p, z] gives a list of cubic splines in the respective intervals given by the list x of independent observations, u being the list of dependent observations at points x, weights list p, and z as the argument of the spline functions. If p->∞, the method leads to usual numerical interpolation while for p->0 it leads to the least squares method."

plotsplines::usage = "plotsplines[x, u, p] produces the plot of the spline function given the list x of independent variables, the list u of dependent variables with weights list p."

```
Clear[splinesJH];
splinesJH[x_, u_, p_, z_] :=
 Module[{pp = p, dz, dyy, yy2, eq1, eq2, eq3, eq4, eq5, variables,
   sol, yyy, yyy2, dyyy, dyyy2, aa, bb, cc, dd, xk, n},
  n = Length[x];
dx = Take[RotateLeft[x] - x, n - 1];
yy = Array[y, n];
dyy = Take[RotateLeft[yy] - yy, n - 1];
yy2 = Array[y2, n]; tt = Array[t, n];
  eq1 = Table[dx[[k - 1]] * yy2[[k - 1]] +
      dx[[k]] * yy2[[k + 1]] + 2 * (dx[[k - 1]] + dx[[k]]) * yy2[[k]] ==
     6 * (dyy[[k]] / dx[[k]] - dyy[[k - 1]] / dx[[k - 1]]), {k, 2, n - 1}];
eq2 = {yy2[[1]] == 0, yy2[[n]] == 0};
eq3 = Table[pp[[k]] * (u[[k]] - yy[[k]]) == tt[[k]], {k, n}];
eq4 = {tt[[1]] == (yy2[[2]] - yy2[[1]]) / dx[[1]],
    tt[[n]] == -((yy2[[n]] - yy2[[n - 1]]) / dx[[n - 1]])};
eq5 = Table[tt[[k]] == (yy2[[k + 1]] - yy2[[k]]) / dx[[k]] -
      (yy2[[k]] - yy2[[k - 1]]) / dx[[k - 1]] ==
     yy2[[k - 1]] / dx[[k - 1]] + yy2[[k + 1]] / dx[[k]] -
      (1 / dx[[k]] + 1 / dx[[k - 1]]) * yy2[[k]], {k, 2, n - 1}];
variables = Join[yy, yy2, tt];
sol = Solve[Join[eq1, eq2, eq3, eq4, eq5], variables];
yyy = Flatten[Partition[yy /. sol, 1]];
yyy2 = Flatten[Partition[yy2 /. sol, 1]];
dyyy = Take[RotateLeft[yyy] - yyy, n - 1];
dyyy2 = Take[RotateLeft[yyy2] - yyy2, n - 1];
aa = dyyy2 / (6 * dx);
bb = 1 / 2 * Take[yyy2, n - 1];
cc = dyyy / dx - 1 / 6 * dx * (4 * bb + Take[RotateLeft[yyy2, 1], n - 1]);
dd = Take[yyy, n - 1]; xk = Take[x, n - 1];
  aa * (z - xk) ^3 + bb * (z - xk) ^2 + cc * (z - xk) + dd]
```

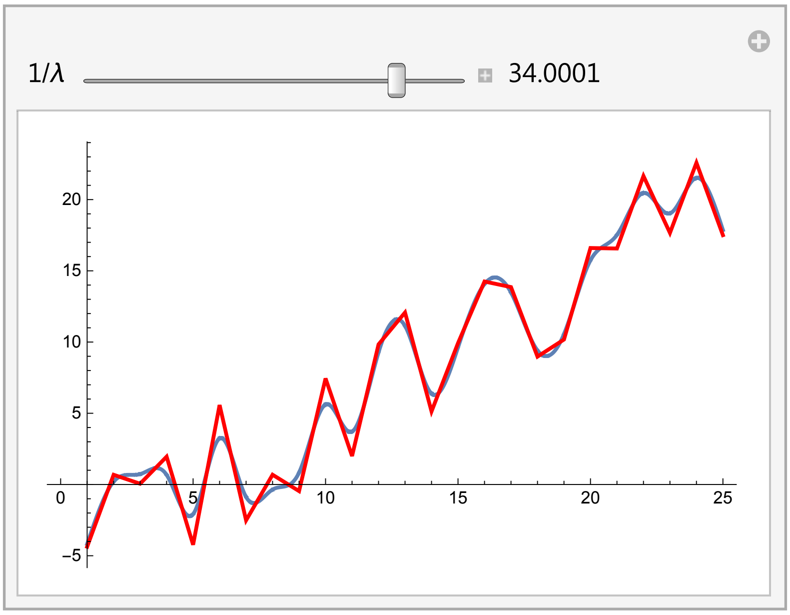
```
Clear[plotsplines];
plotsplines[x_, u_, p_] := Module[{s1, t, x1, tt, intervals, tab1, k, n, plot1, plot2},
  (* Needs["PlotLegends`"];*)
  s1 = splinesJH[x, u, p, t];
  n = Length[x];
  x1 = RotateLeft[x];
  tt = Table[t, {n}];
  intervals = Take[Transpose[{tt, x, x1}], n - 1];
  tab1 = Table[Plot[s1[[k]], Evaluate[intervals[[k]]],
     DisplayFunction → Identity, PlotStyle → Thick], {k, 1, n - 1}];
  (*plot1=Show[tab1,DisplayFunction→$DisplayFunction];*)
  plot2 = ListPlot[Transpose[{x, u}], Joined → True, PlotStyle → {Red, Thick}];
  Show[tab1, plot2, PlotRange → Automatic]]
```

Sorry, the above procedure has been created when the author was junior beginner with *Mathematica.* The system of the equations may be created more effectively by direct symbolic calculation.

Example I

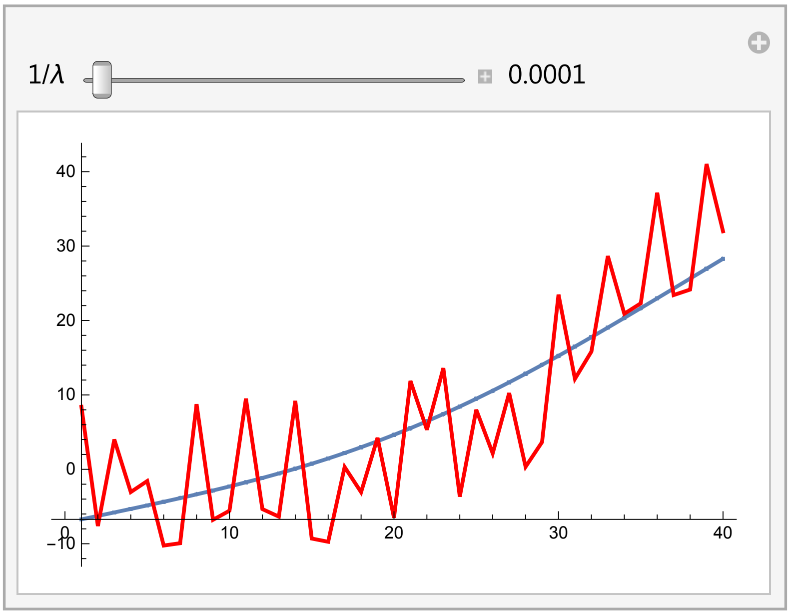
Since the interest rates data are too smooth we will illustrate the procedure on somewhat wilder data. Red are the data, blue the estimated spline function.

```
n = 25;
SeedRandom[13];
uu = Range[n] + 20 Sin[0.5 RandomReal[{0, 1}, n] - 0.5];
xx = Range[n];
ppp = Table[10.500, {n}];
(* plotsplines[xx,uu,ppp]  *)
Manipulate[ppp = Table[p, {Length[xx]}];
 plotsplines[xx, uu, ppp],
 {{p, 0.1, "1/λ"}, 0.0001, 40.0001, 2, Appearance → "Labeled"}, SaveDefinitions → True]
```



Example 2

```
xa = Range[40];
ua = Range[40]^3 / 1000. - 15 Range[40]^2 / 1000 + 10 Sin[ Range[40]^2];
pa = 0.0001;
ppa = Table[pa, {40}];
(* plotsplines[xa,ua,ppa] *)
Manipulate[ppa = Table[pa, {Length[xa]}];
 plotsplines[xa, ua, ppa],
 {{pa, 0.1, "1/λ"}, 0.0001, 100.0001, 5, Appearance → "Labeled"}, SaveDefinitions → True]
```

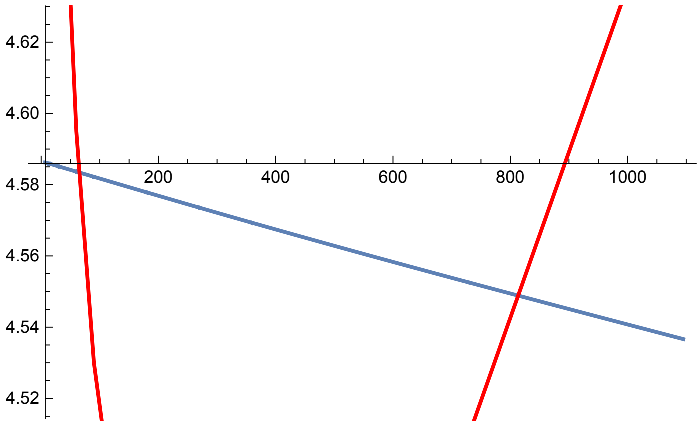


Interest rates data

Since there is almost impossible to see the differences between the fitted curve and the original data if we use all of them, we take just first 10 pairs. The following picture demonstrates that too small weight leads to numerical problems. But we usually do not fit the data by a straight line using cubic splines function.

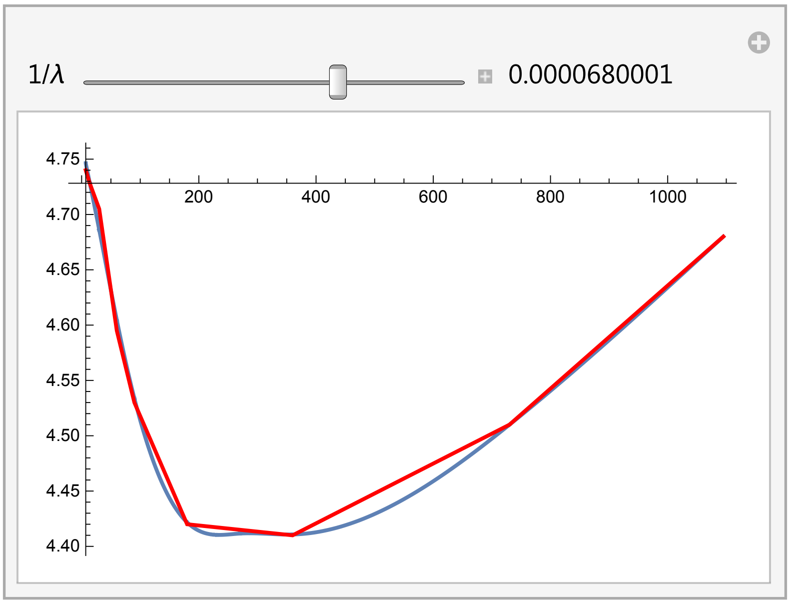
```
x3 = Take[Transpose[ir3]⟦1⟧, 10];
u3 = Take[Transpose[ir3]⟦2⟧, 10];
p3 = 0.00000000010;
pp3 = Table[p3, {Length[x3]}];
plotsplines[x3, u3, pp3]
```

RowReduce: Result for RowReduce of badly conditioned matrix
{{0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 16., 46., 7., 0., 0., 0., 0., 0., 0., 0., −0.375, 1.23214, −0.857143, 0.}, ≪36≫, {0., 0., 0., 0., 0., 0., 0., 0., 0., ≪13≫, 0., 0., 0., 0., 0., 0., 0., 0.}} may contain significant numerical errors.



Next we see that sensitivity of the shape of the fitted curve on the weighting parameter.

```
Manipulate[x3 = Take[Transpose[ir3]⟦1⟧, 10];
 u3 = Take[Transpose[ir3]⟦2⟧, 10];
 pp3 = Table[p3, {Length[x3]}];
 plotsplines[x3, u3, pp3] ,
 {{p3, 0.00005, "1/λ"}, 0.0000000001, 0.0001, 0.000001, Appearance → "Labeled"}]
```



RowReduce: Result for RowReduce of badly conditioned matrix
{{0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 16., 46., 7., 0., 0., 0., 0., 0., 0., 0., −0.375, 1.23214, −0.857143, 0.}, ≪36≫, {0., 0., 0., 0., 0., 0., 0., 0., 0., ≪13≫, 0., 0., 0., 0., 0., 0., 0., 0.}} may contain significant numerical errors.

Using the yeild curves for Užití výnosových křivek pro hodnocení investic

Based on the present value of the cash flow **CF** using spot rates $\boldsymbol{R} = (R_1, \ldots R_T)$ taken from the yield curve

$$\text{PV}(\mathbf{CF}, \boldsymbol{R}) = \sum_{t=0}^{T} \frac{\text{CF}_t}{(1+R_t)^t}$$

Stochastic Models of Interest Rates and Price Developments

General Procedure for Characteristics

In *Mathematica*, the **Normal** and related distributions have the second parameter standard deviation σ and not the variance σ^2!!!

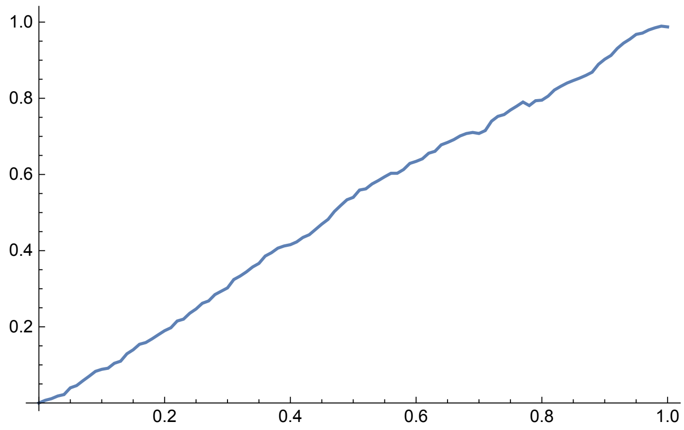
```
Clear[characteristicsofstochasticprocess];
characteristicsofstochasticprocess[process_, s_, t_] := Module[{a}, Text@Grid[
   {{"Process:    ",  Style["  " <> (process // HoldForm // ToString) <> "  ", Bold]},
    {"Mean    ", Mean[process[t]] // FullSimplify // TraditionalForm},
    {"Variance   ", Variance[process[t]] // FullSimplify // TraditionalForm},
    {"Skewness   ", Skewness[process[t]] // FullSimplify // TraditionalForm},
    {"Kurtosis   ", Kurtosis[process[t]] // FullSimplify // TraditionalForm},
    {"Covariance function   ", CovarianceFunction[process, s, t] // FullSimplify //
      TraditionalForm}, {"Slice distribution   ",
     SliceDistribution[process, t] // FullSimplify // TraditionalForm}}, Frame → All]]
```

Wiener process

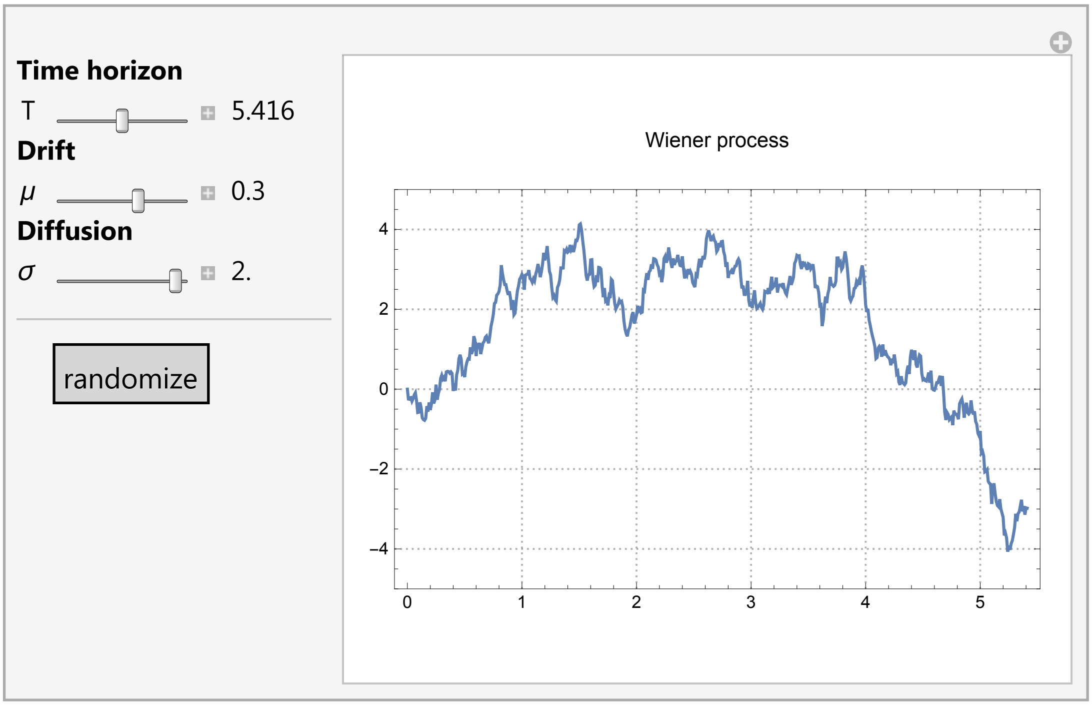
```
Clear[μ, σ, s, t];
characteristicsofstochasticprocess[WienerProcess[μ, σ], s, t]
```

