Exercises in Stochastic Processes I

Petr Čoupek, Michaela Prokešová

February 26, 2020

Contents

Foreword

This collection contains (occasionally solved) exercises on discrete- and continuous-time Markov chains. It was created to accompany the course NMSA334 "Stochastic Processes 1" that is taught at the Faculty of Mathematics and Physics of the Charles University in Prague. This course serves as the first introduction to stochastic processes for students of the General Mathematics undergraduate program.

The aim of this collection is to provide a resource for students that wish to acquaint themselves with Markov chains in a hands-on, practical manner that would complement the theoretical aspects of the course. As such, the content of this collection closely follows the content of the course; however, we have decided to present the results on Markov chains as tools that can be used for modeling real-world phenomena. This means, in practice, that even though we recall theorems and definitions for the reader's convenience, instead of their detailed and rigorous proofs, we offer (hopefully enlightening) comments and sort of a cookbook for their practical application.

The content of this collection is by no means original. The main resources in creating them are the textbook [15] (in Czech) and the monograph [17] (in English) where, apart from most of the theorems with rigorous proofs, many of the exercises collected here can be found. In fact, Markov chains are now a classical subject and there is a huge body of literature devoted to them. In particular, we refer to the excellent books [3, 4, 6, 7, 8, 11, 12, 13, 14, 16, 18, 19] which contain many exercises (or their various versions) that are presented here as well as a large number of other exercise and more advanced results on Markov chains. These books contain a more comprehensive treatment of the subject and they can be recommended to further study. We also present here some exercises from the textbook [2] and some exercises that were created by our colleagues who taught the course over the years, namely, P. Dostál, Š. Hudecová, P. Lachout, Z. Pawlas, and Z. Prášková.

This collection commences with some exercises on sums of count random variables which then leads to the Galton-Watson branching process. Exercises on discrete-time Markov chains can be found in the second chapter and exercises on continuous-time Markov chains are given in the third chapter.

We would be very interested in any comments or suggestions regarding this collection that you might have. In particular, we would greatly appreciate it if you could let us know about any mistakes that you find - Petr Čoupek can be reached at `coupek@karlin.mff.cuni.cz`.

Petr Čoupek and Michaela Prokešová

1 Sums of count random variables

1.1 Generating functions

In this preliminary section, we will deal with models that can be classified as sums of integer-valued random variables. We begin with recalling the notion of (probability) generating functions that are useful when dealing with real sequences and, in particular, with discrete probability distributions.

Definition 1.1. Let $\{a_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers. The function

$$A(s) = \sum_{n=0}^{\infty} a_n s^n$$

defined for all $s \in \mathbb{R}$ for which the sum converges absolutely is called a generating function (GF) of the sequence $\{a_n\}_{n\in\mathbb{N}_0}$.

Exercise 1. Consider a series of Bernoulli trials. Determine the probability that the number of successes in n trials is even.

Solution to Exercise 1. Even though there are many ways how to solve this problem, we will make use of the GFs. Notice that the number of successes in n trials can be even in two ways:

- In the n-th trial, the throw was a success and in $(n-1)$ trials, the number of successes was odd; or
- in the n-th trial, the throw was a failure and in $(n-1)$ trials, the number of successes was even.

Denote now by a_n the probability that in n-trials there will be an even number of successes. By the law of total probability, we obtain that

$$a_n = p(1 - a_{n-1}) + (1-p)a_{n-1}, \quad n \in \mathbb{N},$$

with $a_0 := 1$ (surely in zero trials there are no successes which is an even number) and $p \in (0,1)$ being the probability of a success in one trial. Now, multiplying the whole recurrence equation by s^n and summing from $n = 1$ to ∞ we obtain that

$$\sum_{n=1}^{\infty} a_n s^n = p \sum_{n=1}^{\infty} s^n + (1-2p) \sum_{n=1}^{\infty} a_{n-1} s^n$$

If we denote by $A(s)$ the GF of the sequence of probabilities (not a distribution!) $\{a_n\}_{n=0}^{\infty}$, we can rewrite this (assuming $|s| < 1$) as

$$A(s) - 1 = p \cdot \frac{s}{1-s} + (1-2p)sA(s), \quad |s| < 1,$$

from which we get, using partial fractions, that

$$A(s) = \frac{1-(1-p)s}{(1-s)(1-(1-2p)s)} = \frac{\frac{1}{2}}{1-s} + \frac{\frac{1}{2}}{1-(1-2p)s}, \quad |s| < 1. \tag{1.1}$$

Going back to the definition of $A(s)$ we can see that

$$\sum_{n=0}^{\infty} a_n s^n = \frac{1}{2}\sum_{n=0}^{\infty} s^n + \frac{1}{2}\sum_{n=0}^{\infty}(1-2p)^n s^n = \sum_{n=0}^{\infty} \frac{1}{2}\left[1 + (1-2p)^n\right] s^n.$$

The expansion of the second fraction in (1.1) does not pose additional conditions on s since we have that $|1-2p| < 1$ for arbitrary $p \in (0,1)$. Comparing now the coefficients, we obtain the final formula for a_n:

$$a_n = \frac{1}{2}\left(1 + (1-2p)^n\right), \quad n \in \mathbb{N}_0.$$

△

Exercise 2 (The Matching Problem)**.** At a party, the (possibly tipsy) guests play a game. Each person takes off one of their socks and puts it into a sack. Then all the socks are shuffled and each person chooses one sock at random. Assume there are $n \in \mathbb{N}$ guests at the party. What is the probability p_n that no guest gets their own sock?

Solution to Exercise 2. This is a famous and old problem first considered by Pierre R. de Montmort in 1708 and solved by him in 1713 [1].

Assume that $n \geq 2$. We shall calculate the number of derangements D_n (that is permutations of $\{1, 2, \ldots, n\}$ such that no element appears in its original position). Then, of course, the sought probability p_n satisfies $p_n = \frac{D_n}{n!}$. Let us number the guests by the numbers $1, 2, \ldots, n$ and their socks also by the numbers $1, 2, \ldots, n$. Assume that the guest number 1 chooses sock i (there are $n-1$ ways to make such a choice). The following two situations can occur:

1. Person i does not take sock no. 1. But this is the same situation as if we considered only $(n-1)$ guests with $(n-1)$ socks playing the game.
2. Persion i takes sock no. 1. But this is the same situation as if we considered only $(n-2)$ guests with $(n-2)$ socks playing the game.

Hence, the recurrence relation for D_n is

$$D_n = (n-1)\left(D_{n-1} + D_{n-2}\right), \quad n = 3, 4, \ldots$$

Dividing the last equation by $n!$ yields a formula for p_n:

$$p_n = \frac{D_n}{n!} = (n-1)\left(\frac{1}{n} \cdot \frac{D_{n-1}}{(n-1)!} + \frac{1}{n(n-1)} \cdot \frac{D_{n-2}}{(n-2)!}\right) = \frac{n-1}{n} p_{n-1} + \frac{1}{n} p_{n-2}.$$

Thus, we obtain

$$np_n = (n-1)p_{n-1} + p_{n-2}, \quad n = 3, 4, \ldots$$

If we multiply both sides of the recursion formula by s^n and sum for n from 3 to infinity, we obtain

$$\sum_{n=3}^{\infty} np_n s^n = \sum_{n=3}^{\infty} (n-1)p_{n-1}s^n + \sum_{n=3}^{\infty} p_{n-2}s^n.$$

Now, if we denote by $P(s) = \sum_{n=1}^{\infty} p_n s^n$ the GF of the sequence $\{p_n\}_{n\in\mathbb{N}}$ (this is not a probability distribution!) and if we further notice that $p_1 = 0$ and $p_2 = \frac{1}{2}$, we can rewrite this as

$$s(P'(s) - s) = s^2 P'(s) + s^2 P(s)$$

which yields the differential equation

$$P'(s) = \frac{s}{1-s} P(s) + \frac{s}{1-s}$$

with initial condition $P(0) = 0$ (since $P(0) = p_1 = 0$). By solving this differential equation, the following formula for $P(s)$ is obtained:

$$P(s) = \frac{e^{-s}}{1-s} - 1.$$

Now, we only need to expand $P(s)$ into a power series to compare coefficients. We have

$$P(s) = \frac{1}{1-s}\left(\sum_{n=0}^{\infty} \frac{(-1)^k}{k!} s^k - 1 + s\right) = \left(\sum_{k=0}^{\infty} s^k\right) \cdot \left(\sum_{n=2}^{\infty} \frac{(-1)^k}{k!} s^k\right) = \sum_{k=2}^{\infty} \left(\sum_{j=2}^{k} \frac{(-1)^j}{j!}\right) s^k.$$

This gives

$$p_1 = 0,$$
$$p_n = \sum_{j=2}^{n} \frac{(-1)^j}{j!}, \quad n = 2, 3, \ldots$$

△

Definition 1.2. Let X be a non-negative integer-valued (so-called count) random variable. Its probability distribution is given by the probabilities $\{p_n\}_{n\in\mathbb{N}_0}$ with $p_n := \mathbb{P}(X = n)$. The GF of the sequence $\{p_n\}_{n\in\mathbb{N}_0}$ is called a probability generating function (PGF).

Note that the PGF P_X of a count random variable X that satisfies $\mathbb{P}(X < \infty) = 1$ can be also written as

$$P_X(s) = \mathbb{E}s^X.$$

The following theorem summarizes basic facts about probability generating functions.

Theorem 1.1. Let X be a count random variable, P_X its PGF with the radius of convergence R_X. Then the following holds:

1. $R_X \geq 1$. For all $|s| < R_X$, the derivatives $P_X^{(k)}(s)$ exist and furthermore, the limit
$$\lim_{s\to 1-} P_X^{(k)}(s) = P_X^{(k)}(1-)$$
also exists.
2. $\mathbb{P}(X = k) = \frac{1}{k!}P_X^{(k)}(0)$; and, in particular, $\mathbb{P}(X = 0) = P_X(0)$.
3. $\mathbb{E}X(X-1)\cdots(X-k+1) = P_X^{(k)}(1-)$; and, in particular, $\mathbb{E}X = P_X'(1-)$.

Note that it follows from Theorem 1.1 that if X is a random variable such that $\mathbb{E}X < \infty$, then

$$\operatorname{var} X = P_X''(1-) + P_X'(1-) - [P_X'(1-)]^2 .$$

Exercise 3. Find the GF of the count random variable X and determine its radius of convergence. Using the GF, find the mean and variance of X.

1. X has the Bernoulli distribution with parameter $p \in (0, 1)$.
2. X has the binomial distribution with parameters $m \in \mathbb{N}$ and $p \in (0, 1)$.
3. X has the negative binomial distribution with parameters $m \in \mathbb{N}$ and $p \in (0, 1)$.
4. X has the Poisson distribution with parameter $\lambda > 0$.
5. X has the geometric distribution (on $\mathbb{N}_0$) with parameter $p \in (0, 1)$.

Exercise 4. Let P_X denote the PGF of a count random variable X. Find the probability distribution of X and compute its mean if

1. $P_X(s) = \frac{1}{4-s}$, $|s| < 4$.
2. $P_X(s) = \frac{2}{s^2-5s+6}$, $|s| < 2$.
3. $P_X(s) = \frac{24-9s}{5s^2-30s+40}$, $|s| < 2$.
4. $P_X(s) = \frac{1}{2\min(p,q)}\left(1 - \sqrt{1-4pqs^2}\right)$, $|s| < 1$ with $p \in (0, 1)$ and $q = 1 - p$.

Solution to Exercise 4, part 4. Appealing to Theorem 1.1, part (3), the mean can be computed by taking the derivative $P_X'(s)$ and considering the limit $\lim_{s\to 1-} P_X'(s)$. We obtain

$$\mathbb{E}X = \begin{cases} \frac{2pq}{\min(p,q)|p-q|}, & p \neq q, \\ \infty, & p = q. \end{cases}$$

Now, in order to obtain the probabilities $\mathbb{P}(X = k)$, recall the Taylor expansion of the function $f(x) = \sqrt{1+x}$ at $x = 0$. We have that

$$\sqrt{1+x} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k}{2^{2k}(1-2k)} x^k.$$

Now, take $x := -4pqs^2$ (notice that such x will never be equal to -1 where the Taylor expansion would not exist since the square root is not smooth at zero). Hence, we obtain

$$P_X(s) = \frac{1}{2\min(p,q)} \left(1 + \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(pq)^k}{2k-1} s^{2k} \right).$$

Since we have that

$$P_X(s) = \sum_{k=0}^{\infty} \mathbb{P}(X = k) s^k,$$

comparing the coefficients yields

$$\begin{aligned} \mathbb{P}(X = 0) &= 0, & \\ \mathbb{P}(X = 2k-1) &= 0, & k = 1, 2, \dots \\ \mathbb{P}(X = 2k) &= \tfrac{1}{2\min(p,q)} \tbinom{2k}{k} \tfrac{(pq)^k}{2k-1}, & k = 1, 2, \dots \end{aligned}$$

△

1.2 Sums of count random variables

The following theorem allows us to compute the PGF of a sum of random variables that has a deterministic number of summands.

Theorem 1.2. Let X and Y be two independent count random variables with PGFs P_X (and radius of convergence R_X) and P_Y (with radius of convergence R_Y), respectively. Denote further $Z := X + Y$ and its PGF by P_Z. Then

$$P_Z(s) = P_X(s) P_Y(s), \quad |s| < \min\{R_X, R_Y\}.$$

Exercise 5. Generalize the statement of Theorem 1.2 to n independent count random variables.

Exercise 6. Find the distribution of the sum of n independent random variables $X_1, X_2, \dots, X_n$ where

1. X_i has the Poisson distribution with parameter $\lambda_i > 0$, $i = 1, \dots, n$.
2. X_i has the binomial distribution with parameters $p \in (0, 1)$ and $m_i \in \mathbb{N}$, $i = 1, 2, \dots, n$.

The following theorem allows to compute the PGF of a sum of random variables whose number of summands is random.

Theorem 1.3. Let N be a count random variable and let $X_1, X_2, \dots$ be independent and identically distributed count random variables. Assume that N is also independent of X_i for every $i \in \mathbb{N}$. Set

$$\begin{aligned} S_0 &:= 0, \\ S_N &:= X_1 + \dots + X_N. \end{aligned}$$

If the common probability generating function of X_i's is P_X and the probability generating function of N is P_N, then the probability generating function of S_N, P_{S_N}, satisfies

$$P_{S_N}(s) = P_N(P_X(s)).$$

Exercise 7. Prove Theorem 1.3. Show that, as a corollary, it holds that

1. $\mathbb{E}S_N = \mathbb{E}N \cdot \mathbb{E}X$,
2. $\operatorname{var} S_N = \mathbb{E}N \cdot \operatorname{var} X + \operatorname{var} N \cdot (\mathbb{E}X)^2$,

where X is a generic element of the sequence $\{X_i\}_{i \in \mathbb{N}}$ provided that both N and X are square integrable random variables.