Process:	**WienerProcess[μ, σ]**
Mean	μt
Variance	$\sigma^2 t$
Skewness	0
Kurtosis	3
Covariance function	$\sigma^2 \min(s, t)$
Slice distribution	NormalDistribution$\left[\mu t, \sigma \sqrt{t}\right]$

```
SeedRandom[13 131];
ListLinePlot[RandomFunction[WienerProcess[μ, σ] /. {μ → 1, σ → 0.05}, {0, 1, 0.01}],
 ImageSize → {350, 300}]
```



```
SeedRandom[13 131];
Manipulate[BlockRandom[SeedRandom[r];
  ListLinePlot[RandomFunction[WienerProcess[μ, σ], {0, T, 0.01}],
   ImageSize → {350, 300}, PlotRange → {-5, 5},
   PlotTheme → "Detailed", PlotLabel → "Wiener process\n"]],
 Style["Time horizon", Bold],
 {{T, 1}, 1, 10, 1/250., Appearance → "Labeled", ImageSize → Tiny},
 Style["Drift", Bold],
 {{μ, 0}, -1, 1, 0.1, Appearance → "Labeled", ImageSize → Tiny},
 Style["Diffusion", Bold],
 {{σ, 1}, 0.001, 2, 0.1, Appearance → "Labeled", ImageSize → Tiny},
 Delimiter,
 {{r, 0, ""}, Button["randomize", r = RandomInteger[2^64 - 1]] &},
 SaveDefinitions → True, ControlPlacement → Left]
```

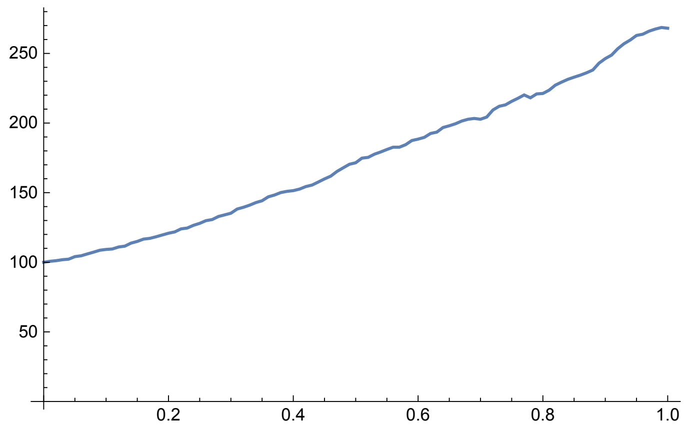


Geometric Brownian Motion Process

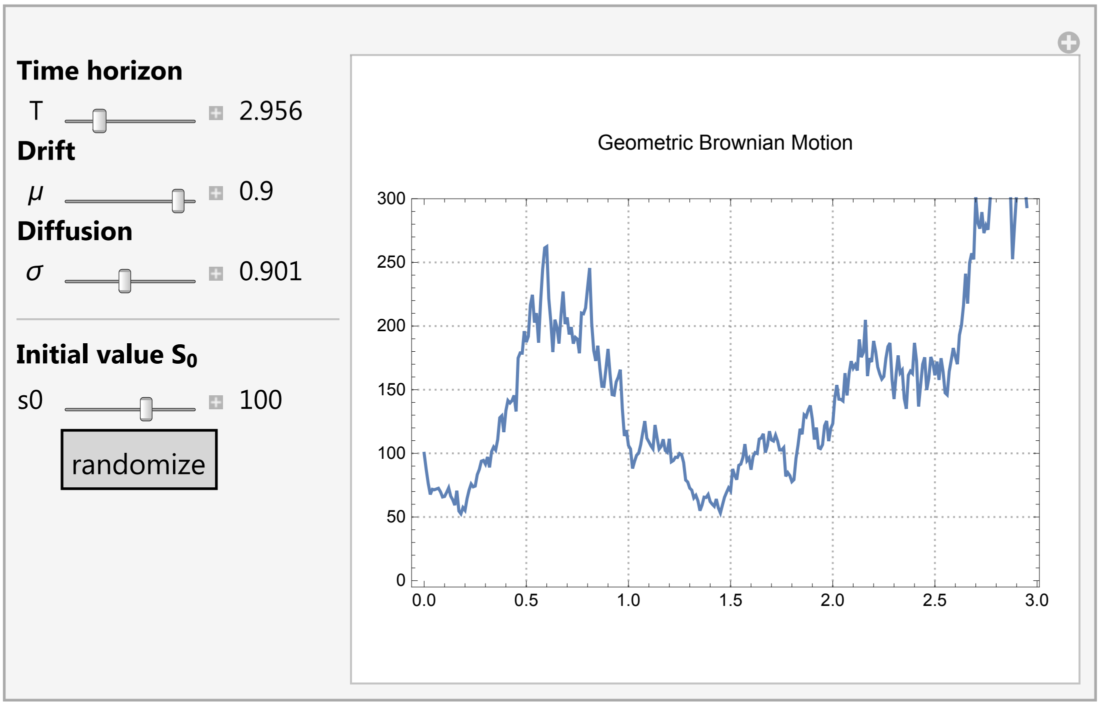
```
Clear[μ, σ, S, s, t];
characteristicsofstochasticprocess[GeometricBrownianMotionProcess[μ, σ, S], s, t]
```

Process:	**GeometricBrownianMotionProcess[μ, σ, S]**
Mean	$S e^{\mu t}$
Variance	$S^2 e^{2 \mu t} \left(e^{\sigma^2 t} - 1\right)$
Skewness	$\sqrt{e^{\sigma^2 t} - 1} \left(e^{\sigma^2 t} + 2\right)$
Kurtosis	$3 e^{2 \sigma^2 t} + 2 e^{3 \sigma^2 t} + e^{4 \sigma^2 t} - 3$
Covariance function	$S^2 e^{\mu (s+t)} \left(e^{\sigma^2 \min(s,t)} - 1\right)$
Slice distribution	$\text{LogNormalDistribution}\left[\log(S) + t\left(\mu - \frac{\sigma^2}{2}\right), \sigma \sqrt{t}\right]$

```
SeedRandom[13 131];
ListLinePlot[RandomFunction[GeometricBrownianMotionProcess[μ, σ, s0] /.
    {μ → 1, σ → 0.05, s0 → 100}, {0, 1, 0.01}], ImageSize → {350, 300}]
```



```
SeedRandom[13 131];
Manipulate[BlockRandom[SeedRandom[r];
  ListLinePlot[RandomFunction[GeometricBrownianMotionProcess[μ, σ, s0], {0, T, 0.01}],
   ImageSize → {350, 300}, PlotRange → {-5, 300}, PlotTheme → "Detailed",
   PlotLabel → "Geometric Brownian Motion\n"]],
 Style["Time horizon", Bold],
 {{T, 1}, 1, 10, 1/250., Appearance → "Labeled", ImageSize → Tiny},
 Style["Drift", Bold],
 {{μ, 0}, -1, 1, 0.1, Appearance → "Labeled", ImageSize → Tiny},
 Style["Diffusion", Bold],
 {{σ, 1}, 0.001, 2, 0.1, Appearance → "Labeled", ImageSize → Tiny},
 Delimiter,
 Style["Initial value S0", Bold],
 {{s0, 100}, 10, 150, 10, Appearance → "Labeled", ImageSize → Tiny},
 {{r, 0, ""}, Button["randomize", r = RandomInteger[2^64 - 1]] &},
 SaveDefinitions → True, ControlPlacement → Left]
```



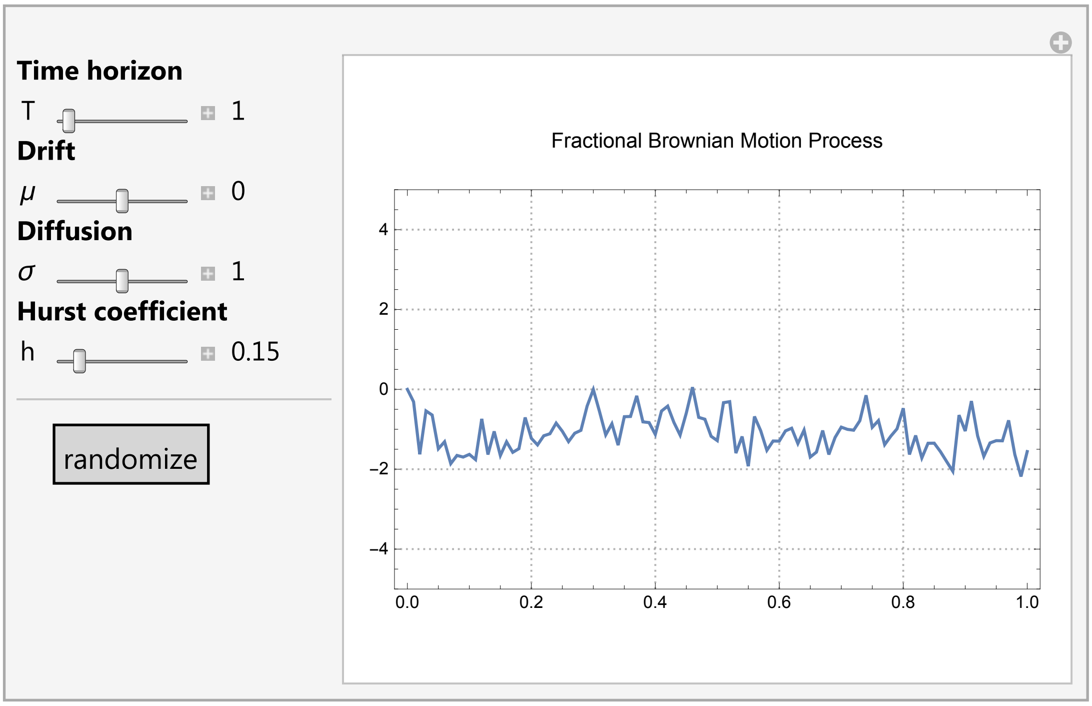
Fractional Brownian Motion

h ... Hurst coefficient

```
Clear[μ, σ, s, t, h];
characteristicsofstochasticprocess[FractionalBrownianMotionProcess[μ, σ, h], s, t]
```

Process:	**FractionalBrownianMotionProcess[μ, σ, h]**
Mean	μt
Variance	$\sigma^2 t^{2h}$
Skewness	0
Kurtosis	3
Covariance function	$\frac{1}{2}\sigma^2\left(-\lvert s-t\rvert^{2h}+s^{2h}+t^{2h}\right)$
Slice distribution	NormalDistribution$[\mu t, \sigma t^h]$

```
SeedRandom[13131];
Manipulate[BlockRandom[SeedRandom[r];
  ListLinePlot[RandomFunction[FractionalBrownianMotionProcess[μ, σ, h], {0, T, 0.01}],
   ImageSize → {350, 300}, PlotRange → {-5, 5}, PlotTheme → "Detailed",
   PlotLabel → "Fractional Brownian Motion Process\n"]],
 Style["Time horizon", Bold],
 {{T, 1}, 1, 10, 1/250., Appearance → "Labeled", ImageSize → Tiny},
 Style["Drift", Bold],
 {{μ, 0}, -1, 1, 0.1, Appearance → "Labeled", ImageSize → Tiny},
 Style["Diffusion", Bold],
 {{σ, 1}, 0.001, 2, 0.1, Appearance → "Labeled", ImageSize → Tiny},
 Style["Hurst coefficient", Bold],
 {{h, 0.5}, 0.05, .95, 0.1, Appearance → "Labeled", ImageSize → Tiny},
 Delimiter,
 {{r, 0, ""}, Button["randomize", r = RandomInteger[2^64 - 1]] &},
 SaveDefinitions → True, ControlPlacement → Left]
```



Jump Diffusion Processes, Illustration of CLT, 3σ Rule

Poisson Process

Let $\{N(t),\ t \geq 0\}$ be a homogeneous Poisson process with intensity λ, i. e., a stochastic process with independent increments, $N(0) = 0)$ w. p. 1, and $N(t) - N(s) \approx \text{Poisson}(\lambda\ |t - s|)$. The trajectories are nondecreasing with jumps of magnitude 1.

```
Clear[λ, s, t];
characteristicsofstochasticprocess[PoissonProcess[λ], s, t]
```

Process:	**PoissonProcess[λ]**
Mean	λt
Variance	λt
Skewness	$\frac{1}{\sqrt{\lambda t}}$
Kurtosis	$\frac{1}{\lambda t} + 3$
Covariance function	$\lambda \min(s, t)$
Slice distribution	PoissonDistribution[λt]

```
SeedRandom[13131];
Manipulate[BlockRandom[SeedRandom[r];
  ListLinePlot[RandomFunction[PoissonProcess[λ], {0, T, 0.01}],
   ImageSize → {350, 300}, PlotRange → {0, 2 λ T}, PlotTheme → "Detailed",
   PlotStyle → Thick, PlotLabel → "Poisson Process\n", InterpolationOrder → 0]],
 Style["Time horizon", Bold],
 {{T, 1}, 1, 10, 1/250., Appearance → "Labeled", ImageSize → Tiny},
 Style["Intensity λ", Bold],
 {{λ, 20}, 1, 100, 10, Appearance → "Labeled", ImageSize → Tiny},
 Delimiter,
 {{r, 0, ""}, Button["randomize", r = RandomInteger[2^64 - 1]] &},
 SaveDefinitions → True, ControlPlacement → Left]
```

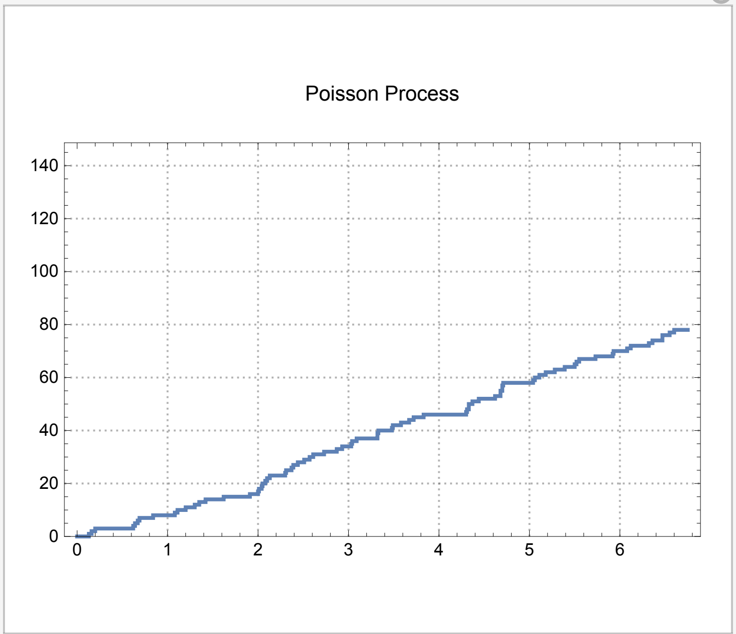
Time horizon

T 6.756

Intensity λ

λ 11

randomize



Compound Poisson Process

Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson process with intensity λ, independent of iid $Z_1, Z_2, \ldots$. Then

$$\{X(t) := \sum_{i=1}^{N(t)} Z_i\,,\ t \geq 0\}$$

is a *compound Poisson process*. Note that Z_i's may be of arbitrary sign.