Solution to Exercise 7. By using the definition of S_N and conditioning on the number of summands we obtain

$$\begin{aligned} P_{S_N}(s) &= \sum_{n=0}^{\infty} \mathbb{P}(S_N = n) s^n \\ &= \sum_{n=0}^{\infty} s^n \sum_{k=0}^{\infty} \mathbb{P}(S_N = n | N = k) \mathbb{P}(N = k) \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \mathbb{P}(S_N = n | N = k) s^n \right) \mathbb{P}(N = k) \\ &= \sum_{k=0}^{\infty} (P_X(s))^k \mathbb{P}(N = k) \\ &= P_N\left(P_X(s)\right). \end{aligned}$$

The interchange of the sums is possible due to Tonelli's theorem and since S_k is a deterministic sum of independent count random variables, we have that

$$P_{S_k}(s) = \sum_{n=0}^{\infty} \mathbb{P}(S_N = n | N = k) s^n = (P_X(s))^k$$

by Theorem 1.2. The corollary follows by differentiating the function P_{S_N} and taking the limit as $s \to 1-$ of the derivative. We obtain

$$\mathbb{E}S_N = \lim_{s \to 1-} P'_{S_N}(s) = \lim_{s \to 1-} P'_N\left(P_X(s)\right) P'_X(s) = P'_N(1-) P'_X(1-) = \mathbb{E}N \cdot \mathbb{E}X_1.$$

Similarly for the variance. △

Exercise 8.

1. Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of independent, identically distributed random variables with Poisson distribution with parameter $\alpha > 0$. Let N be a random variable with Poison distribution with parameter $\lambda > 0$ that is independent of the sequence $\{X_i\}_{i \in \mathbb{N}}$. Consider the sum $S_0 = 0, S_N = \sum_{i=1}^{N} X_i$. Find the probability distribution of S_N, its mean and variance.
2. The number of eggs a sea turtle lays on the beach, N, has Poisson distribution with parameter $\lambda > 0$. Each egg hatches with probability $p \in (0, 1)$ independently of the other eggs and each hatched little sea turtle reaches the sea with probability $q \in (0, 1)$ independently of the other turtles. Find the probability distribution of the number of sea turtles that hatch. Find the probability distribution of the number of sea turtles that hatch and reach the sea.

1.3 Galton-Watson branching process

Our aim is to analyse the evolution of a population of identical organisms, each of which lives for the same length of time (one generation) and as it dies, it gives birth to a random number of new organisms, thus producing a new generation. As an example, one can think of the males that carry the family name. In fact, this was the original motivation of Francis Galton and Henry W. Watson who introduced the process for their investigation of the probability of extinction of aristocratic surnames in their article [5].

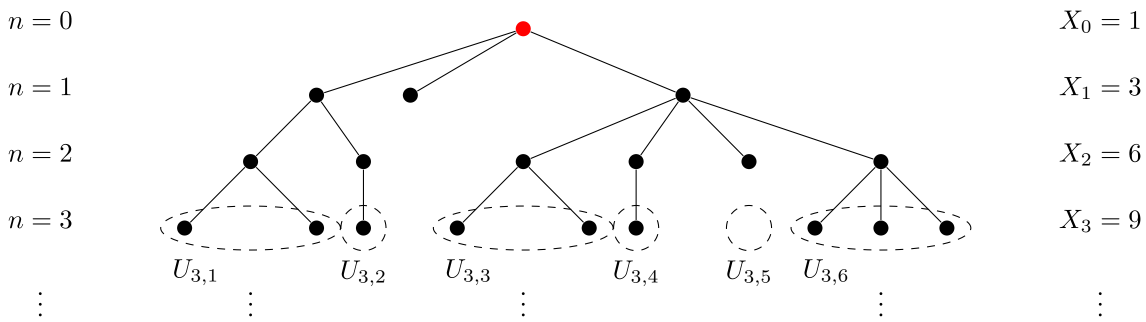
The evolution of the population is described by the random sequence $X = (X_n, n \in \mathbb{N}_0)$ where

$$\begin{aligned} X_0 &:= 1, \\ X_n &:= \sum_{i=1}^{X_{n-1}} U_{n,i} \end{aligned}$$

if $X_{n-1} \neq 0$ and $X_n := 0$ if $X_{n-1} = 0$, $n = 1, 2, \ldots$ Here, $(U_{n,i}, n, i \in \mathbb{N})$, are independent count random variables which are identically distributed with probabilities

$$\mathbb{P}(U = k) = p_k, \quad k = 0, 1, 2, \ldots$$

The random variables $U_{n,i}$ represent the number of offspring of the i-th member of the $(n-1)$-th generation. A neat way how to visualise one realisation of the Galton-Watson branching process is via random trees as in the following picture.



Since $U_{n,i}$ are independent (of each other with respect to i and n and also of X_{n-1}), by using Theorem 1.3 we obtain that the probability generating function P_{X_n} can be expressed as

$$P_{X_n}(s) = P_{X_{n-1}}\left(P_U(s)\right), \quad n = 1, 2, \dots \tag{1.2}$$

Using the same argument and Theorem 1.1 we also have (cf. Exercise 7) that

$$\mathbb{E}X_n = P'_{X_n}(1-) = P'_{X_{n-1}}\left(P_U(1-)\right) \cdot P'_U(1-) = P'_{X_{n-1}}(1-) \cdot P'_U(1-) = \mathbb{E}X_{n-1} \cdot \mathbb{E}U$$

and, denoting $\mathbb{E}U =: \mu$ and $\operatorname{var} U =: \sigma^2$, we obtain $\mathbb{E}X_n = \mu^n$. In a similar manner, the formula for $\operatorname{var} X_n$ can be obtained. Indeed, from Exercise 7, part (2), we have that

$$\begin{aligned}\operatorname{var} X_n &= (\mathbb{E}X_1)^2 \operatorname{var} X_{n-1} + \mathbb{E}X_{n-1} \operatorname{var} X_1 \\ &= (\mathbb{E}U)^2 \operatorname{var} X_{n-1} + \mu^{n-1} \operatorname{var} U \\ &= \mu^2 \operatorname{var} X_{n-1} + \mu^{n-1}\sigma^2\end{aligned}$$

Hence, we obtain a linear difference equation which can be solved by recursion. We obtain

$$\operatorname{var} X_n = \sigma^2 \mu^{n-1}\left(1 + \mu + \mu^2 + \ldots + \mu^{n-1}\right)$$

with the last term being a partial geometric series. We have arrived at the following result.

Theorem 1.4. Let $X = (X_n, n \in \mathbb{N}_0)$ be the Galton-Watson process with the number of children in a family U having the mean μ and variance σ^2. Then

$$\begin{aligned}\mathbb{E}X_n &= \mu^n \\ \operatorname{var} X_n &= \begin{cases} n\sigma^2, & \mu = 1, \\ \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1}, & \mu \neq 1. \end{cases}\end{aligned}$$

Denote $e_n := \mathbb{P}(X_n = 0)$, i.e. the probability that the Galton-Watson branching process X is extinct by the n-th generation. Clearly $\{e_n\}_{n \in \mathbb{N}_0}$ is a bounded and non-decreasing sequence (because if $X_n = 0$ for some $n \in \mathbb{N}_0$, then also $X_{n+1} = 0$) and therefore there is the limit $e := \lim_{n \to \infty} e_n$. The number e is called the probability of ultimate extinction of the process X.

Theorem 1.5. Suppose that $p_0 > 0$. Then $e = 1$ if and only if $\mu \leq 1$.