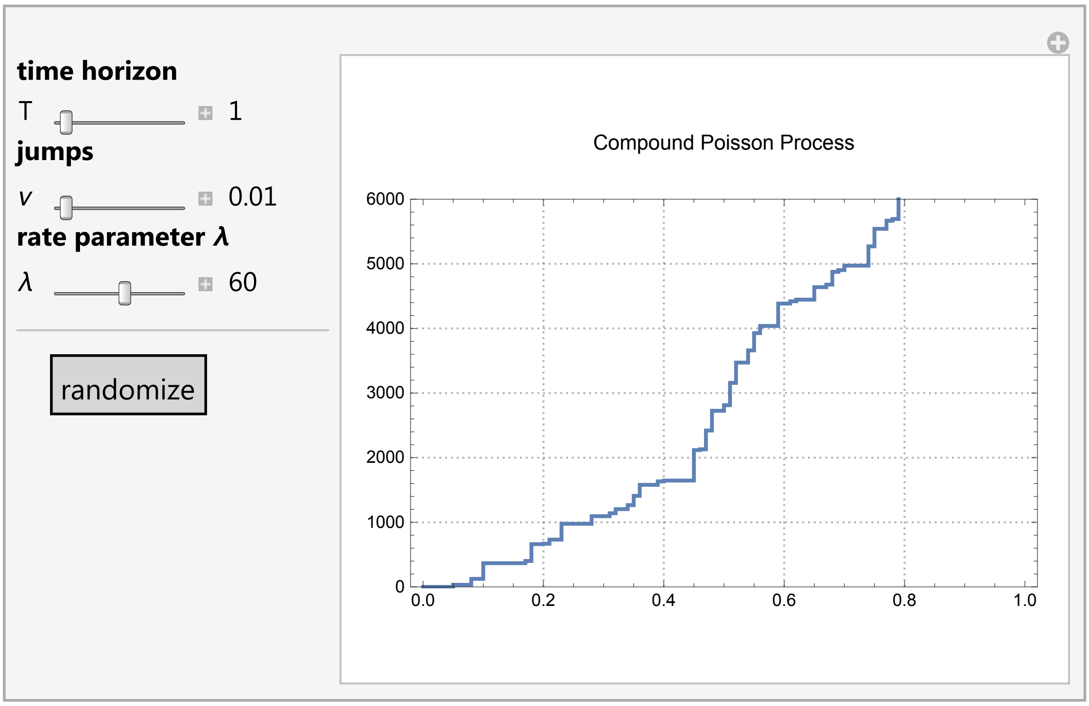
Built – in function : CompoundPoissonProcess[λ, distribution of jumps]

Example

```
Clear[λ, μ, ν, s, t];
characteristicsofstochasticprocess[
 CompoundPoissonProcess[λ, ExponentialDistribution[ν]], s, t]
```

Process:	**CompoundPoissonProcess[λ, ExponentialDistribution[ν]]**
Mean	$\frac{\lambda t}{\nu}$
Variance	$\frac{2 \lambda t}{\nu^2}$
Skewness	$\frac{3}{\sqrt{2} \sqrt{\lambda t}}$
Kurtosis	$\frac{6}{\lambda t} + 3$
Covariance function	CovarianceFunction[CompoundPoissonProcess[λ, ExponentialDistribution[ν]], s, t]
Slice distribution	CompoundPoissonDistribution[λ t, ExponentialDistribution[ν]]

```
SeedRandom[13131];
Manipulate[BlockRandom[SeedRandom[r];
  ListLinePlot[RandomFunction[CompoundPoissonProcess[λ, ExponentialDistribution[ν]],
    {0, T, 0.01}], ImageSize → {350, 300},
   PlotRange → {0, 100 λ T}, PlotTheme → "Detailed", PlotStyle → Thick,
   PlotLabel → "Compound Poisson Process\n", InterpolationOrder → 0]],
 Style["time horizon", Bold],
 {{T, 1}, 1, 10, 1/250., Appearance → "Labeled", ImageSize → Tiny},
 Style["jumps", Bold],
 {{ν, 0.01}, 0.01, 0.2, 0.01, Appearance → "Labeled", ImageSize → Tiny},
 Style["rate parameter λ", Bold],
 {{λ, 20}, 10, 100, 10, Appearance → "Labeled", ImageSize → Tiny},
 Delimiter,
 {{r, 0, ""}, Button["randomize", r = RandomInteger[2^64 - 1]] &},
 SaveDefinitions → True, ControlPlacement → Left]
```



Insurance Example (see *Mathematica* Help)

Jumps follow iid Pareto type II distribution with the density function

```
PDF[ParetoDistribution[k, α, μ], x] // TraditionalForm
```

$$\begin{cases} \frac{\alpha \left(\frac{k-\mu+x}{k}\right)^{-\alpha-1}}{k} & x \geq \mu \\ 0 & \text{True} \end{cases}$$

For $\alpha > 2$

```
Text@Row@{"Mean:   ",
   Mean[ParetoDistribution[k, α, μ]][[1, 1, 1]] // TraditionalForm, Spacer[20],
   "Variance:    ", Variance[ParetoDistribution[k, α, μ]][[1, 1, 1]] // TraditionalForm}
```

Mean: $\frac{k}{\alpha-1}+\mu$ Variance: $\frac{\alpha k^2}{(\alpha-2)(\alpha-1)^2}$

Aggregate claims from a risk follow a compound Poisson process with Poisson parameter 200.
The claim amount distribution is a Pareto distribution with minimum value parameter 300,
shape parameter 3, and location parameter 0. The insurer has effected excess of loss
reinsurance with retention level 300. Simulate the claims process for four years :

The mean of the corresponding Pareto distribution is

```
Mean[TransformedDistribution[Max[0, x - 300], x ≈ ParetoDistribution[300, 3, 0.]]]
```

37.5

and the standard deviation

```
StandardDeviation[
 TransformedDistribution[Max[0, x - 300], x ≈ ParetoDistribution[300, 3, 0.]]]
```

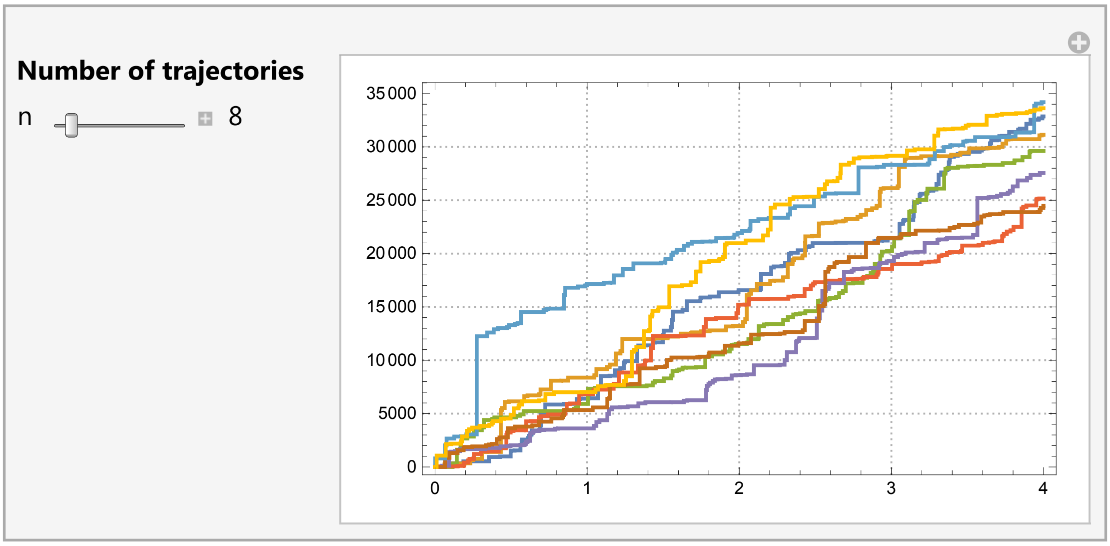
208.791

```
claimsProcess = CompoundPoissonProcess[200,
   TransformedDistribution[Max[0, x - 300], x ≈ ParetoDistribution[300, 3, 0]]];
```

```
characteristicsofstochasticprocess[claimsProcess, s, t]
```

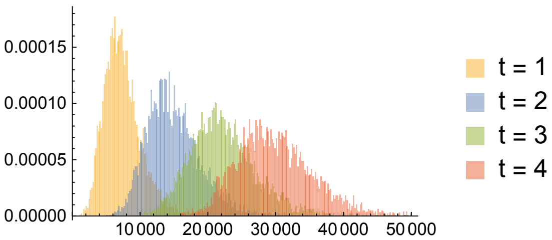
Process:	**CompoundPoissonProcess[200, TransformedDistribution[Max[0, –300 + x], x ≈ ParetoDistribution[300, 3, 0]]]**
Mean	$7500\,t$
Variance	$9\,000\,000\,t$
Skewness	Skewness[CompoundPoissonDistribution[200 *t*, TransformedDistribution[max(0, *x* – 300), *x* ≈ ParetoDistribution[300, 3, 0]]]]
Kurtosis	Kurtosis[CompoundPoissonDistribution[200 *t*, TransformedDistribution[max(0, *x* – 300), *x* ≈ ParetoDistribution[300, 3, 0]]]]
Covariance function	CovarianceFunction[CompoundPoissonProcess[200, TransformedDistribution[max(0, *x* – 300), *x* ≈ ParetoDistribution[300, 3, 0]]], *s*, *t*]
Slice distribution	CompoundPoissonDistribution[200 *t*, TransformedDistribution[max(0, *x* – 300), *x* ≈ ParetoDistribution[300, 3, 0]]]

```
SeedRandom[13 131];
Manipulate[ListLinePlot[RandomFunction[claimsProcess, {0, 4}, n],
   InterpolationOrder → 0, PlotStyle → Thick, PlotTheme → "Detailed"],
 Style["Number of trajectories", Bold],
 {{n, 8}, 2, 100, 2, Appearance → "Labeled", ImageSize → Tiny},
 SaveDefinitions → True, ControlPlacement → Left]
```



Slice distributions for the first four years:

```
Histogram[
 Table[RandomVariate[TruncatedDistribution[{0, 50000}, claimsProcess[t]], 5 × 10^3],
  {t, 4}], 150, "PDF", ChartLegends → (StringJoin["t = ", ToString[#]] & /@ Range[4])]
```



Mean and standard deviation of the reinsurer's aggregate claims for the first four years:

```
Table[Mean[claimsProcess[t]], {t, 4}]
```

```
{7500, 15000, 22500, 30000}
```

```
Table[StandardDeviation[claimsProcess[t]], {t, 4}] // N
```

```
{3000., 4242.64, 5196.15, 6000.}
```

Illustration of the Central Limit Theorem

Define a random number generator ($n = 12$ in practise)

```
Clear[generator];
generator[n_] :=
 Module[{i}, (Sum[RandomVariate[UniformDistribution[{0, 1}]], {i, 1, n}] - n/2) / Sqrt[n/12]]
```

$$\text{Module}\left[\{i\}, \left(\sum_{i=1}^{n} \text{RandomVariate}[\text{UniformDistribution}[\{0,1\}]] - n/2\right) \Big/ \sqrt{n/12}\right]$$

Functional programming approach, results are the same but this generator is much more faster:

```
Clear[generator2];
generator2[n_?(IntegerQ[#] && Positive[#] &)] :=
  (Total[RandomReal[{0, 1}, n]] - n / 2) / √(n / 12)
```

Illustration:

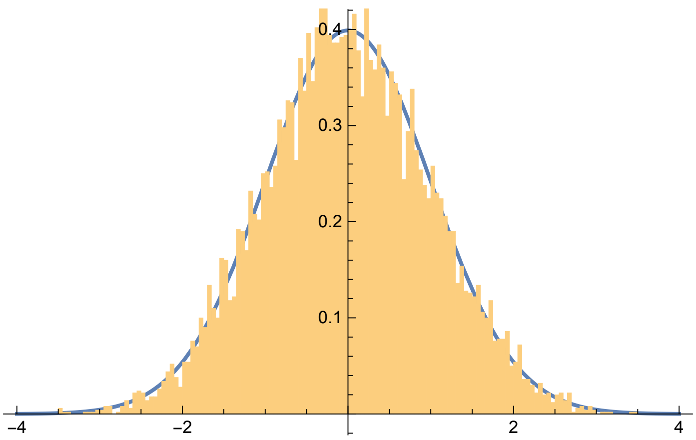
```
SeedRandom[13131];
{generator[12], generator[12], generator[12]}
SeedRandom[13131];
{generator2[12], generator2[12], generator2[12]}
```

```
{0.356074, -1.25641, -0.494065}
```

```
{0.356074, -1.25641, -0.494065}
```

```
SeedRandom[13131];
tab10000 = Table[generator2[12], {10000}]; // Timing
Show[{Plot[PDF[NormalDistribution[], x], {x, -4, 4}, PlotStyle → Thick],
  Histogram[tab10000, 100, "PDF"]}]
Text@Row@{"p-value:   ", DistributionFitTest[tab10000]}
```

```
{0.0780005, Null}
```



p-value: 0.100351

```
{Mean[tab10000], StandardDeviation[tab10000], Skewness[tab10000], Kurtosis[tab10000]}
```

```
{-0.00334335, 0.997614, 0.0214018, 2.88722}
```

More generally for more samples (numberOfSamples) of the same size (sizeOfSample). Implicitely, random seed is set to 13131.

```
tableOfpValues[sizeOfSample_, numberOfSamples_, seedrandom_: 13131] :=
  (SeedRandom[seedrandom];
   Map[DistributionFitTest,
    Table[Table[generator2[12], {sizeOfSample}], {numberOfSamples}]])
```

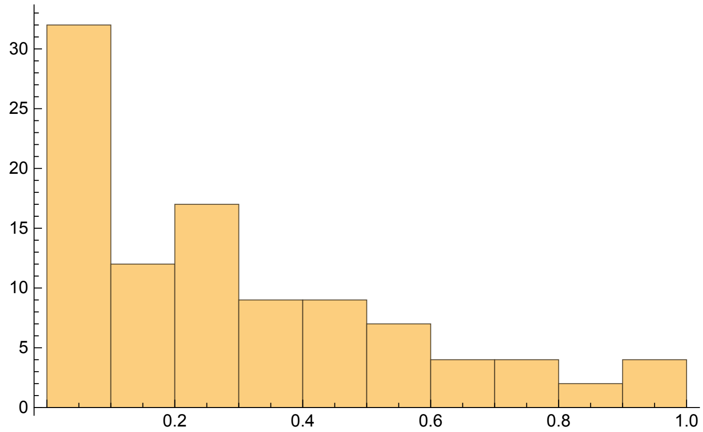
```
(* 10 samples of size 10000 *)
tableOfpValues[10000, 10]
```

```
{0.100351, 0.229417, 0.147909, 0.0574672,
 0.22432, 0.0613655, 0.271885, 0.583139, 0.13578, 0.996401}
```

```
(* 100 samples of size 10000 *)
pValues100 = tableOfpValues[10000, 100]
```

```
{0.100351, 0.229417, 0.147909, 0.0574672, 0.22432, 0.0613655, 0.271885, 0.583139,
 0.13578, 0.996401, 0.458817, 0.142894, 0.27753, 0.525878, 0.918257, 0.484734,
 0.0290423, 0.467584, 0.00120941, 0.511306, 0.0242014, 0.013217, 0.266528,
 0.0729311, 0.864206, 0.275771, 0.0167355, 0.727202, 0.0382003, 0.0583544, 0.174548,
 0.0252473, 0.374546, 0.00535984, 0.343915, 0.252778, 0.000851989, 0.0218259,
 0.44301, 0.0707303, 0.378544, 0.168946, 0.64155, 0.0154996, 0.115459, 0.0179169,
 0.0384796, 0.367638, 0.0150481, 0.615701, 0.265767, 0.321751, 0.174624, 0.830323,
 0.0300878, 0.213665, 0.0132979, 0.551005, 0.462489, 0.0965705, 0.066547, 0.733872,
 0.710477, 0.232664, 0.247759, 0.393102, 0.719248, 0.305912, 0.572589, 0.597077,
 0.262646, 0.435651, 0.0765035, 0.285626, 0.48381, 0.249925, 0.222732, 0.200692,
 0.199055, 0.013974, 0.97108, 0.110828, 0.0000822816, 0.000146541, 0.459935,
 0.196544, 0.459464, 0.645762, 0.0475796, 0.572611, 0.363473, 0.672794, 0.0873876,
 0.0355776, 0.00766174, 0.208426, 0.0551239, 0.185204, 0.958755, 0.324656}
```

```
Histogram[pValues100, 10]
Text@Row@{"Minimum of p-values:  ", Min[pValues100],
   Spacer[20], "Maximum of p-values:   ", Max[pValues100]}
(* Number of p-values ≤ 0.05 and ≥0.95 *)
Text@Row@{"Out of ", Length[pValues100], "  p-values ",
   Count[pValues100, _?(# ≤ 0.05 &)], " are less or equal 0.05 and ",
   Count[pValues100, _?(# ≥ 0.95 &)], "  p-values are greater or equal 0.95."}
```



Minimum of p-values: 0.0000822816   Maximum of p-values: 0.996401

Out of 100 p-values 22 are less or equal 0.05 and 3 p-values are greater or equal 0.95.

```
pos = Position[pValues100, Min[pValues100]]
```

```
{{83}}
```

3σ Rule and 2σ Rule

```
Probability[Abs[X - μ] < 3 σ, X ≈ NormalDistribution[μ, σ]] // N
```

```
0.9973
```

```
Probability[Abs[X - μ] < 2 σ, X ≈ NormalDistribution[μ, σ]] // N
```

```
0.9545
```

Compound Poisson Process with drift

Let $\{N(t),\ t \geq 0\}$ be a homogeneous Poisson process with intensity λ, independent of iid Z_1, Z_2, ... Then

$$\{X(t) := \sum_{i=1}^{N(t)} Z_i\,,\ t \geq 0\}$$

is a *compound Poisson process* and

$$Y(t) = y_0 + c\,t + X(t)$$

is a *compound Poisson process with drift* c, y_0 an initial value.

Example

```
Clear[c, λ, x, μ, s, t];
CompoundPoissonProcessWithDrift[c_, λ_, μ_, y0_] := TransformedProcess[
  y0 + c t + x[t], x ≈ CompoundPoissonProcess[λ, ExponentialDistribution[μ]], t]
```

```
characteristicsofstochasticprocess[CompoundPoissonProcessWithDrift[c, λ, μ, y0], s, t]
```

Process:	**TransformedProcess[c t + y0 + x[t], x ≈ CompoundPoissonProcess[λ, ExponentialDistribution[μ]], t]**
Mean	$c\,t + \frac{\lambda t}{\mu} + \text{y0}$
Variance	$\frac{2 \lambda t}{\mu^2}$
Skewness	$\frac{3}{\sqrt{2}\,\mu \sqrt{\frac{\lambda t}{\mu^2}}}$
Kurtosis	$\frac{6}{\lambda t} + 3$
Covariance function	CovarianceFunction[TransformedProcess[$c\,t + x(t)$ + y0, x ≈ CompoundPoissonProcess[λ, ExponentialDistribution[μ]], t], s, t]
Slice distribution	TransformedProcess[$c\,t + x(t)$ + y0, x ≈ CompoundPoissonProcess[λ, ExponentialDistribution[μ]], t][t]

Skewness and kurtosis:

```
{Skewness[CompoundPoissonProcessWithDrift[c, λ, μ, y0][t]],
  Kurtosis[CompoundPoissonProcessWithDrift[c, λ, μ, y0][t]]} //
 FullSimplify[#, Assumptions → μ > 0 ∧ λ > 0 ∧ t > 0] &
```

$$\left\{\frac{3}{\sqrt{2}\,\sqrt{t\,\lambda}},\ 3 + \frac{6}{t\,\lambda}\right\}$$

Gauss-Poisson Process

$$Y(t) = W(t) + N(t)$$

where $\{W(t),\ t > 0\}$ is a standard Wiener process and $\{N(t),\ t \geq 0\}$ is a homogeneous Poisson process, W and N independent.

Slice distribution:

$$P(Y(t) \le x) = \sum_{k=1}^{\infty} P(Y(t) \le x,\ N(t) = k) = \sum_{k=1}^{\infty} P(W(t) \le x - k,\ N(t) = k) =$$
$$\sum_{k=1}^{\infty} P(W(t) \le x - k)\, P(N(t) = k) = \sum_{k=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!}\, \Phi(x - k)$$

Gauss-Compound-Poisson Jump Process with Drift

$$Y(t) = y_0 + \mu t + \sigma W(t) + X(t)$$

where $\{W(t),\ t \ge 0\}$ is a standard Wiener process, $\{X(t),\ t \ge 0\}$ a compound Poisson process, $Z_1, Z_2, \ldots$ iid independent of $\{N(t),\ t \ge 0\}$, W and X independent. Recall that Z_i's may be of arbitrary sign.

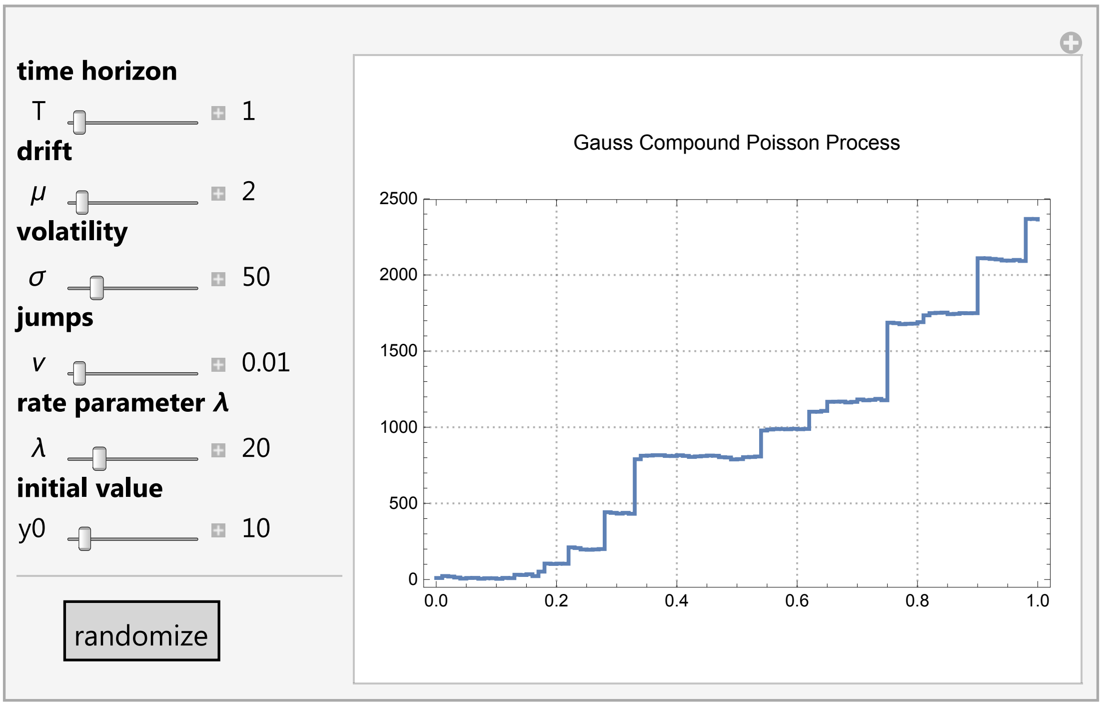
Example

```
Clear[λ, y0, μ, σ, s, t];
GaussCompoundPoissonProcessWithDrift[λ_, μ_, σ_, ν_, y0_] :=
 TransformedProcess[y0 + μ t + σ w[t] + x[t],
  {w ≈ WienerProcess[], x ≈ CompoundPoissonProcess[λ, ExponentialDistribution[ν]]}, t]
```

```
characteristicsofstochasticprocess[
 GaussCompoundPoissonProcessWithDrift[λ, μ, σ, ν, y0], s, t]
```

Process:	**TransformedProcess[y0 + t μ + σ p1[t] + p2[t], {p1 ≈ WienerProcess[0, 1], p2 ≈ CompoundPoissonProcess[λ, ExponentialDistribution[ν]]}, t]**
Mean	$t\left(\frac{\lambda}{\nu} + \mu\right) + \text{y0}$
Variance	$t\left(\frac{2\lambda}{\nu^2} + \sigma^2\right)$
Skewness	$\frac{6\lambda t}{\nu^3\left(t\left(\frac{2\lambda}{\nu^2}+\sigma^2\right)\right)^{3/2}}$
Kurtosis	$\frac{24\lambda}{t(2\lambda+\nu^2\sigma^2)^2} + 3$
Covariance function	CovarianceFunction[TransformedProcess[μ t + σ p1(t) + p2(t) + y0, {p1 ≈ WienerProcess[0, 1], p2 ≈ CompoundPoissonProcess[λ, ExponentialDistribution[ν]]}, t], s, t]
Slice distribution	TransformedProcess[μ t + σ p1(t) + p2(t) + y0, {p1 ≈ WienerProcess[0, 1], p2 ≈ CompoundPoissonProcess[λ, ExponentialDistribution[ν]]}, t][t]

```
SeedRandom[13131];
Manipulate[BlockRandom[SeedRandom[r];
  ListLinePlot[RandomFunction[GaussCompoundPoissonProcessWithDrift[λ, μ, σ, ν, y0],
    {0, T, 0.01}], ImageSize → {350, 300}, PlotRange → All
   (*{-10 λ T,100 λ T}*), PlotTheme → "Detailed", PlotStyle → Thick,
   PlotLabel → "Gauss Compound Poisson Process\n", InterpolationOrder → 0]],
 Style["time horizon", Bold],
 {{T, 1}, 1, 10, 1/250., Appearance → "Labeled", ImageSize → Tiny},
 Style["drift", Bold],
 {{μ, 2}, 0.1, 100, 0.1, Appearance → "Labeled", ImageSize → Tiny},
 Style["volatility", Bold],
 {{σ, 50}, 0.1, 300, 0.1, Appearance → "Labeled", ImageSize → Tiny},
 Style["jumps", Bold],
 {{ν, 0.01}, 0.01, 0.2, 0.01, Appearance → "Labeled", ImageSize → Tiny},
 Style["rate parameter λ", Bold],
 {{λ, 20}, 1, 100, 10, Appearance → "Labeled", ImageSize → Tiny},
 Style["initial value", Bold],
 {{y0, 10}, 1, 200, 1, Appearance → "Labeled", ImageSize → Tiny},
 Delimiter,
 {{r, 0, ""}, Button["randomize", r = RandomInteger[2^64 - 1]] &},
 SaveDefinitions → True, ControlPlacement → Left]
```



Ornstein-Uhlenbeck Process, Vasicek Model

The state $X(t)$ of a Ornstein–Uhlenbeck process satisfies an Ito differential equation

$$dX(t) = \theta(\mu - X(t))\,dt + \sigma\,dW(t), \tag{4}$$

with long-term mean μ, volatility σ, and mean reversion speed θ. Part $\mu - X(t)$ is the mean reverting drift pulled to a level μ at rate θ. Here $\theta > 0$, $\sigma > 0$, $\mu \in \mathbb{R}$. A drawback of the solution is that $X(t)$ may attain negative values.

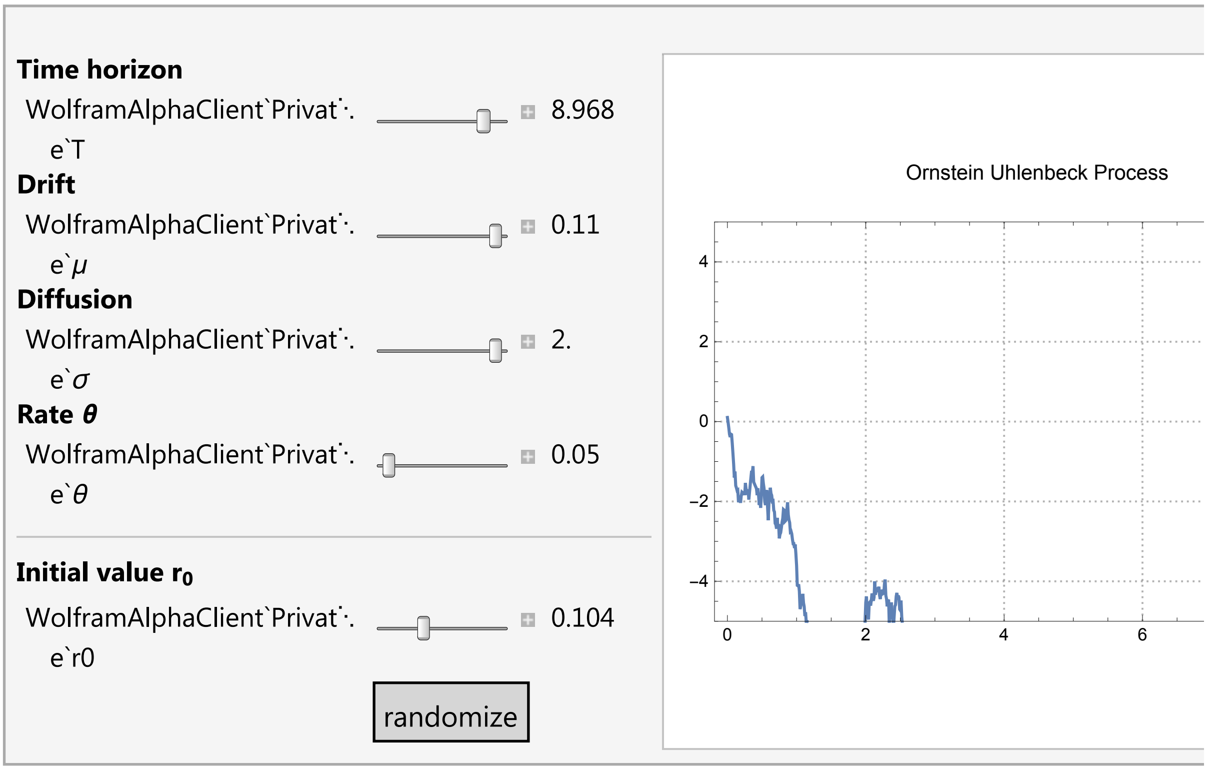
Oldrich Alfons Vasicek (1977): spot rate $r(t)$, $\alpha > 0$

$$dr = \alpha (\gamma - r)\, dt + \rho\, dz \tag{5}$$

In contrast to the random walk (the Wiener process), which is an unstable process and after a long time will diverge to infinite values, the Ornstein-Uhlenbeck process possesses a stationary distribution. The instantaneous drift $\alpha (\gamma - r)$ represents a force that keeps pulling the process towards its long-term mean γ with magnitude proportional to the deviation of the process from the mean.The stochastic element, which has a constant instantaneous variance ρ^2, causes the process to fluctuate around the level γ in an erratic, but continuous, fashion. The conditional expectation and variance level are ...

```
Clear[μ, σ, s, t, θ, r0(* initial value of arbitrary sign *)];
characteristicsofstochasticprocess[OrnsteinUhlenbeckProcess[μ, σ, θ, r0], s, t]
characteristicsofstochasticprocess[OrnsteinUhlenbeckProcess[μ, σ, θ, r0], s, t]
```

```
SeedRandom[13131];
Manipulate[BlockRandom[SeedRandom[r];
  ListLinePlot[RandomFunction[OrnsteinUhlenbeckProcess[μ, σ, θ, r0], {0, T, 0.01}],
   ImageSize → {350, 300}, PlotRange → {-5, 5}, PlotTheme → "Detailed",
   PlotLabel → "Ornstein Uhlenbeck Process\n"]],
 Style["Time horizon", Bold],
 {{T, 1}, 1, 10, 1/250., Appearance → "Labeled", ImageSize → Tiny},
 Style["Drift", Bold],
 {{μ, 0.04}, 0.01, 0.11, 0.01, Appearance → "Labeled", ImageSize → Tiny},
 Style["Diffusion", Bold],
 {{σ, 1}, 0.01, 2, 0.01, Appearance → "Labeled", ImageSize → Tiny},
 Style["Rate θ", Bold],
 {{θ, 0.2}, 0.05, 0.5, 0.05, Appearance → "Labeled", ImageSize → Tiny},
 Delimiter,
 Style["Initial value r₀", Bold],
 {{r0, 0.02}, 0.01, 0.3, 0.001, Appearance → "Labeled", ImageSize → Tiny},
 {{r, 0, ""}, Button["randomize", r = RandomInteger[2^64 - 1]] &},
 SaveDefinitions → True, ControlPlacement → Left]
```



Cox-Ingersoll-Ross Process (CIR)

`CoxIngersollRossProcess`

The state $X(t)$ of a Cox-Ingersoll-Ross process (CIR) satisfies an Ito differential equation

$$dX(t) = \theta(\mu - X(t))\,dt + \sigma\sqrt{X(t)}\,dW(t), \quad (6)$$

with long-term mean μ, volatility σ, speed of adjustment θ, and (possible) initial condition $x_0 > 0$. Here $\sigma > 0$, $\theta\mu > 0$, usually $\mu > 0$. The last condition means that both θ and μ must be nonzero and of the same sign. As compared with the Ornstein-Uhlenbeck, CIR can never attain negative values.

It can be shown that $X(t) > 0$ a. s. if $\sigma^2 \leq 2\,\theta\mu$.

```
Clear[μ, σ, s, t, θ, r0(* initial value positive *)];
characteristicsofstochasticprocess[CoxIngersollRossProcess[μ, σ, θ, r0], s, t]
```

Process:	**CoxIngersollRossProcess[μ, σ, θ, r0]**
Mean	$\mu + (\text{r0} - \mu)\, e^{\theta(-t)}$
Variance	$\frac{\sigma^2 e^{-2\theta t}(e^{\theta t}-1)(2\,\text{r0}+\mu(e^{\theta t}-1))}{2\theta}$
Skewness	$\frac{\sqrt{2}\,(3\,\text{r0}+\mu(e^{\theta t}-1))}{(2\,\text{r0}+\mu(e^{\theta t}-1))\sqrt{\frac{\theta\left(\mu+\frac{2\,\text{r0}}{e^{\theta t}-1}\right)}{\sigma^2}}}$
Kurtosis	$\frac{3\sigma^2\left(1-\frac{4\,\text{r0}^2}{(2\,\text{r0}+\mu(e^{\theta t}-1))^2}\right)}{\theta\mu} + 3$
Covariance function	$\frac{1}{\theta}\sigma^2 e^{\theta(-(s+t))}\left(e^{\theta\min(s,t)}-1\right)\left(\frac{1}{2}\mu\left(e^{\theta\min(s,t)}-1\right)+\text{r0}\right)$
Slice distribution	CoxIngersollRossProcess[μ, σ, θ, r0][t]

Limit of the variance:

```
Limit[σ^2 ⅇ^(-2 θ t) (ⅇ^(θ t) - 1) (2 r0 + μ (ⅇ^(θ t) - 1)) / (2 θ), t → ∞, Assumptions → θ > 0 && r0 > 0 && μ > 0]
```