Theorem 1.5 says that if $p_0 > 0$, then the process X will almost surely extinct if the mean number of offspring is smaller than or equal to one. Let us briefly comment on the assumption $p_0 > 0$. If $p_0 = 0$, then it would mean that there will always be at least one member of the next generation in one family U. This would mean that $\mathbb{P}(X_n = 0) = 0$ for all $n \in \mathbb{N}_0$ and consequently the process would never extinct (i.e. $e = 0$). The following theorem shows how the probability of ultimate extinction can be found.

> **Theorem 1.6.** If $\mu > 1$, then the probability of ultimate extinction e is the unique non-negative solution to the fixed point equation $P_U(x) = x$ that is smaller than 1.

Theorems 1.5 and 1.6 give us a cookbook how to compute the probability of ultimate extinction of the Galton-Watson branching process. First compute $\mu = \mathbb{E}U$ and if $\mathbb{E}U \leq 1$, we immediately know that $e = 1$. If $\mathbb{E}U > 1$, we have to solve the fixed point equation $x = P_U(x)$.

Exercise 9. Let $X = (X_n, n \in \mathbb{N}_0)$ be the Galton-Watson branching process with U having the distribution $\{p_n\}_{n\in\mathbb{N}_0}$. Find the probability of ultimate extinction e and the expected number of members of the n-the generation.

1. $p_0 = \frac{1}{5}$, $p_1 = \frac{1}{5}$, $p_2 = \frac{3}{5}$ and $p_k = 0$ for $k \geq 3$;
2. $p_0 = \frac{1}{12}$, $p_1 = \frac{5}{12}$, $p_2 = \frac{1}{2}$ and $p_k = 0$ for $k \geq 3$;
3. $p_0 = \frac{1}{10}$, $p_1 = \frac{2}{5}$, $p_2 = \frac{1}{2}$ and $p_k = 0$ for $k \geq 3$;
4. $p_0 = \frac{1}{2}$, $p_1 = 0$, $p_2 = 0$, $p_3 = \frac{1}{2}$ and $p_k = 0$ for $k \geq 4$;
5. $p_k = \left(\frac{1}{2}\right)^{k+1}$ for $k \geq 0$;
6. $p_k = pq^k$ for $k \geq 0$ with $p \in (0,1)$ and $q = 1 - p$.

Solution to Exercise 9, part 1. First notice, that $\mathbb{E}U = \frac{1}{5}\cdot 0 + \frac{1}{5}\cdot 1 + \frac{3}{5}\cdot 2 = \frac{7}{5}$. Hence $\mathbb{E}X_n = \left(\frac{7}{5}\right)^n$. We further have that $\mathbb{E}U = \mu > 1$ and therefore it holds that $e < 1$ by Theorem 1.5. We thus have to solve the fixed point equation $P_U(s) = s$ by Theorem 1.6. In this case,

$$P_U(s) = \frac{1}{5}s^0 + \frac{1}{5}s^1 + \frac{3}{5}s^2, \quad s \in \mathbb{R}.$$

This means that we need to solve a quadratic equation, namely,

$$\frac{1}{5} + \frac{1}{5}s + \frac{3}{5}s^2 = s.$$

We will always have the root $s_1 = 1$ but the second one is $s_2 = \frac{1}{3}$. Since we need to take the smallest non-negative root of the fixed point equation, we obtain $e = \frac{1}{3}$. △

Exercise 10. Suppose that the family sizes U have geometric distribution on $\{0, 1, 2, \ldots\}$ with parameter $p \in (0,1)$. Find the distribution of X_n, i.e. the probabilities $\mathbb{P}(X_n = j)$ for $j = 0, 1, 2, \ldots$.

Solution to Exercise 10. The starting point is to determine the probability generating function P_{X_n}. Using Theorem 1.3 iteratively, we obtain

$$P_{X_n}(s) = \underbrace{P_U(P_U(\ldots(s)\ldots))}_{n \text{ times}}.$$

We know that the PGF of the geometric distribution on $\mathbb{N}_0$ is given by

$$P_U(s) = \frac{p}{1-qs} = \frac{1}{1+\mu-\mu s}, \quad |s| < \frac{1}{q},$$

where μ is the mean $\mu = \mathbb{E}U = \frac{p}{q}$. Writing down the first few iterates of P_U, we can guess the pattern

$$\begin{aligned}
P_{X_1}(s) &= \frac{1}{1+\mu-\mu s} \\
P_{X_2}(s) &= \frac{1}{1+\mu-\mu\frac{1}{1+\mu-\mu s}} = \frac{(1+\mu)-\mu s}{(1+\mu+\mu^2)-s(\mu+\mu^2)} \\
&\vdots \\
P_{X_n}(s) &= \frac{\sum_{k=0}^{n-1}\mu^k - \left(\sum_{k=1}^{n-1}\mu^k\right)s}{\sum_{k=0}^{n}\mu^k - \left(\sum_{k=1}^{n}\mu^k\right)s}
\end{aligned}$$

which can be proved by induction on n. Furthermore, for $n \geq 3$, we can simplify the formula as

$$P_{X_n}(s) = \begin{cases} \frac{(\mu^{n-1}-1)-(\mu^{n-2}-1)\mu s}{(\mu^{n}-1)-(\mu^{n-1}-1)\mu s} & \mu \neq 1 \\ \frac{n-(n-1)s}{n+1-ns} & \mu = 1 \end{cases}$$

The case $\mu = 1$: Now we only need to expand the formula above into a power series and compare coefficients. This is easily done by dividing the nominator by the denominator to obtain

$$\begin{aligned} P_{X_n}(s) &= \frac{n-(n-1)s}{(n+1)-ns} \\ &= 1 - \frac{1}{n} + \frac{\frac{1}{n}}{(n+1)-ns} \\ &= 1 - \frac{1}{n} + \frac{1}{n(n+1)} \cdot \frac{1}{1-\frac{n}{n+1}s} \\ &= 1 - \frac{1}{n} + \frac{1}{n(n+1)} \cdot \sum_{k=0}^{\infty} \left(\frac{n}{n-1}\right)^k s^k \end{aligned}$$

Which gives

$$\begin{aligned} \mathbb{P}(X_n = 0) &= 1 - \frac{1}{n+1} \\ \mathbb{P}(X_n = k) &= \frac{1}{n(n+1)} \left(\frac{n}{n-1}\right)^k, \quad k = 1, 2, \ldots \end{aligned}$$

The case $\mu \neq 1$: Similarly as in the previous case, we have to write $P_{X_n}(s)$ as a power series and compare coefficients. We have

$$\begin{aligned} P_{X_n}(s) &= \frac{(\mu^{n-1}-1)-(\mu^{n-2}-1)\mu s}{(\mu^{n}-1)-(\mu^{n-1}-1)\mu s} \\ &= \frac{\mu^{n-2}-1}{\mu^{n-1}-1} + \frac{(\mu^{n-1}-1)-(\mu^{n}-1)\frac{\mu^{n-2}-1}{\mu^{n-1}-1}}{(\mu^{n}-1)-(\mu^{n-1}-1)\mu s} \\ &= \frac{\mu^{n-2}-1}{\mu^{n-1}-1} + \left(\frac{\mu^{n-1}-1}{\mu^{n}-1} - \frac{\mu^{n-2}-1}{\mu^{n-1}-1}\right) \cdot \frac{1}{1-\frac{\mu^{n-1}-1}{\mu^{n}-1}\mu s} \\ &= \frac{\mu^{n-2}-1}{\mu^{n-1}-1} + \left(\frac{\mu^{n-1}-1}{\mu^{n}-1} - \frac{\mu^{n-2}-1}{\mu^{n-1}-1}\right) \sum_{k=0}^{\infty} \left(\frac{\mu^{n-1}-1}{\mu^{n}-1}\mu\right)^k s^k \end{aligned}$$

which gives

$$\begin{aligned} \mathbb{P}(X_n = 0) &= \frac{\mu^{n-1}-1}{\mu^{n}-1} \\ \mathbb{P}(X_n = k) &= \frac{\mu^{n-2}-1}{\mu^{n-1}-1} + \left(\frac{\mu^{n-1}-1}{\mu^{n}-1} - \frac{\mu^{n-2}-1}{\mu^{n-1}-1}\right) \left(\frac{\mu^{n-1}-1}{\mu^{n}-1}\mu\right)^k \end{aligned}$$

△

1.4 Answers to exercises

Answer to Exercise 3. The answers are as follows:

1. $P_X(s) = 1 - p + ps$ for $s \in \mathbb{R}$, $\mathbb{E}X = p$, $\operatorname{var} X = p(1-p)$.
2. $P_X(s) = (1 - p + ps)^m$ for $s \in \mathbb{R}$, $\mathbb{E}X = mp$, $\operatorname{var} X = mp(1-p)$.
3. $P_X(s) = \left(\frac{ps}{1-(1-p)s}\right)^m$, for $|s| < \frac{1}{1-p}$, $\mathbb{E}X = \frac{pm}{1-p}$, $\operatorname{var} X = \frac{pm}{(1-p)^2}$.
4. $P_X(s) = e^{\lambda(s-1)}$ for $s \in \mathbb{R}$, $\mathbb{E}X = \lambda$, $\operatorname{var} X = \lambda$.
5. $P_X(s) = \frac{p}{1-(1-p)s}$ for $|s| < \frac{p}{1-p}$, $\mathbb{E}X = \frac{1-p}{p}$ and $\operatorname{var} X = \frac{1-p}{p^2}$.

Answer to Exercise 4. The answers are as follows:

1. $\mathbb{E}X = \frac{1}{3}$ and $\mathbb{P}(X = k) = \frac{3}{4} \cdot \left(\frac{1}{4}\right)^k$, $k \in \mathbb{N}_0$.
2. $\mathbb{E}X = \frac{3}{2}$ and $\mathbb{P}(X = k) = \frac{1}{2^k} - \frac{2}{3} \cdot \frac{1}{3^k}$, $k \in \mathbb{N}_0$.
3. $\mathbb{E}X = \frac{11}{15}$ and $\mathbb{P}(X = k) = \frac{3}{10} \cdot \frac{1}{2^k}\left(1 + \frac{1}{2^k}\right)$, $k \in \mathbb{N}_0$.
4. The mean is given by
$$\mathbb{E}X = \begin{cases} \frac{2pq}{\min(p,q)|p-q|}, & p \neq q, \\ \infty, & p = q. \end{cases}$$
and the distribution is given by
$$\begin{aligned} \mathbb{P}(X = 0) &= 0 \\ \mathbb{P}(X = 2k-1) &= 0, & k = 1, 2, \dots \\ \mathbb{P}(X = 2k) &= \frac{1}{2\max(p,q)}\binom{2k}{k}\frac{(pq)^k}{2k-1}, & k = 1, 2, \dots \end{aligned}$$

Answer to Exercise 5. Let $X_1, \dots, X_n$ be independent count random variables with probability generating functions $P_{X_1}, \dots, P_{X_n}$ and the corresponding radii of convergence $R_1, \dots, R_n$. Let us denote $Z := \sum_{k=1}^n X_i$ and P_Z its probability generating function. Then

$$P_Z(s) = \prod_{i=1}^n P_{X_i}(s), \quad |s| < \min_{i=1,2,\dots,n} R_i.$$

Answer to Exercise 6. Denote $Z = \sum_{i=1}^n X_i$.

1. Z has the Poisson distribution with parameter $\sum_{i=1}^n \lambda_i$.
2. Z has the binomial distribution with parameter p and $\sum_{i=1}^n m_i$.

Answer to Exercise 8. The answers are as follows:

1. $P_{S_N}(s) = e^{\lambda(e^{\alpha(s-1)}-1)}$ for $s \in \mathbb{R}$, $\mathbb{E}S_N = \lambda\alpha$, $\operatorname{var} S_N = \lambda\alpha(1+\alpha)$.
2. The number of sea turtles that hatch has the Poisson distribution with parameter λp. The number of sea turtles that hatch and reach the sea has the Poisson distribution with parameter λpq.

Answer to Exercise 9. The probabilities of ultimate extinction and the mean of the size of the population in the n^{th} generation are as follows:

1. $e = \frac{1}{3}$, $\mathbb{E}X_n = \left(\frac{7}{5}\right)^n$.
2. $e = \frac{1}{6}$, $\mathbb{E}X_n = \left(\frac{17}{12}\right)^n$.
3. $e = \frac{1}{5}$, $\mathbb{E}X_n = \left(\frac{7}{5}\right)^n$.
4. $e = \frac{\sqrt{5}-1}{2}$, $\mathbb{E}X_n = \left(\frac{3}{2}\right)^n$.
5. $e = 1$, $\mathbb{E}X_n = 1$.
6. If $p < \frac{1}{2}$ then $e = \frac{1}{\mu} = \frac{p}{1-p}$. If $p \geq \frac{1}{2}$, then $e = 1$. $\mathbb{E}X_n = \left(\frac{1-p}{p}\right)^n$.

2 Discrete-time Markov chains

2.1 Markov property and time homogeneity

Let us recall the definition of a discrete-time Markov chain.

Definition 2.1. A $\mathbb{Z}$-valued random sequence $X = (X_n, n \in \mathbb{N})$ is called a discrete-time Markov chain with a state space S if

1. $S = \{i \in \mathbb{Z} : \text{ there exists } n \in \mathbb{N}_0 \text{ such that } \mathbb{P}(X_n = i) > 0\}$,
2. and it holds that

$$\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j | X_n = i)$$

for all times $n \in \mathbb{N}_0$ and all states $i, j, i_{n-1}, \ldots, i_0 \in S$ such that

$$\mathbb{P}(X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) > 0.$$

Roughly speaking, the first condition says that the state space contains all the possible values of the stochastic process X but no other values. The second condition, called the Markov property, says that given the current state, the past and the future of the process X are independent.

For a discrete-time Markov chain X, the probability

$$p_{ij}(n, n+1) := \mathbb{P}(X_{n+1} = j | X_n = i)$$

is called the transition probability from state i at time n to the state j at time $n+1$. With these, we can create a stochastic matrix[1] $\boldsymbol{P(n, n+1)} = (p_{ij}(n, n+1))_{i,j \in S}$ which is called the transition probability matrix of X from time n to time $n+1$. Similarly, we can define (for $k \in \mathbb{N}_0$)

$$p_{ij}(n, n+k) := \mathbb{P}(X_{n+k} = j | X_n = i)$$

and a stochastic matrix $\boldsymbol{P(n, n+k)} = (p_{ij}(n, n+k))_{i,j \in S}$. Although it is possible to define matrix multiplication for stochastic matrices of the type $\mathbb{Z} \times \mathbb{Z}$ rigorously, whenever we will multiply infinite matrices, we will assume that $S \subset \mathbb{N}_0$ (thus excluding the case $S = \mathbb{Z}$).

Definition 2.2. If there is a stochastic matrix $\boldsymbol{P}$ such that the equality $\boldsymbol{P(n, n+1)} = \boldsymbol{P}$ holds for all $n \in \mathbb{N}_0$, then we say that X is a (time) homogeneous discrete-time Markov chain.