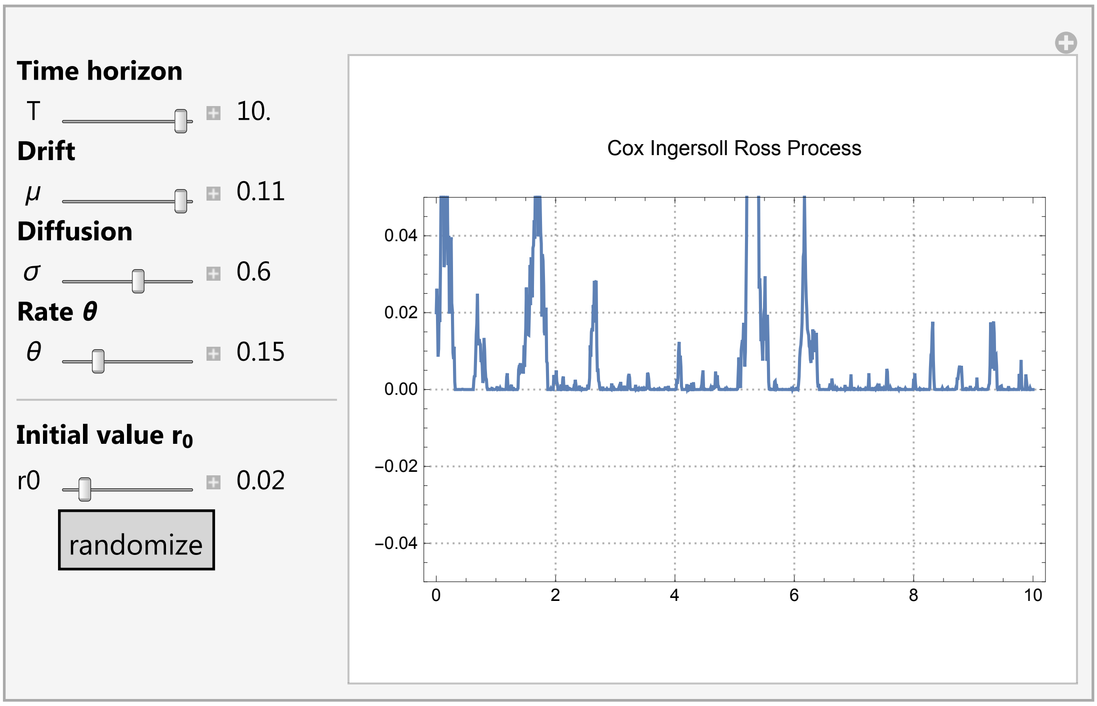
$$\frac{\mu\,\sigma^2}{2\,\theta}$$

Skewness and kurtosis:

```
Text@Row@{"The limits of skewness:  ",
  Limit[Sqrt[2] (3 r0 + μ (ⅇ^(θ t) - 1)) / ((2 r0 + μ (ⅇ^(θ t) - 1)) Sqrt[θ (μ + 2 r0/(ⅇ^(θ t) - 1)) / σ^2]), t → ∞, Assumptions → θ > 0 && r0 > 0 && μ > 0] //
   TraditionalForm, Spacer[20], "and kurtosis:   ",
  Limit[3 σ^2 (1 - 4 r0^2/(2 r0 + μ (ⅇ^(θ t) - 1))^2) / (θ μ) + 3, t → ∞, Assumptions → θ > 0 && r0 > 0 && μ > 0]}
```

The limits of skewness: $\frac{\sqrt{2}}{\sqrt{\frac{\theta\mu}{\sigma^2}}}$ and kurtosis: $3 + \frac{3\sigma^2}{\theta\mu}$

```
SeedRandom[13131];
Manipulate[BlockRandom[SeedRandom[r];
  ListLinePlot[RandomFunction[CoxIngersollRossProcess[μ, σ, θ, r0], {0, T, 0.01}],
   ImageSize → {350, 300}, PlotRange → {-0.05, 0.05},
   PlotTheme → "Detailed", PlotLabel → "Cox Ingersoll Ross Process\n"]],
 Style["Time horizon", Bold],
 {{T, 1}, 1, 10, 1/250., Appearance → "Labeled", ImageSize → Tiny},
 Style["Drift", Bold],
 {{μ, 0.06}, 0.01, 0.11, 0.01, Appearance → "Labeled", ImageSize → Tiny},
 Style["Diffusion", Bold],
 {{σ, 0.05}, 0.01, 1, 0.01, Appearance → "Labeled", ImageSize → Tiny},
 Style["Rate θ", Bold],
 {{θ, 0.15}, 0.05, 0.5, 0.05, Appearance → "Labeled", ImageSize → Tiny},
 Delimiter,
 Style["Initial value r0", Bold],
 {{r0, 0.02}, 0.01, 0.1, 0.001, Appearance → "Labeled", ImageSize → Tiny},
 {{r, 0, ""}, Button["randomize", r = RandomInteger[2^64 - 1]] &},
 SaveDefinitions → True, ControlPlacement → Left]
```



Ito process I

`ItoProcess`

$$dX(t) = a(X(t), t)\,dt + b(X(t), t).dW(t) \tag{7}$$

is represented as ItoProcess[{a, b}, x, t]]. It may be multidimensional.

Alternatively, it may be defined by a stochastic differential equation (SDE): ItoProcess[SDEquations, expr, x, t, $w \approx$ dproc] represents an Ito process specified by a stochastic differential equation(s) *sdeqns*, output expression *expr*, with state x and time t, driven by w following the stochastic process *dproc*.

```
Row@{Hyperlink[Button["Ito Process"], "paclet:ref/ItoProcess"],
  Spacer[20], Hyperlink[Button["Stochastic Differential Equation Processes"],
   "paclet:guide/StochasticDifferentialEquationProcesses"], Spacer[20],
  Hyperlink[Button["Heston Model"], "paclet:example/HestonModel"]}
```

Ito Process | Stochastic Differential Equation Processes | Heston Model

Built-in functions ending with “Process”:

```
? *Process
```

▼ System`

ARCHProcess	ItoProcess
ARIMAProcess	KillProcess
ARMAProcess	MAProcess
ARProcess	OrnsteinUhlenbeckProcess
BernoulliProcess	PoissonProcess
BinomialProcess	ProfileHiddenMarkovProcess
BrownianBridgeProcess	QueueingNetworkProcess
CompoundPoissonProcess	QueueingProcess
CompoundRenewalProcess	RandomWalkProcess
ContinuousMarkovProcess	RenewalProcess
CoxIngersollRossProcess	RunProcess
DiscreteMarkovProcess	SARIMAProcess
EstimatedProcess	SARMAProcess
FARIMAProcess	StartProcess
FractionalBrownianMotionProcess	StratonovichProcess
FractionalGaussianNoiseProcess	TelegraphProcess
GARCHProcess	TransformedProcess
GeometricBrownianMotionProcess	WhiteNoiseProcess
HiddenMarkovProcess	WienerProcess

ARMAProcess[{$a_1, \ldots, a_p$}, {$b_1, \ldots, b_q$}, v] represents a weakly stationary autoregressive moving-average process with AR coefficients a_i, MA coefficients b_j, and normal white noise variance v.

ARMAProcess[{$a_1, \ldots, a_p$}, {$b_1, \ldots, b_q$}, Σ] represents a weakly stationary vector ARMA process with coefficient matrices a_i and b_j and covariance matrix Σ.

ARMAProcess[{$a_1, \ldots, a_p$}, {$b_1, \ldots, b_q$}, v, *init*] represents an ARMA process with initial data *init*.

ARMAProcess[c, …] represents an ARMA process with a constant c. ≫

```
? *PoissonProcess
```

▼ System`

CompoundPoissonProcess	PoissonProcess

CompoundPoissonProcess[λ, *jdist*] represents a
compound Poisson process with rate parameter λ and jump size distribution *jdist*. >>

Example 1

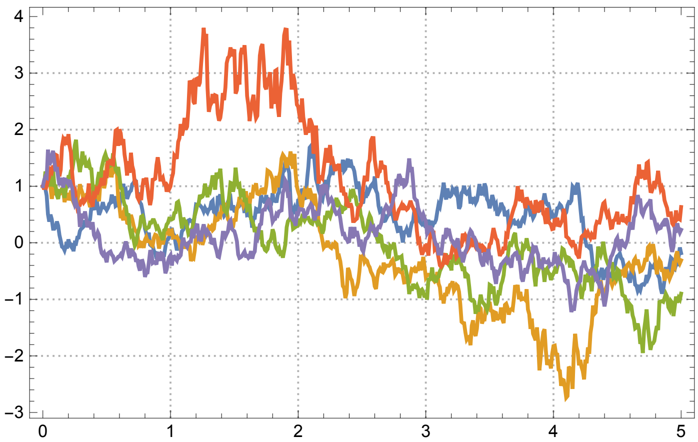
```
example1ItoProcess =
 ItoProcess[dx[t] == -x[t] dt + √(1 + x[t]^2) dw[t], x[t], {x, 1}, t, w ≈ WienerProcess[]]
```

```
ItoProcess[{{-x[t]}, {{√(1 + x[t]^2)}}, x[t]}, {{x}, {1}}, {t, 0}]
```

```
characteristicsofstochasticprocess[example1ItoProcess, s, t]
```

Process:	2 ItoProcess[{{-x[t]}, {{Sqrt[1 + x[t]]}}, x[t]}, {{x}, {1}}, {t, 0}]
Mean	e^{-t}
Variance	$1 - e^{-2t}$
Skewness	$\frac{2 e^t \sqrt{1-e^{-2t}} (2 e^t+1)}{(e^t+1)^2}$
Kurtosis	$4 e^{2t} + \frac{4 \sinh(t)+\cosh(t)-3}{\cosh(t)+1}$
Covariance function	$e^{-s-t} (e^{2 \min(s,t)} - 1)$
Slice distribution	$\text{ItoProcess}\left[\left\{\{-x(t)\}, \left(\sqrt{x(t)^2+1}\right), x(t)\right\}, \begin{pmatrix} x \\ 1 \end{pmatrix}, \{t, 0\}\right][t]$

```
SeedRandom[131 311];
ListLinePlot[Table[RandomFunction[example1ItoProcess, {0., 5., 0.01}], {5}],
 PlotStyle → Thick, PlotTheme → "Detailed", PlotRange → All]
```



```
example1SliceDistribution = SliceDistribution[example1ItoProcess, t]
```

```
ItoProcess[{{-x[t]}, {{√(1 + x[t]^2)}}, x[t]}, {{x}, {1}}, {t, 0}][t]
```

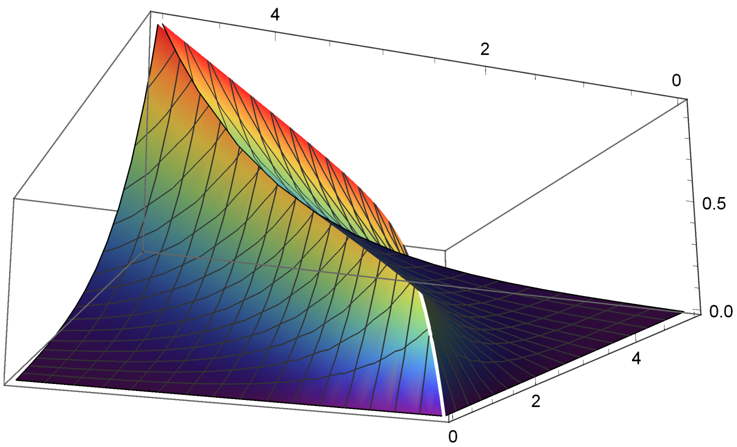
```
Mean[example1SliceDistribution]
```

e^{-t}

```
CovarianceFunction[example1ItoProcess, s, t]
```

$e^{-s-t}\left(-1+e^{2\,\text{Min}[s,t]}\right)$

```
Plot3D[CovarianceFunction[example1ItoProcess, s, t] // Evaluate,
 {s, 0, 5}, {t, 0, 5}, ColorFunction → "Rainbow"]
```



SDE: deterministic solutions via DSolve

See: S. M. Iacus (University of Milan), Chicago, R/Finance 2011, April 29th

Statistical data analysis of financial time series and option pricing in R

Geometric Brownian motion

```
gBm
```

```
DSolve[{x'[t] == μ x[t], x[0] == x0}, x[t], t]
```

$\{\{x[t] \to e^{t\mu}\, x0\}\}$

Cox-Ingersoll-Ross (CIR)

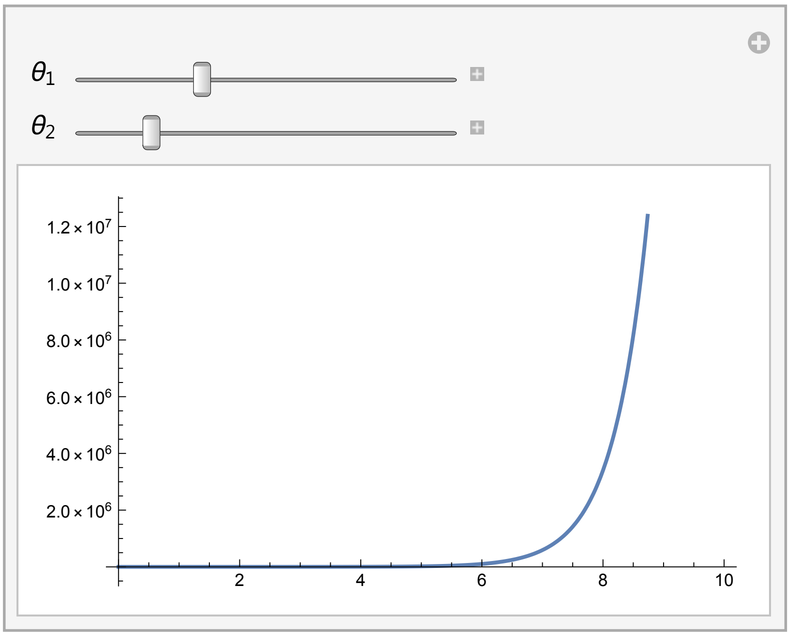
```
Clear[y];
y[t_, θ1_, θ2_, x0_] =
 x[t] /. DSolve[{x'[t] == (θ1 + θ2 x[t]), x[0] == x0}, x[t], t] // FullSimplify // First
```

$$\frac{-\theta1 + e^{t\,\theta2}\left(\theta1 + x0\,\theta2\right)}{\theta2}$$

```
y[t, θ₁, θ₂, 1]
```

$$\frac{-\theta_1 + e^{t\,\theta_2}\left(\theta_1 + \theta_2\right)}{\theta_2}$$

```
Manipulate[Plot[(y[t, θ1, θ2, 1]) // Evaluate, {t, 0, 10}, PlotStyle → Thick],
 {θ1, 0.1, 10}, {θ2, 0.1, 10}, SaveDefinitions → True]
```



Chan-Karolyi-Longstaff-Sanders (CKLS)

The same as CIR, the difference is only in the stochastic term.

Nonlinear mean reversion (Aït-Sahalia)

```
DSolve[{x'[t] == α_{-1}/x[t] + α0 + α1 x[t] + α2 x[t]^2, x[0] == x0}, x[t], t] // ToRadicals
```

Solve::nsmet : This system cannot be solved with the methods available to Solve. ≫

Solve[

$$\Bigg(\mathrm{Log}\Big[\frac{\alpha_1}{3\,\alpha_2}+\left(2^{1/3}\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\left(3\,\alpha_2\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right)-\frac{1}{3\times2^{1/3}\,\alpha_2}\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}+x[t]\Big]$$

$$\left(-\frac{\alpha_1}{3\,\alpha_2}-\left(2^{1/3}\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\left(3\,\alpha_2\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right)+\frac{1}{3\times2^{1/3}\,\alpha_2}\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right)\Bigg)\Big/$$

$$\Bigg(\alpha_0+2\,\alpha_1\left(-\frac{\alpha_1}{3\,\alpha_2}-\left(2^{1/3}\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\left(3\,\alpha_2\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right)+\frac{1}{3\times2^{1/3}\,\alpha_2}\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right)+$$

$$3\,\alpha_2 \left(-\frac{\alpha_1}{3\,\alpha_2} - \left(2^{1/3}\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)\right) \Big/ \left(3\,\alpha_2 \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2 + \right.\right.\right.$$

$$\left.\left.\sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right) +$$

$$\frac{1}{3 \times 2^{1/3}\,\alpha_2}\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \right.}\right.$$

$$\left.\left.\left.\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3}\right)^2\right) +$$

$$\left(\text{Log}\left[\frac{\alpha_1}{3\,\alpha_2} - \left(\left(1 + \mathbb{i}\sqrt{3}\right)\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)\right) \Big/ \left(3 \times 2^{2/3}\,\alpha_2 \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\right.\right.\right.\right.$$

$$\left.\left.\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right) +$$

$$\frac{1}{6 \times 2^{1/3}\,\alpha_2}\left(1 - \mathbb{i}\sqrt{3}\right)\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \right.}\right.$$

$$\left.\left.\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3} + x[t]\Big]$$

$$\left(-\frac{\alpha_1}{3\,\alpha_2} + \left(\left(1 + \mathbb{i}\sqrt{3}\right)\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)\right) \Big/ \left(3 \times 2^{2/3}\,\alpha_2 \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\right.\right.\right.$$

$$\left.\left.\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right) -$$

$$\frac{1}{6 \times 2^{1/3}\,\alpha_2}\left(1 - \mathbb{i}\sqrt{3}\right)\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \right.}\right.$$

$$\left.\left.\left.\left.\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3}\right)\right) \Big/$$

$$\left(\alpha_0 + 2\,\alpha_1 \left(-\frac{\alpha_1}{3\,\alpha_2} + \left(\left(1 + \mathbb{i}\sqrt{3}\right)\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)\right) \Big/ \left(3 \times 2^{2/3}\,\alpha_2 \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - \right.\right.\right.\right.$$

$$\left.\left.27\,\alpha_{-1}\,\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right) -$$

$$\frac{1}{6 \times 2^{1/3}\,\alpha_2}\left(1 - \mathbb{i}\sqrt{3}\right)\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \right.}\right.$$

$$\left.\left.\left.\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3}\right) +$$

$$3\,\alpha_2 \left(-\frac{\alpha_1}{3\,\alpha_2} + \left(\left(1 + \mathbb{i}\sqrt{3}\right)\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)\right) \Big/ \left(3 \times 2^{2/3}\,\alpha_2 \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - \right.\right.\right.$$

$$\left.\left.27\,\alpha_{-1}\,\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right) -$$

$$\frac{1}{6 \times 2^{1/3}\,\alpha_2}\left(1 - \mathbb{i}\sqrt{3}\right)\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \right.}\right.$$

$$\left.\left.\left.\left.\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3}\right)^2\right) +$$

$$\left(\text{Log}\left[\frac{\alpha_1}{3\,\alpha_2} - \left(\left(1 - \mathbb{i}\sqrt{3}\right)\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)\right) \Big/ \left(3 \times 2^{2/3}\,\alpha_2 \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\right.\right.\right.\right.$$

$$\left.\left.\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right) +$$

$$\frac{1}{6 \times 2^{1/3}\,\alpha_2}\left(1 + \mathbb{i}\sqrt{3}\right)\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \right.}\right.$$

$$\left.\left.\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3} + x[t]\Big]$$

$$\left(-\frac{\alpha_1}{3\,\alpha_2} + \left(\left(1 - \mathbb{i}\sqrt{3}\right)\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)\right) \Big/ \left(3 \times 2^{2/3}\,\alpha_2 \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\right.\right.\right.$$

$$\left.\left.\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right) -$$

$$\frac{1}{6 \times 2^{1/3}\,\alpha_2}\left(1 + \mathbb{i}\sqrt{3}\right)\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \right.}\right.$$

$$\left.\left.\left.\left.\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3}\right)\right) \Big/$$

$$\left(\alpha_0 + 2\,\alpha_1 \left(-\frac{\alpha_1}{3\,\alpha_2} + \left(\left(1 - \mathbb{i}\sqrt{3}\right)\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)\right) \Big/ \left(3 \times 2^{2/3}\,\alpha_2 \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - \right.\right.\right.\right.$$

$$\left.\left.27\,\alpha_{-1}\,\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right) -$$

$$\frac{1}{6 \times 2^{1/3}\,\alpha_2}\left(1 + \mathbb{i}\sqrt{3}\right)\left(-2\,\alpha_1^3 + 9\,\alpha_0\,\alpha_1\,\alpha_2 - 27\,\alpha_{-1}\,\alpha_2^2 + \sqrt{\left(4\left(-\alpha_1^2 + 3\,\alpha_0\,\alpha_2\right)^3 + \right.}\right.$$

$$\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\;+$$

$$3\,\alpha_2\,\Big(-\frac{\alpha_1}{3\,\alpha_2}+\Big(\left(1-\mathrm{i}\,\sqrt{3}\right)\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\Big)\Big/\Big(3\times 2^{2/3}\,\alpha_2\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-$$

$$27\,\alpha_{-1}\,\alpha_2^2+\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\;-$$

$$\frac{1}{6\times 2^{1/3}\,\alpha_2}\,\left(1+\mathrm{i}\,\sqrt{3}\right)\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+$$

$$\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)^2\Big)\;==$$

$$t+\Big(\mathrm{Log}\Big[x0+\frac{\alpha_1}{3\,\alpha_2}+\left(2^{1/3}\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\Big(3\,\alpha_2\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+$$

$$\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\;-$$

$$\frac{1}{3\times 2^{1/3}\,\alpha_2}\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+$$

$$\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big]$$

$$\Big(-\frac{\alpha_1}{3\,\alpha_2}-\left(2^{1/3}\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\Big(3\,\alpha_2\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+$$

$$\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\;+$$

$$\frac{1}{3\times 2^{1/3}\,\alpha_2}\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+$$

$$\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\Big)\Big/$$

$$\Big(\alpha_0+2\,\alpha_1\,\Big(-\frac{\alpha_1}{3\,\alpha_2}-\left(2^{1/3}\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\Big(3\,\alpha_2\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+$$

$$\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\;+$$

$$\frac{1}{3\times 2^{1/3}\,\alpha_2}\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+$$

$$\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\;+$$

$$3\,\alpha_2\,\Big(-\frac{\alpha_1}{3\,\alpha_2}-\left(2^{1/3}\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\Big(3\,\alpha_2\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+$$

$$\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\;+$$

$$\frac{1}{3\times 2^{1/3}\,\alpha_2}\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+$$

$$\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)^2\Big)\;+$$

$$\Big(\mathrm{Log}\Big[x0+\frac{\alpha_1}{3\,\alpha_2}-\Big(\left(1+\mathrm{i}\,\sqrt{3}\right)\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\Big)\Big/\Big(3\times 2^{2/3}\,\alpha_2\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27$$

$$\alpha_{-1}\,\alpha_2^2+\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\;+$$

$$\frac{1}{6\times 2^{1/3}\,\alpha_2}\,\left(1-\mathrm{i}\,\sqrt{3}\right)\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+$$

$$\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big]$$

$$\Big(-\frac{\alpha_1}{3\,\alpha_2}+\Big(\left(1+\mathrm{i}\,\sqrt{3}\right)\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\Big)\Big/\Big(3\times 2^{2/3}\,\alpha_2\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}$$

$$\alpha_2^2+\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\;-$$

$$\frac{1}{6\times 2^{1/3}\,\alpha_2}\,\left(1-\mathrm{i}\,\sqrt{3}\right)\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+$$

$$\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\Big)\Big/$$

$$\Big(\alpha_0+2\,\alpha_1\,\Big(-\frac{\alpha_1}{3\,\alpha_2}+\Big(\left(1+\mathrm{i}\,\sqrt{3}\right)\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\Big)\Big/\Big(3\times 2^{2/3}\,\alpha_2\,\Big(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-$$

$$27\,\alpha_{-1}\,\alpha_2^2+\surd\Big(4\,\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\Big)\Big)^{1/3}\Big)\;-$$

$$\frac{1}{6\times 2^{1/3}\,\alpha_2}\left(1-i\sqrt{3}\right)\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\right.}\right.$$

$$\left.\left.\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3}\Bigg)+$$

$$3\,\alpha_2\left(-\frac{\alpha_1}{3\,\alpha_2}+\left(\left(1+i\sqrt{3}\right)\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\left(3\times 2^{2/3}\,\alpha_2\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-\right.\right.\right.$$

$$\left.\left.27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right)-$$

$$\frac{1}{6\times 2^{1/3}\,\alpha_2}\left(1-i\sqrt{3}\right)\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\right.}\right.$$

$$\left.\left.\left.\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3}\right)^2\Bigg)+$$

$$\Bigg(\mathrm{Log}\Big[x0+\frac{\alpha_1}{3\,\alpha_2}-\left(\left(1-i\sqrt{3}\right)\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\left(3\times 2^{2/3}\,\alpha_2\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\right.\right.$$

$$\left.\left.\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right)+$$

$$\frac{1}{6\times 2^{1/3}\,\alpha_2}\left(1+i\sqrt{3}\right)\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\right.}\right.$$

$$\left.\left.\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3}\Big]$$

$$\left(-\frac{\alpha_1}{3\,\alpha_2}+\left(\left(1-i\sqrt{3}\right)\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\left(3\times 2^{2/3}\,\alpha_2\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\right.\right.\right.$$

$$\left.\left.\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right)-$$

$$\frac{1}{6\times 2^{1/3}\,\alpha_2}\left(1+i\sqrt{3}\right)\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\right.}\right.$$

$$\left.\left.\left.\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3}\right)\Big)\Big/$$

$$\Bigg(\alpha_0+2\,\alpha_1\left(-\frac{\alpha_1}{3\,\alpha_2}+\left(\left(1-i\sqrt{3}\right)\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\left(3\times 2^{2/3}\,\alpha_2\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-\right.\right.\right.$$

$$\left.\left.27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right)-$$

$$\frac{1}{6\times 2^{1/3}\,\alpha_2}\left(1+i\sqrt{3}\right)\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\right.}\right.$$

$$\left.\left.\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3}\Bigg)+$$

$$3\,\alpha_2\left(-\frac{\alpha_1}{3\,\alpha_2}+\left(\left(1-i\sqrt{3}\right)\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)\right)\Big/\left(3\times 2^{2/3}\,\alpha_2\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-\right.\right.\right.$$

$$\left.\left.27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)}\right)^{1/3}\right)-$$

$$\frac{1}{6\times 2^{1/3}\,\alpha_2}\left(1+i\sqrt{3}\right)\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2+\sqrt{\left(4\left(-\alpha_1^2+3\,\alpha_0\,\alpha_2\right)^3+\right.}\right.$$

$$\left.\left.\left.\left(-2\,\alpha_1^3+9\,\alpha_0\,\alpha_1\,\alpha_2-27\,\alpha_{-1}\,\alpha_2^2\right)^2\right)\right)^{1/3}\right)^2\Bigg),\ x[t]\Big]$$

Double Well potential (bimodal behaviour, highly nonlinear)

```
DSolve[{x'[t] == (x[t] - x[t]^3), x[0] == x0}, x[t], t]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$$\left\{\left\{x[t]\to-\frac{e^{t}}{\sqrt{-1+e^{2t}+\frac{1}{x0^2}}}\right\},\left\{x[t]\to\frac{e^{t}}{\sqrt{-1+e^{2t}+\frac{1}{x0^2}}}\right\}\right\}$$

Test

$$D\left[-\frac{e^t}{\sqrt{-1+e^{2t}+\frac{1}{x0^2}}}, t\right] == -\frac{e^t}{\sqrt{-1+e^{2t}+\frac{1}{x0^2}}} - \left(-\frac{e^t}{\sqrt{-1+e^{2t}+\frac{1}{x0^2}}}\right)^3$$

```
True
```

Jacobi diffusion (political polarization)

$$\text{DSolve}\left[\left\{x'[t] == -\theta\left(x[t]-\frac{1}{2}\right), x[0] == x0\right\}, x[t], t\right] \text{ // FullSimplify}$$

$$\left\{\left\{x[t] \to \frac{1}{2}\left(1+e^{-t\theta}\left(-1+2\,x0\right)\right)\right\}\right\}$$

Ohrenstein-Uhlenbeck (OU)

It differs from gBm only in the stochastic term.

Radical Ohrenstein-Uhlenbeck

$$\text{DSolve}\left[\left\{x'[t] == \frac{\theta}{x[t]} - x[t], x[0] == x0\right\}, x[t], t\right] \text{ // FullSimplify}$$

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\left\{\left\{x[t] \to -\sqrt{e^{-2t}\left(x0^2-\theta\right)+\theta}\right\}, \left\{x[t] \to \sqrt{e^{-2t}\left(x0^2-\theta\right)+\theta}\right\}\right\}$$

Hyperbolic diffusion (dynamics of sand)

$$\text{DSolve}\left[\left\{x'[t] == \frac{\theta}{x[t]} - x[t], x[0] == x0\right\}, x[t], t\right] \text{ // FullSimplify}$$

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\left\{\left\{x[t] \to -\sqrt{e^{-2t}\left(x0^2-\theta\right)+\theta}\right\}, \left\{x[t] \to \sqrt{e^{-2t}\left(x0^2-\theta\right)+\theta}\right\}\right\}$$

Ito process 2

Example 2

```
μ, σ, θ, r0
```

$$\text{example2ItoProcess}[\mu\_, \sigma\_, \theta\_, r0\_] := \text{ItoProcess}\left[dx[t] == \theta\left(\mu - x[t]\right) dt + \sigma\sqrt{x[t]}\, dw[t], x[t], \{x, x0\}, t, w \approx \text{WienerProcess}[]\right]$$

Exercise. What is the name of this process?

```
example2SliceDistribution = SliceDistribution[example2ItoProcess[μ, σ, θ, r0], t]
```

```
ItoProcess[{{θ μ - θ x[t]}, {{σ √x[t] }}, x[t]}, {{x}, {x0}}, {t, 0}][t]
```

```
Mean[example2SliceDistribution]
```

$e^{-t\theta}\left(x0+\left(-1+e^{t\theta}\right)\mu\right)$

```
characteristicsofstochasticprocess[example2ItoProcess[μ, σ, θ, r0], s, t]
```

Process:	**ItoProcess[{{θ μ – θ x[t]}, {{σ Sqrt[x[t]]}}, x[t]}, {{x}, {x0}}, {t, 0}]**
Mean	$\mu + e^{\theta(-t)}(x0-\mu)$
Variance	$\frac{\sigma^2 e^{-2\theta t}(e^{\theta t}-1)(\mu(e^{\theta t}-1)+2\,x0)}{2\theta}$
Skewness	$\left(\sqrt{2}\,\sigma^4 e^{-3\theta t}(e^{\theta t}-1)^2(\mu(e^{\theta t}-1)+3\,x0)\right)/\left(\theta^2\left(\frac{1}{\theta}\sigma^2 e^{-2\theta t}(e^{\theta t}-1)(\mu(e^{\theta t}-1)+2\,x0)\right)^{3/2}\right)$
Kurtosis	$\frac{3\sigma^2\left(1-\frac{4\,x0^2}{(\mu(e^{\theta t}-1)+2\,x0)^2}\right)}{\theta\mu}+3$
Covariance function	$\frac{1}{2\theta}\sigma^2 e^{\theta(-(s+t))}(e^{\theta\min(s,t)}-1)(\mu(e^{\theta\min(s,t)}-1)+2\,x0)$
Slice distribution	$\text{ItoProcess}\left[\{\{\theta(\mu-x(t))\},\ (\sigma\sqrt{x(t)}),\ x(t)\},\ \binom{x}{x0},\ \{t,0\}\right][t]$

Ho-Lee model

$$dX(t) = \theta(t)\,dt + \sigma\,dW(t)$$

Hull-White model

$$dX(t) = (\theta(t) - \alpha(t)\,X(t))\,dt + \sigma(t)\,X^{\beta}(t)\,dW(t)$$

The mean reversion level is given by $\theta(t)/\alpha(t)$. For Vasicek model $\beta = 0$, other parameters constant. For CIR, $\beta = 1/2$.

Black-Karasinski model

$$d\log X(t) = (\theta(t) - \alpha(t)\log X(t))\,dt + \sigma(t)\,dW(t)$$

In practise often $\alpha(t)$ and $\sigma(t)$ are supposed to be constant so that the model reads:

$$d\log X(t) = (\theta(t) - \alpha)\,dt + \sigma\,dW(t)$$

Black-Derman-Toy model

$$d\log X(t) = (\theta(t) - \alpha(t)\log X(t))\,dt + \sigma(t)\,dW(t)$$

is a special case of Black-Karasinski where $\alpha(t) = \sigma'(t)/\sigma(t)$.

Financial Risk

Financial Risk

The procedures as well as numbered references in