For a homogeneous discrete-time Markov chain X it is possible to define a transition matrix after $k \in \mathbb{N}_0$ steps by $\boldsymbol{P(k)} := \boldsymbol{P(n, n+k)}$. It holds that

$$\boldsymbol{P(k)} = \boldsymbol{P}^k$$

and, as a corollary, we obtain the famous Chapman-Kolmogorov equation

$$\boldsymbol{P(m+n)} = \boldsymbol{P(m)} \cdot \boldsymbol{P(n)}, \quad m, n \in \mathbb{N}_0.$$

If we are interested in the probability distribution of the random variable X_n, denoted by $\boldsymbol{p(n)}$, we can compute it by using the formula

$$\boldsymbol{p(n)}^T = \boldsymbol{p(0)}^T \boldsymbol{P}^n, \quad n \in \mathbb{N}_0.$$

where $\boldsymbol{p(0)}$ denotes the initial distribution of X, i.e. the vector that contains the probabilities $\mathbb{P}(X_0 = k)$ for $k \in S$.

Exercise 11. Let X be the Galton-Watson branching process with $X_0 = 1$ and $p_0 > 0$.

1. Show that X is a homogeneous discrete-time Markov chain with the state space $S = \mathbb{N}_0$.
2. Find its transition matrix $\boldsymbol{P}$.

[1] A stochastic matrix is a matrix $\boldsymbol{P} = (p_{ij})_{i,j \in S}$ such that each row is a probability distribution on S.

3. Compute the distribution of X_2 in the case when the distribution of U is $p_0 = \frac{1}{5}$, $p_1 = \frac{1}{5}$, $p_2 = \frac{3}{5}$, $p_k = 0$ for all $k = 3, 4, \ldots$

Solution to Exercise 11. Let X be the Galton-Watson process for which $X_0 = 1$ and $p_0 > 0$.

1. We will proceed in three steps: first we find the effective states S, then we show the Markov property (these two are the key requirement for X to be a Markov chain) and then we will show homogeneity.

   State space S: Since $X_0 = 1$, we immediately have that $1 \in S$. Since $p_0 > 0$, then it can happen (with probability p_0) that $X_1 = 0$. Hence also $0 \in S$ and we know that $\{0, 1\} \subset S$. The case when $\{0, 1\} = S$ is only possible if U takes either only the value 0 or takes values in the set $\{0, 1\}$. If there is $k \in \mathbb{N} \setminus \{1\}$ such that $p_k > 0$, then we have that $S = \mathbb{N}_0$.

   Markov property: Now that we have the set of effective states S, take some $n \in \mathbb{N}_0$ and some states $i, j, i_{n-1}, \ldots, i_1 \in S$ such that $\mathbb{P}(X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = 1) > 0$ and consider

$$\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = 1).$$

   If $i = 0$, this probability is 1 if $j = 0$ and 0 if $j \neq 0$. This is, however, the same as $\mathbb{P}(X_{n+1} = j | X_n = i)$. If $i \neq 0$, we have that

$$\begin{aligned}
\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = 1) &= \\
&= \mathbb{P}\left(\sum_{k=1}^{X_n} U_{n,k} = j | X_n = i, \ldots, X_0 = 1\right) \\
&= \mathbb{P}\left(\sum_{k=1}^{X_n} U_{n,k} | X_n = i\right)
\end{aligned}$$

   since the sum depends only on X_n and $U_{n,i}$ are independent of $X_{n-1}, \ldots, X_0$. Hence, the Markov property (2) of Definition 2.1 also holds and we can infer, that X is a discrete-time Markov chain.

   Homogeneity: We have that

$$p_{ij}(n, n+1) = \mathbb{P}\left(\sum_{k=1}^{X_n} U_{n,k} | X_n = i\right) = \mathbb{P}\left(Z := \sum_{k=1}^{i} U_{n,k} = j\right)$$

   We need to show, that this number does not depend on n. Notice, that Z represents a sum of a deterministic number of independent, identically distributed count random variables $U_k^{(n)}$. Hence, we can use Theorem 1.2. The PGF of the sum is $P_Z(s) = P_U(s)^i$ and the sought probability is

$$p_{ij}(n, n+1) = \mathbb{P}\left(\sum_{k=1}^{X_n} U_{n,k} | X_n = i\right) = \mathbb{P}\left(S := \sum_{k=1}^{i} U_{n,k} = j\right) = [P_U(s)^i]_j$$

   where $[R(s)]_k$ denotes the coefficient at s^k. This, however, does not depend on n.

2. The transition matrix is $\boldsymbol{P} := (p_{ij})_{i,j \in S}$ with $p_{ij} := [P_U(s)^i]_j$.

3. Consider now the particular case when U has the distribution $p_0 = \frac{1}{5}$, $p_1 = \frac{1}{5}$, $p_2 = \frac{3}{5}$ and $p_k = 0$ for all $k = 3, 4, \ldots$ Then we have

$$P_U(s) = \frac{1}{5} + \frac{1}{5}s + \frac{3}{5}s^2$$

   and

$$p_{ij} = [P_U(s)^i]_j = \left[\left(\frac{1}{5} + \frac{1}{5}s + \frac{3}{5}s^2\right)^i\right]_j.$$

In particular,

$$\begin{aligned}
P_U(s)^0 &= 1 \\
P_U(s)^1 &= \frac{1}{5} + \frac{1}{5}s + \frac{3}{5}s^2 \\
P_U(s)^2 &= \frac{1}{25} + \frac{2}{25}s + \frac{7}{25}s^2 + \frac{6}{25}s^3 + \frac{9}{25}s^4 \\
P_U(s)^3 &= \frac{1}{125} + \frac{3}{125}s + \frac{12}{125}s^2 + \frac{19}{125}s^3 + \frac{36}{125}s^4 + \frac{27}{125}s^5 + \frac{27}{125}s^6 \\
&\vdots
\end{aligned}$$

and

$$\boldsymbol{P} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{25} & \frac{2}{25} & \frac{7}{25} & \frac{6}{25} & \frac{9}{25} & 0 & 0 & 0 & \cdots \\
\frac{1}{125} & \frac{3}{125} & \frac{12}{125} & \frac{19}{125} & \frac{36}{125} & \frac{27}{125} & \frac{27}{125} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Since we know that the initial distribution is given by $\boldsymbol{p(0)}^T = (0, 1, 0, \ldots)$ (since $X_0 = 1$), we have that $\boldsymbol{p(2)}^T = \boldsymbol{p(0)}^T \boldsymbol{P}^2$ which will give

$$\boldsymbol{p(2)}^T = \begin{pmatrix} \frac{33}{125}, & \frac{11}{125}, & \frac{36}{125}, & \frac{18}{125}, & \frac{27}{125}, & 0, & \ldots \end{pmatrix}$$

△

Exercise 12 (Symmetric random walk). Let $(Y_n, n \in \mathbb{N})$ be a sequence of independent identically distributed random variables such that $\mathbb{P}(Y = -1) = \mathbb{P}(Y = +1) = \frac{1}{2}$ where Y is a generic element of the sequence $(Y_n, n \in \mathbb{N})$. Define $X_0 := 0$ and $X_n := \sum_{i=1}^{n} Y_i$ for $n \in \mathbb{N}$. Decide whether the process $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete-time Markov chain or not.

Hint: Notice that we can write $X_n = \sum_{i=1}^{n-1} Y_i + Y_n = X_{n-1} + Y_n$.