```
Hyperlink[Button[" Risk measures "],
 "http://www.karlin.mff.cuni.cz/~hurt/
   WTC2010JanHurtRiskMeasuresRevisited2withReference.nb"]
```

Risk measures

S_0, S_1, ... prices of a financial asset (random variables), rate of return (ROR, míra výnosnosti) or shortly return (výnos):

$$\rho_t = \frac{S_{t+1}-S_t}{S_t}$$

loss (ztráta):

$$L_t = -\rho_t$$

In literature we can often find very misleading assumptions that returns or losses are always nonnegative, hidden in the assumptions about lognormal distribution of returns (losses) &c. It is obviously nonsense.

In what follows we alternatively denote loss as L, L_t, X, ...

Definitions of risk measures

Standard deviation (směrodatná odchylka)

$$\sigma = \sqrt{\operatorname{var} \rho} = \sqrt{\operatorname{var} L}$$

Value at Risk at level 1 – α (VaR, hodnota v riziku na úrovni 1-α)

The definitions of Value at Risk differ. Particularly, what concerns the significance level. Logic of VaR is to determine the value over which the loss will exceed with a small probability. Thus, in any case, our interest is in the **right-hand tails** of the the distributions of **losses,** or in the left-hand tails of profits (rates of return). In this course we suppose the significance level $1-\alpha$ but in other sources often you can find α instead of $1-\alpha$.

Here is an example of the VaR in terms of profits. The example comes from Nagy Gergely, the Graduate from Charles University, Financial & Insurance Mathematics. Licensed users may download the source code and take a benefit from the technics and tricks in *Mathematica* coding.

http://demonstrations.wolfram.com/ValueAtRisk/

Value at risk at level $1-\alpha$ is simply the α-quantile of X, sometimes denoted by q_α.

$$\text{VaR}_\alpha(X) = F^{-1}(\alpha) = \inf\{x : F(x) \geq \alpha\}.$$

The usual interpretation is that loss will exceed $\mathrm{VaR}_\alpha(X)$ with a (small) probability $1-\alpha$ with typical values of $\alpha = 0.95,\ 0.99$ or in some extreme cases even $\alpha = 0.999$. In practice the time horizons are one day or one week but often longer time horizons are of interest but they need a special care.

Conditional Value at Risk (CVaR)

Conditional value at risk (also called Expected Shortfall, Tail Conditional Expectation, Mean Excess Loss, Tail VaR) is the mean of the truncated distribution of X truncated at point $\mathrm{VaR}_\alpha(X)$ from the left:

$$\mathrm{CVaR}_\alpha(X) = E\{X \mid X \geq \mathrm{VaR}_\alpha(X)\}.$$

Note that sometimes an analogue to the mean residual lifetime is considered:

$$\mathrm{CVaR}_\alpha(X) - \mathrm{VaR}_\alpha(X) = E\{X - \mathrm{VaR}_\alpha(X) \mid X \geq \mathrm{VaR}_\alpha(X)\}.$$

For absolutely continuous distributions we have

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_{\mathrm{Var}_\alpha(X)}^{\infty} y\, f(y)\, dy.$$

Average Value at Risk at level $1-\alpha$ (AVaR)

Average value at risk is defined as

$$\mathrm{AVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_p(X)\, dp = \frac{1}{1-\alpha} \int_\alpha^1 F^{-1}(p)\, dp.$$

After substitution $y = F^{-1}(p)$ we get

$$\mathrm{AVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_{\mathrm{VaR}_\alpha(X)}^{\infty} y\, dF(y) = \mathrm{CVaR}_\alpha(X).$$

Hence both characteristics coincide but they differ in interpretation.

Weighted VaR (WVaR)

$$\mathrm{WAVaR}_\mu(X) = \int_0^1 \mathrm{AVaR}_\lambda(X)\, \mu(d\lambda),$$

where μ is a probability measure on [0, 1].

Median Value at Risk (Median Shortfall, Tail Conditional Median)

Median value at risk is defined as the median of the conditional distribution given that the loss exceeded value at risk.

$$\mathrm{MVaR}_\alpha(X) = \mathrm{Median}\{X \mid X \geq \mathrm{VaR}_\alpha(X)\}.$$

Therefore, $\mathrm{MVaR}_\alpha(X)$ is a value satisfying

$$P(X \geq \mathrm{MVaR}_\alpha(X) \mid X \geq \mathrm{VaR}_\alpha(X)) = \frac{P(X \geq \mathrm{MVaR}_\alpha(X),\, X \geq \mathrm{VaR}_\alpha(X))}{P(X \geq \mathrm{VaR}_\alpha(X))} = \frac{1}{2}$$

or

$$P(X \geq \mathrm{MVaR}_\alpha(X), X \geq \mathrm{VaR}_\alpha(X)) = \frac{1}{2}(1-\alpha).$$

Since $P(X \geq \mathrm{VaR}_\alpha(X)) = 1 - \alpha > \frac{1}{2}(1-\alpha)$, $\mathrm{MVaR}_\alpha(X) >= \mathrm{VaR}_\alpha(X)$, and therefore

$$P(X \geq \mathrm{MVaR}_\alpha(X), X \geq \mathrm{VaR}_\alpha(X)) = P(X \geq \mathrm{MVaR}_\alpha(X)) = \frac{1}{2}(1-\alpha)$$

so that $P(X < \mathrm{MVaR}_\alpha(X)) = \frac{1+\alpha}{2}$ and

$$\mathrm{MVaR}_\alpha(X) = \mathrm{VaR}_{(1+\alpha)/2}(X).$$

Sometimes median residual lifetime is considered:

$$\mathrm{MVaR}_\alpha(X) - \mathrm{VaR}_\alpha(X) = \mathrm{Median}\,\{X - \mathrm{VaR}_\alpha(X) \mid X \geq \mathrm{VaR}_\alpha(X)\}.$$

Quantile Value at Risk

Quantile Value at Risk (Quantile Shortfall, Tail Conditional Quantile) is defined as

$$\mathrm{QVaR}_{\alpha,\beta}(X) = \beta - \text{quantile of the conditional distribution}\,\{X \mid X \geq \mathrm{VaR}_\alpha(X)\}.$$

It means that the probability of loss greater than QVaR given that it exceeded VaR will be $1-\beta$.

Thus $\mathrm{QVaR}_{\alpha,\beta}(X)$ satisfies the relation

$$P(X \geq \mathrm{QVaR}_{\alpha,\beta}(X) \mid X \geq \mathrm{VaR}_\alpha(X)) = \frac{P(X \geq \mathrm{QVaR}_{\alpha,\beta}(X), X \geq \mathrm{VaR}_\alpha(X))}{P(X \geq \mathrm{VaR}_\alpha(X))} = 1-\beta,$$

and, since it has sense to consider $\mathrm{QVaR}_\alpha(X) \geq \mathrm{VaR}_\alpha(X)$ only, we have

$$P(X \geq \mathrm{QVaR}_{\alpha,\beta}(X)) = (1-\beta)(1-\alpha),$$

so that (due to $1-(1-\beta)(1-\alpha) = \alpha + \beta - \alpha\beta$)

$$P(X < \mathrm{QVaR}_{\alpha,\beta}(X)) = \alpha + \beta - \alpha\beta.$$

It follows

$$\mathrm{QVaR}_{\alpha,\beta}(X) = F^{-1}(\alpha + \beta - \alpha\beta) = (\alpha + \beta - \alpha\beta) - \text{quantile of } X.$$

Spectral measures

Suppose that ϕ is a non-decreasing non-negative real function defined on [0, 1], $\int_0^1 \phi(p)\,dp = 1$, sometimes called *spectrum*. (ϕ is simply non-decreasing PDF with support [0, 1].The quantity

$$M_\phi(X) = \int_0^1 F^{-1}(p)\,\phi(p)\,dp$$

is called *spectral risk measure*. If F is absolutely continuous with the probability density function f and strictly increasing on its support then after substitution $F^{-1}(p) = y$ we obtain

$$M_\phi(X) = \int_{-\infty}^{\infty} y\,\phi(F(y))\,f(y)\,dy.$$

In fact, ϕ is a weighting function giving higher weights to higher losses.

Examples of spectra

Dual-power risk aversion function

```
ϕdual[u_, ν_] := ν u^(ν-1) (* ν≥1 *)
```

Proportional Hazard risk aversion function

```
ϕhazard[u_, γ_] := 1/γ (1 - u)^(1/γ-1) (* γ≥1 *)
```

Wang's risk aversion function

```
ϕWang[u_, δ_] := Exp[-δ Quantile[NormalDistribution[0, 1], u] - δ^2 / 2] (* δ>0 *)
```

Exponential risk aversion function

```
ϕArrowPratt[u_, k_] := k Exp[-k (1 - u)] / (1 - Exp[-k])
```

???

where *k* is the Arrow-Pratt degree of absolute risk aversion (ARA, see also Wolfram Demonstration by [3] Chandler):

```
- D[ϕArrowPratt[u, k], {u, 2}] / D[ϕArrowPratt[u, k], u]
```

```
-k
```

???

Expectiles

We start with motivation given in [22] Yao (1996) despite the idea originates in [18] Newey and Powell (1987). First we give an alternative approach for defining quantiles through an optimization problem. Define

$$R_\alpha(x) = (1-\alpha)\,|\,x\,|\,, \quad x \le 0,$$
$$R_\alpha(x) = \alpha\,|\,x\,|\,, \qquad x > 0.$$

The α-quantile q_α of X may be defined as

$$q_\alpha = \arg\min_a E R_\alpha(X-a).$$

Obviously $\alpha = P(X \le q_\alpha)$. In order to have an analogy with expectiles, write

$$\alpha = \frac{E I\{X \le q_\alpha\}}{E 1},$$

where I means the indicator function.

To define expectiles, let us define

$$Q_\omega(x) = (1-\omega)\,x^2\,, \quad x \le 0,$$
$$Q_\omega(x) = \omega\,x^2\,, \qquad x > 0.$$

The ω-expectile of X is defined as the minimizer

$$\tau_\omega = \arg\min_a E Q_\omega(X-a), \tag{8}$$

see [22] Yao (1996), e.g. For $\omega = \frac{1}{2}$ we have $\tau_{1/2} = EX$. Since $Q_\omega(.)$ has a continuous first derivative, τ_ω satisfies the equation

$E L_\omega(X - \tau_\omega) = 0$

where

$L_\omega(x) = (1 - \omega)\, x\,, \quad x \le 0,$

$L_\omega(x) = \omega\, x\,, \qquad\quad x > 0.$

Hence τ_ω satisfies

$$\omega = \frac{E[\,|\,X - \tau_\omega\,|\; I\{X \le \tau_\omega\}]}{E\,|\,X - \tau_\omega\,|}. \tag{9}$$

Distribution Specific Calculations

Normal distribution

VaR

```
VaRNormal[α_, μ_, σ_] = VaR[NormalDistribution[μ, σ], α]
```

$\mu + \sqrt{2}\;\sigma\,\text{InverseErf}[-1 + 2\,\alpha]$

CVaR

```
CVaRNormal[α_, μ_, σ_] = CVaR[NormalDistribution[μ, σ], α, Re[σ^2] > 0]
```

$$\frac{2\,\mu - 2\,\alpha\,\mu + e^{-\text{InverseErf}[-1+2\,\alpha]^2}\sqrt{\frac{2}{\pi}}\;\sigma}{2 - 2\,\alpha}$$

AVaR

```
AVaRNormal[α_, μ_, σ_] = AVaR[NormalDistribution[μ, σ], α] // PowerExpand // Simplify
```

$$\frac{\mu - \alpha\,\mu + \frac{e^{-\text{InverseErf}[-1+2\,\alpha]^2}\,\sigma}{\sqrt{2\,\pi}}}{1 - \alpha}$$

Spectral measure

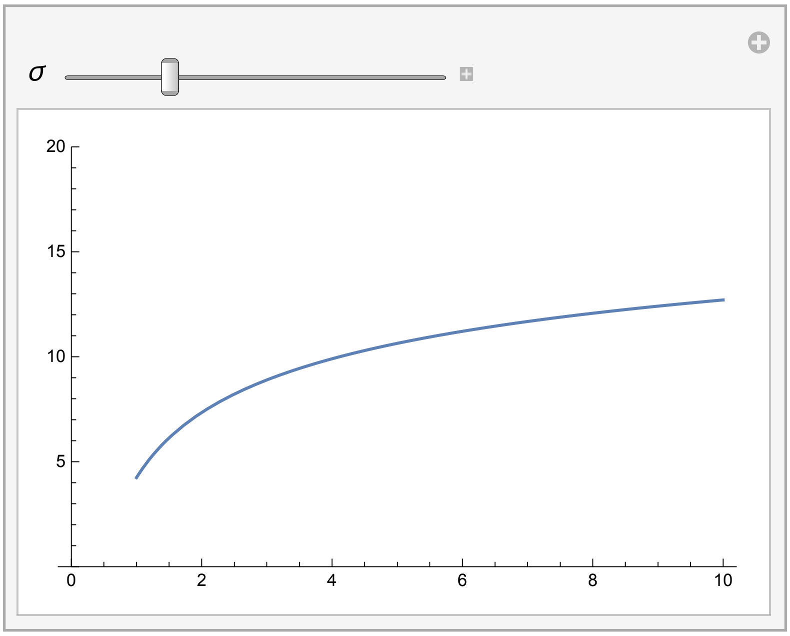
Recall that throughout the paper if we speak on specific distributions, we consider distributions with mean and standard deviation equal to

```
{3 √2, √34} // N
```

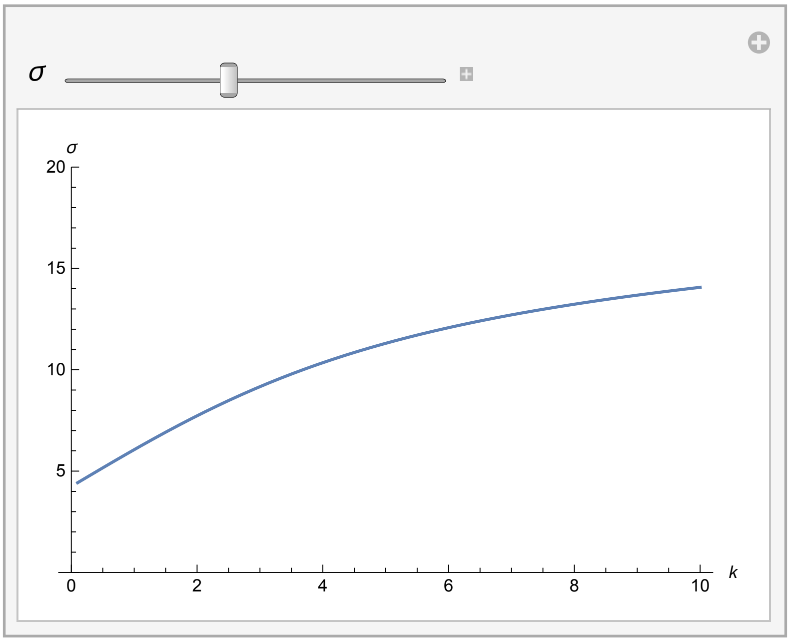
{4.24264, 5.83095}

We start with the spectral risk for a normal distribution.

```
Manipulate[Plot[NSpectralRisk[NormalDistribution[3. √2, σ], ϕdual[#, ν] &], {ν, 1, 10},
   AxesOrigin → {0, 0}, PlotRange → {0, 20}, AxesLabel → {"ν", "Spectral risk"}],
 {{σ, 5.83}, 4, 10}, SaveDefinitions → True]
```




```
Manipulate[Plot[NSpectralRisk[NormalDistribution[3. √2, σ], ϕArrowPratt[#, k] &],
   {k, 0.1, 10}, AxesOrigin → {0, 0}, PlotRange → {0, 20}, AxesLabel → {k, "σ"}],
 {{σ, 5.83}, 4, 10}, SaveDefinitions → True]
```



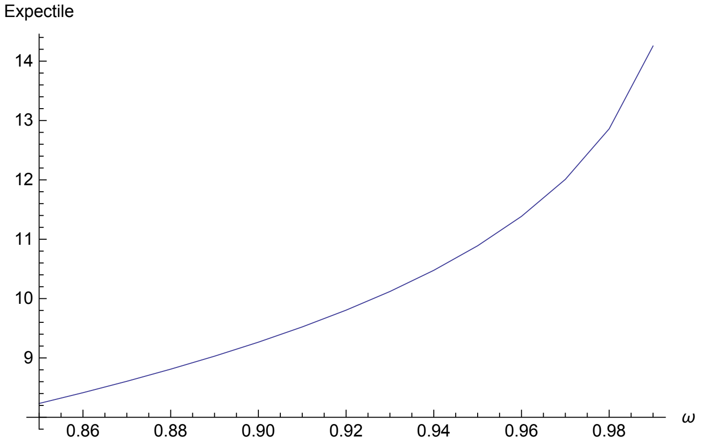
The reader will simply discover that such plots are rather boring. More interesting and more important are the ratios of the risk measures.

Expectiles of the normal distribution

```
Text@Grid[Prepend[tableExpectile2Normal =
    Table[{ω, Expectile2[NormalDistribution[3. √2, 5.83], ω]}, {ω, 0.85, 0.99, 0.01}],
   {"ω", "Expectile2"}], Frame → All]
```

ω	Expectile2
0.85	8.23314
0.86	8.41456
0.87	8.6066
0.88	8.81094
0.89	9.02974
0.9	9.26572
0.91	9.52248
0.92	9.80482
0.93	10.1195
0.94	10.4761
0.95	10.8898
0.96	11.3853
0.97	12.0085
0.98	12.8604
0.99	14.2553

```
ListPlot[tableExpectile2Normal, Joined → True,
 AxesOrigin → {0.85, 8}, AxesLabel → {"ω", "Expectile"}]
```



Laplace distribution

VaR

```
VaRLaplace[α_, μ_, β_] = VaR[LaplaceDistribution[μ, β], α, β > 0 && 1/2 < α < 1]
```

$\mu - \beta \, \text{Log}[2 - 2\,\alpha]$

CVaR

```
CVaRLaplace[α_, μ_, β_] = CVaR[LaplaceDistribution[μ, β], α, β > 0 && (1/2 < α < 1)]
```

$$\frac{\int_{\mu-\beta \text{Log}[2-2\alpha]}^{\infty} \frac{e^{-\frac{(y-\mu)\,\text{Sign}[y-\mu]}{\beta}}\, y}{2\beta}\, dy}{1-\alpha}$$

For $\frac{1}{2} < \alpha < 1$ we have $2 - 2\alpha < 1$ so that $\mu - \beta \text{Log}[2 - 2\alpha] > \mu$ and thus

```
CVaRLaplace[α_, μ_, β_] = Assuming[β > 0, (∫_{μ-β Log[2-2α]}^∞ (e^(-(y-μ)/β) y)/(2β) dy)/(1 - α)] // FullSimplify
```

$\beta + \mu - \beta \text{Log}[2 - 2\alpha]$

Observe that direct calculation of the AVaR (which coincides with CVaR) is straightforward and does need any manual adjustment.

AVaR

```
AVaRLaplace[α_, μ_, β_] = AVaR[LaplaceDistribution[μ, β], α, β > 0 && (1/2 < α < 1)]
```

$\beta + \mu - \beta \text{Log}[2 - 2\alpha]$

Spectral measure

We choose the parameters to have the same mean and standard deviation as in the normal case. We have

```
{Mean[LaplaceDistribution[μ, β]], StandardDeviation[LaplaceDistribution[μ, β]]}
```

$\{\mu, \sqrt{2}\,\beta\}$

so that a solution to these equations returns the equivalent parameters.

```
Solve[{μ == 3. √2, √2 β == 5.83}, {μ, β}]
```

{{μ → 4.24264, β → 4.12243}}

For ϕdual the computation mostly takes longer time than for ϕArrowPratt, e.g. However, the absolute exponential risk aversion is, fortunately, most popular in practice. Thus it would be convenient to give some examples with the Arrow-Pratt spectrum only.

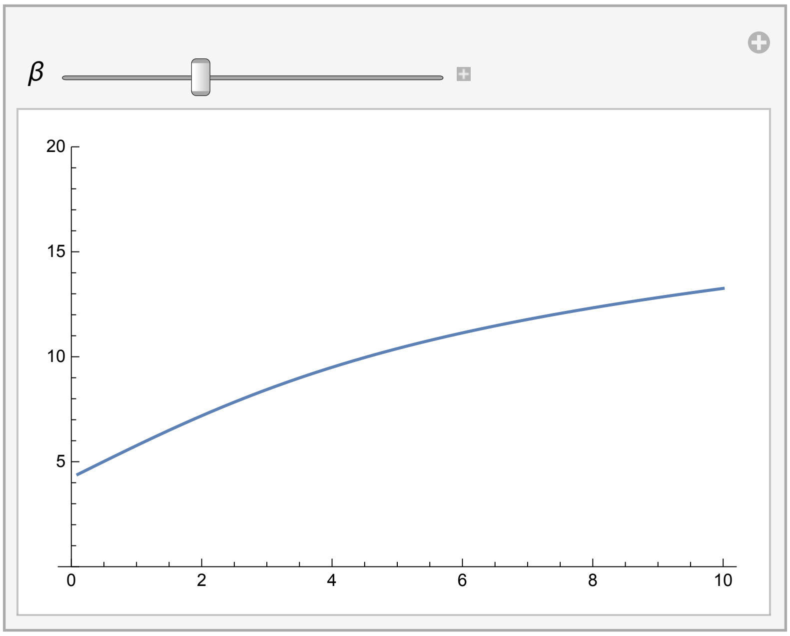
Nevertheless, we start with ϕdual:

```
Text@Grid[Prepend[
    Table[{ν, NSpectralRisk[LaplaceDistribution[4.24264, 4.12243], ϕdual[#, ν] &]},
     {ν, 1, 10}], {"ν", "Spectral Risk   "}] // Transpose, Frame → True]
```

ν	1	2	3	4	5	6	7	8	9	10
Spectral Risk	4.24264	7.33446	8.88037	9.95392	10.7913	11.4827	12.0731	12.589	13.0473	13.4596

And now for the absolute exponential risk aversion.

```
Manipulate[Plot[NSpectralRisk[LaplaceDistribution[4.24264, β], ϕArrowPratt[#, k] &],
  {k, 0.1, 10}, AxesOrigin → {0, 0}, PlotRange → {0, 20}],
 {{β, 4.12243}, 1, 10}, SaveDefinitions → True]
```



Expectiles

```
Expectile2[LaplaceDistribution[4.24264, 4.12243], 0.75] // Timing
```

```
{414.75, 6.58065}
```

On an old HP notebook the following calculation appeared to be too ambitious.

```
tableExpectile2Laplace =
 Table[{ω, Expectile2[LaplaceDistribution[4.24264, 4.12243], ω]}, {ω, 0.85, 0.99, 0.01}]
```

```
$Aborted
```

Aborted after 5349 seconds.

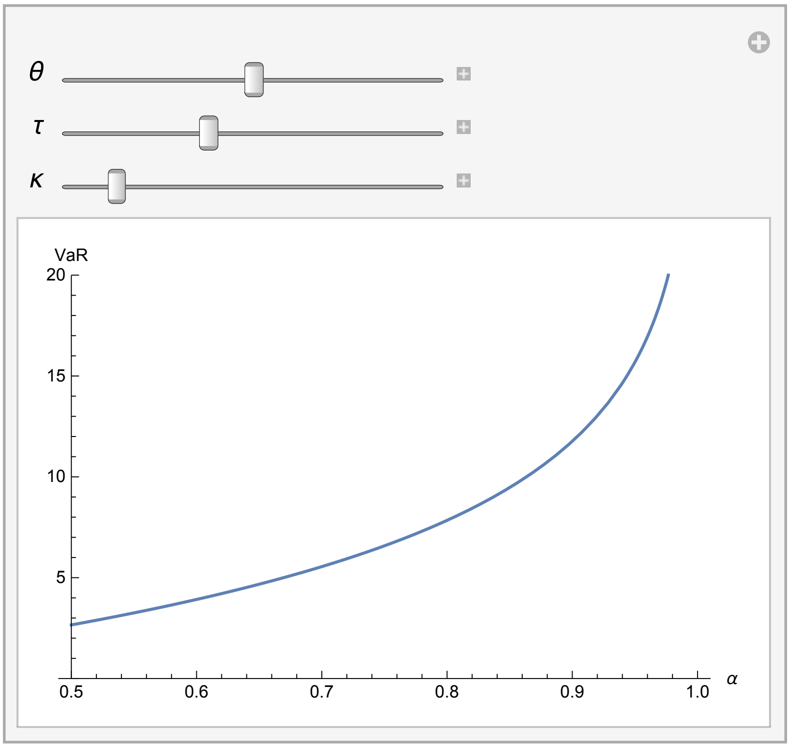
Asymmetric Laplace distribution (ALD)

VaR

```
VaRALD[α_, θ_, τ_, κ_] = VaR[AsymmetricLaplaceDistribution[θ, τ, κ], α]
```

$$\text{Which}\left[0 < \alpha \le \frac{\kappa^2}{1+\kappa^2},\ \theta + \frac{(\kappa\,\tau)\ \text{Log}\left[\alpha\left(1+\frac{1}{\kappa^2}\right)\right]}{\sqrt{2}},\ \frac{\kappa^2}{1+\kappa^2} < \alpha < 1,\ \theta - \frac{\tau\ \text{Log}\left[-\left(-1+\alpha\right)\left(1+\kappa^2\right)\right]}{\sqrt{2}\ \kappa}\right]$$

```
Manipulate[Plot[VaRALD[α, θ, τ, κ], {α, 1/2, 1}, AxesOrigin → {1/2, 0},
  PlotRange → {0, 20}, AxesLabel → {"α", "VaR"}], {{θ, 0}, -3, 3},
 {{τ, 4}, 1/2, 10}, {{κ, 1/2}, 0.1, 4}, SaveDefinitions → True]
```

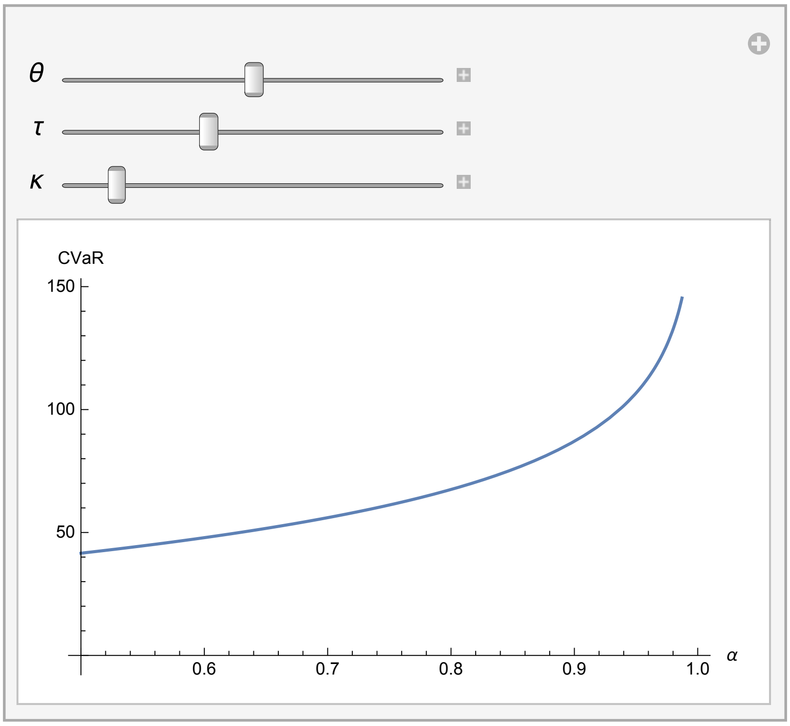


CVaR

See Section Calculations for explanation.

```
CVaRALD[α_, θ_, τ_, κ_] = Which[0 < α ≤ κ^2/(1 + κ^2),
   1/(1 - α) 1/2 (1 + κ^2) τ (-2 (-1 + α) θ + 1/κ √2 τ (1 + κ^2 (-1 + α - α (Log[α] + Log[1 + 1/κ^2])))),
   κ^2/(1 + κ^2) < α < 1, - 1/(1 - α) 1/(√2 κ) (-1 + α) (1 + κ^2) τ (√2 θ κ + τ - τ Log[-(-1 + α) (1 + κ^2)])];
```

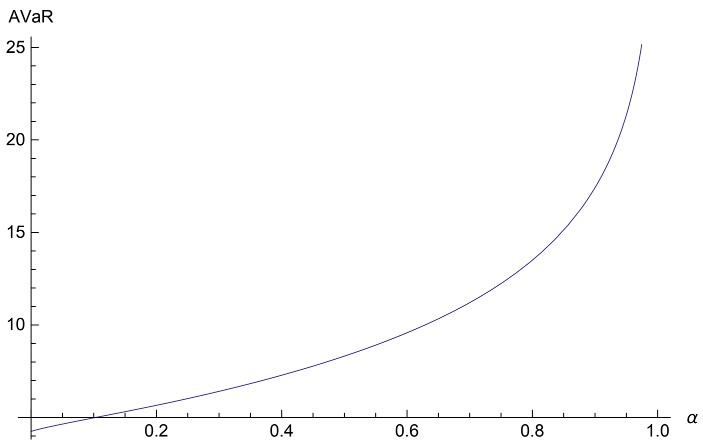
```
Manipulate[Plot[CVaRALD[α, θ, τ, κ], {α, 1/2, 1},
  AxesOrigin → {1/2, 0}, AxesLabel → {"α", "CVaR"}], {{θ, 0}, -3, 3},
 {{τ, 4}, 1/2, 10}, {{κ, 1/2}, 0.1, 4}, SaveDefinitions → True]
```



AVaR

See Section Calculations for details.

```
AVaRALD[α_, θ_, τ_, κ_] =
  Which[0 < α ≤ κ^2/(1 + κ^2), -((2 θ κ - 2 α θ κ + Sqrt[2] τ - Sqrt[2] κ^2 τ + Sqrt[2] α κ^2 τ -
        Sqrt[2] α κ^2 τ (Log[α] + Log[1 + 1/κ^2]))/(2 (-1 + α) κ)),
   κ^2/(1 + κ^2) < α < 1, 1/(2 κ) (2 θ κ + Sqrt[2] τ - Sqrt[2] τ Log[1 - α] - Sqrt[2] τ Log[1 + κ^2])];
Plot[AVaRALD[α, 0, 4, 1/2], {α, 0, 1}, AxesLabel → {"α", "AVaR"}]
```



Spectral measure

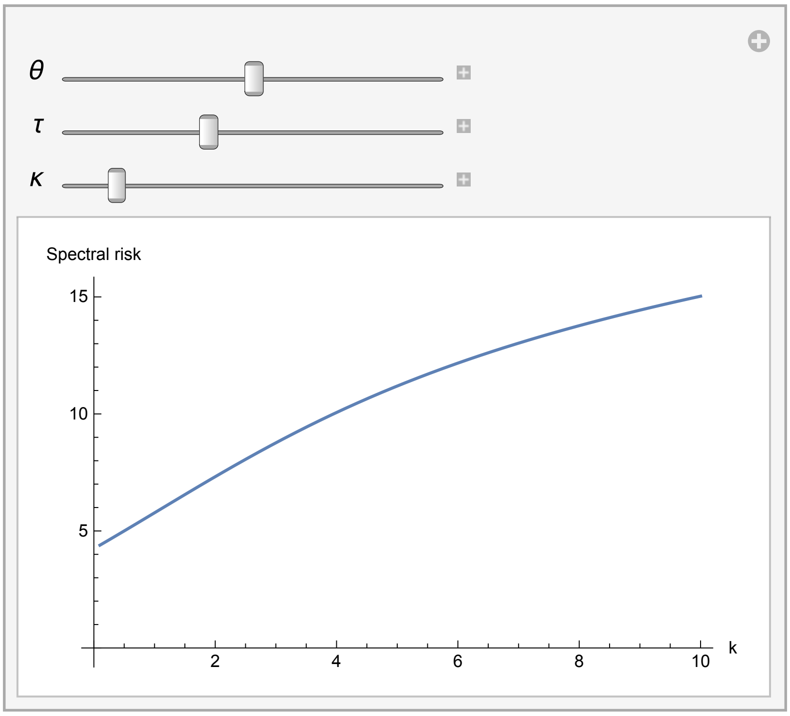
```
NSpectralRisk[AsymmetricLaplaceDistribution[0, 4, 1/2], ϕdual[#, 1] &] // Timing
```

```
{0.032, 4.24264}
```

```
Table[NSpectralRisk[AsymmetricLaplaceDistribution[0, 4, 1/2], ϕArrowPratt[#, k] &],
 {k, 0.1, 10}]
```

```
{4.3918, 5.93074, 7.46799, 8.90228, 10.1813, 11.2959, 12.2617, 13.1022, 13.8406, 14.4964}
```

```
Manipulate[
 Plot[NSpectralRisk[AsymmetricLaplaceDistribution[θ, τ, κ], ϕArrowPratt[#, k] &],
  {k, 0.1, 10}, AxesOrigin → {0, 0}, AxesLabel → {"k", "Spectral risk"}],
 {{θ, 0}, -3, 3}, {{τ, 4}, 1/2, 10}, {{κ, 1/2}, 0.1, 4}, SaveDefinitions → True]
```



Expectiles

First illustration:

```
Expectile2[AsymmetricLaplaceDistribution[0, 4, 1/2], 0.75] // Timing
```

```
{16.782, 6.91046}
```

```
(tableExpectile2ALD =
  Table[{ω, Expectile2[AsymmetricLaplaceDistribution[0., 4., 1./2], ω]} // Flatten,
   {ω, 0.85, 0.99, 0.01}])
```

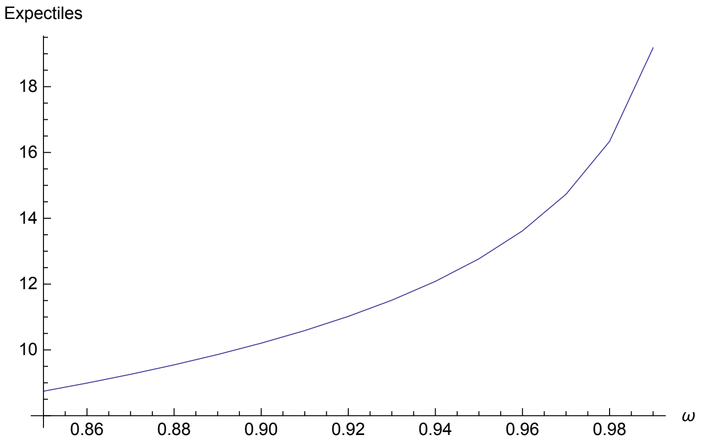
```
{310.281, {{0.85, 8.74414}, {0.86, 8.99135}, {0.87, 9.2573},
  {0.88, 9.54518}, {0.89, 9.85904}, {0.9, 10.2041}, {0.91, 10.5874},
  {0.92, 11.0183}, {0.93, 11.5102}, {0.94, 12.0831}, {0.95, 12.7679},
  {0.96, 13.6173}, {0.97, 14.7312}, {0.98, 16.3377}, {0.99, 19.181}}}
```

```
Text@Grid[Prepend[{{0.87`, 9.257299332722239`}, {0.88`, 9.54517970636458`},
    {0.89`, 9.859036271596631`}, {0.9`, 10.204110341986302`},
    {0.91`, 10.587353105732982`}, {0.9199999999999999`, 11.018251652751681`},
    {0.9299999999999999`, 11.51021881284805`}, {0.94`, 12.08307975792722`},
    {0.95`, 12.767884865996166`}, {0.96`, 13.617254460402796`},
    {0.97`, 14.731159451739403`}, {0.98`, 16.337714446735852`},
    {0.99`, 19.181023514893155`}}, {"ω", "Expectile2"}], Frame → True
 ]
```

ω	Expectile2
0.87	9.2573
0.88	9.54518
0.89	9.85904
0.9	10.2041
0.91	10.5874
0.92	11.0183
0.93	11.5102
0.94	12.0831
0.95	12.7679
0.96	13.6173
0.97	14.7312
0.98	16.3377
0.99	19.181

The plot of the expectiles for the reference distribution

```
ListPlot[tableExpectile2ALD,
 AxesLabel → {"ω", "Expectiles"}, AxesOrigin → {0.85, 8}, Joined → True]
```



Value at Risk (VaR, hodnota v riziku)

at confidence level $1-\alpha$ is the α-quantile of the loss distribution of L, shortly $\mathrm{VaR}_\alpha(L)$:

$$P(L \le \mathrm{VaR}_\alpha(L)\ ) = \alpha$$

so that

$$P(L > \mathrm{VaR}_\alpha(L)) = 1-\alpha \qquad (10)$$

Typically $\alpha = 0.95,\ \ 0.99$ &c., so that the probability in the above formula is small.

Parametric VaR (Parametrická hodnota v riziku)

Nonparametric VaR (Neparametrická hodnota v riziku)

Conditional VaR (CVaR, podmíněná hodnota v riziku)

Expected shortfall (očekávaná extrémní ztráta), Tailed VaR

$$\text{CVaR}_\alpha(L) = E(L \mid L \geq \text{VaR}_\alpha(L))$$

Nonparametric estimate of CVaR

Similar risk measures (příbuzné míry rizika)

Average VaR

Weighted Average VaR

Median VaR

Quantile VaR

***Spectral measure of risk