Exercise 13 (Running maximum). Let $(Y_n, n \in \mathbb{N})$ be a sequence of independent identically distributed integer-valued random variables and define $X_n := \max\{Y_1, \ldots, Y_n\}$ for $n \in \mathbb{N}$. Decide whether the process $X = (X_n, n \in \mathbb{N})$ is a homogeneous discrete-time Markov chain or not.

Hint: Can you find a formula for X_n which uses only X_{n-1} and Y_n?

Exercise 14 (Recursive character of Markov chains). Let $Y = (Y_n, n \in \mathbb{N})$ be a sequence of independent identically distributed integer-valued random variables. Let X_0 be an S-valued random variable, $S \subset \mathbb{Z}$, which is independent of the sequence Y and consider a measurable function $f : S \times \mathbb{Z} \to S$. Define

$$X_{n+1} = f(X_n, Y_{n+1}), \quad n \in \mathbb{N}_0.$$

Decide whether the process $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete-time Markov chain or not.

If X is a homogeneous discrete-time Markov chain, then it holds that

$$\begin{aligned}
&\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \\
&\qquad\qquad = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = j_{n-1}, \ldots, X_0 = j_0)
\end{aligned}$$

for all $n \in \mathbb{N}_0$ and all possible trajectories $(i, i_{n-1}, \ldots, i_0)$ and $(i, j_{n-1}, \ldots, j_0)$ of $(X_n, \ldots, X_0)$. Thus, to show that a stochastic process with discrete time is not a Markov chain, it suffices to find $n \in \mathbb{N}$ and different possible trajectories of length n such that the equality between the above conditional probabilities does not hold. Notice that i and j are the same for both considered trajectories.

Exercise 15. Let $(Y_n, n \in \mathbb{N}_0)$ be a sequence of independent, identically distributed random variables with a discrete uniform distribution on $\{-1, 0, 1\}$. Set $X_n := Y_n + Y_{n+1}$ for $n \in \mathbb{N}_0$. Decide whether $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete-time Markov chain or not.

Solution to Exercise 15. By the defining relation for X_n, we have that

$$X_0 = Y_0 + Y_1$$
$$X_1 = Y_1 + Y_2$$
$$X_2 = Y_2 + Y_3$$

We will show that the values $(2, 0, 2)$ and $(-2, 0, 2)$ of a random vector (X_0, X_1, X_2) have different probabilities. Observe that

$$(X_0, X_1, X_2) = (-2, 0, 2) \leftrightarrow (Y_0, Y_1, Y_2, Y_3) = (-1, -1, 1, 1)$$
$$(X_0, X_1, X_2) = (2, 0, 2) \leftrightarrow (Y_0, Y_1, Y_2, Y_3) = \text{ not possible}$$

Hence, for the corresponding probabilities, we obtain

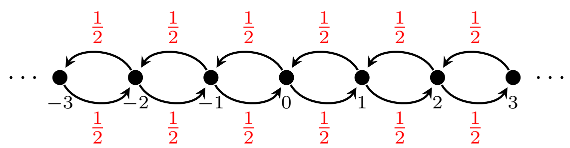
$$\mathbb{P}(X_2 = 2 | X_1 = 0, X_0 = 2) = \frac{\mathbb{P}(X_0 = 2, X_1 = 0, X_2 = 2)}{\mathbb{P}(X_0 = 2, X_1 = 0)} = 0$$
$$\mathbb{P}(X_2 = 2 | X_1 = 0, X_0 = -2) \frac{\mathbb{P}(Y_0 = -1, Y_1 = -1, Y_2 = 1, Y_3 = 1)}{\mathbb{P}(Y_0 = -1, Y_1 = -1, Y_2 = 1)} = \frac{1}{3}.$$

△

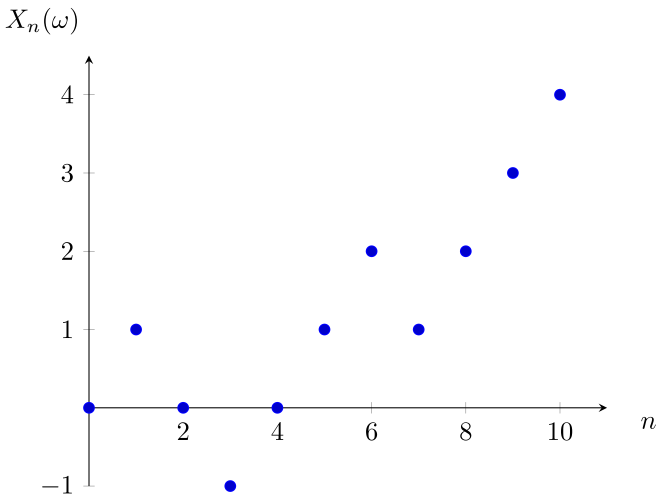
Imagine now that we perform the following experiment. We put a board game figurine on a board which looks like this:

⋯	-3	-2	-1	0	1	2	3	⋯

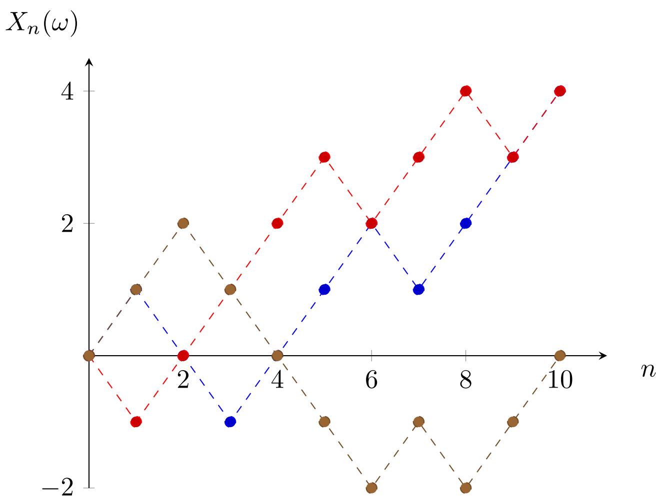
Then, we flip a coin. If heads, the figurine steps to the right and if tails, it moves to the left. Then we flip a coin again and so on. This can be modelled by the symmetric random walk X from Example 12. The board can be visualized as an infinite graph whose vertices are labelled by integers (i.e. corresponding to the state space $S = \mathbb{Z}$). If $\boldsymbol{P}$ is the transition matrix of X, then whenever $p_{ij} > 0$, we will draw an arrow from i to j. These arrows will be the edges. So we obtain the so-called transition diagram:



The transition diagram will be very useful when analysing the irreducibility of the chain X which will be needed for classification of states in the next section. A different way of visualizing a random process that is helpful for intuition is to draw the so-called sample path of X. For this reason, flip a coin ten times, move the figurine on the board and note down the number of the cell in which it ended. The aim is to draw an honest graph of a function. Put the numbers of cells on the y axis and the number of flips on the x axis. Since we start on the yellow cell at the 0th flip, we put a black dot at $(0, 0)$. Now flip a coin and put the next dot at either $(1, 1)$ or $(1, -1)$ depending on whether we got heads or tails. Flip a coin again and so on. We can obtain, for example, the following:



Repeating the experiment again and again, we can always obtain a different graph. For example, repeating it three times, we could obtain the following:



Each color in the above graph corresponds to one experiment that we performed. The dots of the same color (the results of one experiment) comprise the so-called sample path of the random process X.

Definition 2.3. Let $X : \mathbb{N}_0 \times \Omega \to S$ be a random process. Fix $\omega \in \Omega$. Then the function (of one variable) $X(\cdot, \omega) : \mathbb{N}_0 \to S$ is called a trajectory or sample path of X.

Unfortunately, having one trajectory of a process does not tell us much about it since even two sample paths can be dramatically different. Thus, when analysing a stochastic process, we typically want to know what (almost) all its trajectories have in common.

2.2 Classification of states

From no on, we only consider homogeneous discrete-time Markov chains. For notational simplicity, we will use the following two symbols:

$$\mathbb{P}_i(\cdot) := \mathbb{P}(\,\cdot\,|X_0 = i), \quad \mathbb{E}_i(\cdot) := \int_\Omega (\cdot)\mathrm{d}\mathbb{P}_i$$

Let us recall how states a homogeneous discrete-time Markov chains are classified based on how often the chain visits them.

Definition 2.4. Let $X = (X_n, n \in \mathbb{N}_0)$ be a homogeneous discrete-time Markov chain with a state space S.

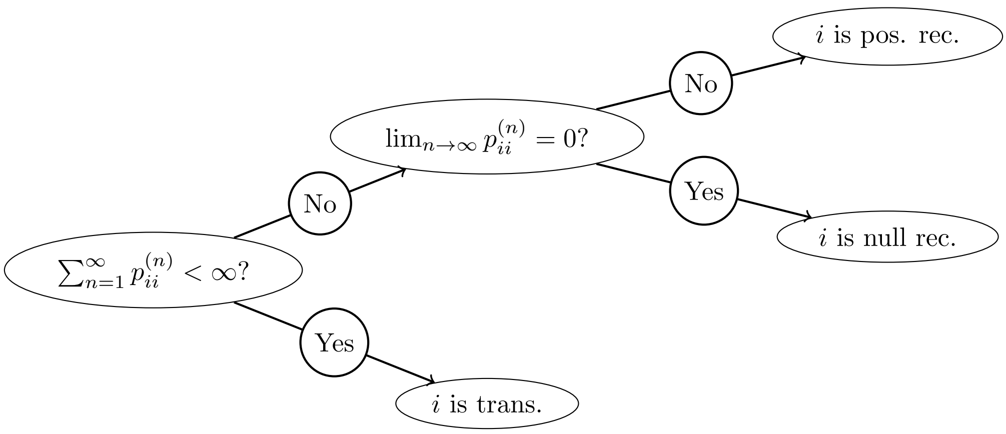
1. A transient state is a state $i \in S$ such that if X starts from i, then it may never return to it (i.e. $\mathbb{P}_i(\forall n \in \mathbb{N}_0 : X_n \neq i) > 0$).
2. A recurrent state is a state $i \in S$ such that if X starts from i, then it will visit it again almost surely (i.e. $\mathbb{P}_i(\exists n \in \mathbb{N}_0 : X_n = i) = 1$.)
   (a) Let
   $$\tau_i(1) := \inf\{n \in \mathbb{N} : X_n = i\}.$$
   the first return time to the state i. A recurrent state i is called null recurrent, if the mean value of the first return time to i is infinite, i.e. if $\mathbb{E}_i\tau_i(1) = \infty$.
   (b) A recurrent state $i \in S$ is called positive recurrent if $\mathbb{E}_i\tau_i(1) < \infty$.
3. Denote by $p_{ij}^{(n)}$ the elements of a stochastic matrix $\boldsymbol{P(n)}$ (i.e. the transition probabilities from a state $i \in S$ to a state $j \in S$ in n steps). If a state $i \in S$ is such that
   $$D_i := \{n \in \mathbb{N}_0 : p_{ii}^{(n)} > 0\} \neq \emptyset,$$
   we can define
   $$d_i := GCD(D_i)$$
   and if $d_i > 1$, we say that i is periodic with a period d_i and aperiodic if $d_i = 1$.

Recall that the transition matrix $\boldsymbol{P(n)}$ can be obtained as the n-th power of the transition matrix $\boldsymbol{P}$ of the chain X since we only deal with homogeneous discrete-time Markov chains.

Theorem 2.1. Denote the elements of the transition matrix of a homogeneous discrete-time Markov chain X in n steps $\boldsymbol{P(n)}$ by $p_{ij}^{(n)}$. The following characterizations hold:

1. A state $i \in S$ is recurrent if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$.
2. A recurrent state $i \in S$ is null-recurrent if and only if $\lim_{n\to\infty} p_{ii}^{(n)} = 0$.

Theorem 2.1 provides a cookbook for classification of states based on the limiting behaviour of $p_{ii}^{(n)}$.



It is clear, that for a successful application of these criteria, we need to be able to know the behaviour of $p_{ii}^{(n)}$ when n tends to infinity and in some cases, $p_{ii}^{(n)}$ can be even computed explicitly.

Exercise 16. Let $X = (X_n, n \in \mathbb{N}_0)$ be the symmetric random walk considered in Exercise 12. Find the probabilities $p_{ii}^{(n)}$ and classify the states of X.

Solution to Exercise 16. *Transient or recurrent states?* Our aim is to use the cookbook above. For this, we need to know the limiting behaviour of $p_{ii}^{(n)}$ as $n \to \infty$. Suppose we make an even number $n = 2k$ of steps. Then in order to return to the original position (say i) we have to make k steps to the right and k steps to the left. On the other hand, if we make an odd number $n = 2k + 1$ of steps, then we will never be able to get back to the original position. Hence,

$$p_{ii}^{(n)} = \begin{cases} \binom{2k}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^k, & n = 2k \\ 0, & n = 2k+1 \end{cases}$$

Now, we need to decide if $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$ or not. In order to do so, notice that, using the Stirling's formula ($m! \approx \sqrt{2\pi m} m^m e^{-m}$), we have

$$\binom{2k}{k} \frac{1}{2^{2k}} = \frac{(2k)!}{(k!)^2} \cdot \frac{1}{2^{2k}} \approx \frac{1}{\sqrt{\pi k}}, \quad k \to \infty,$$

and, using the comparison test, we have that the sum $\sum_{n=1}^{\infty} p_{ii}^{(n)}$ converges if and only if the sum $\sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}}$ converges. This is, however, not true and hence we can infer that

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty.$$

This means that the state i is recurrent and we need to say if it is null recurrent of positive recurrent. But this is easy since $\frac{1}{\sqrt{\pi k}}$ goes to 0 (which corresponds to $n = 2k$) and also 0 goes to 0 trivially (which corresponds to $n = 2k+1$). Hence, the state i is null recurrent. At the beginning, i was arbitrary and our reasoning is valid for any $i \in \mathbb{N}_0$. Hence, all the states of the symmetric random walk are null recurrent.

Periodicity: For every i, we have that for all k, $p_{ii}^{(2k+1)} = 0$ and $p_{ii}^{2k} > 0$. Hence, we have that

$$\{n \in \mathbb{N}_0 : p_{ii}^{(n)} > 0\} = \{2k, k \in \mathbb{N}_0\} = \{0, 2, 4, \ldots\}.$$

The greatest common divisor of the above set is 2 and hence, $d_i = 2$ for every i. Altogether, the symmetric random walk has 2-periodic, null recurrent states. △

Exercise 17. Let $(Y_n, n \in \mathbb{N})$ be a sequence of independent, identically distributed random variables with a discrete uniform distribution on $\{-1, 0, 1\}$. Define $X_n := \max\{Y_1, \ldots, Y_n\}$. Then $X = (X_n, n \in \mathbb{N})$ is a homogeneous discrete-time Markov chain (see Exercise 13). Find the probabilities $p_{ii}^{(n)}$ and classify the states of X.

Hint: Use similar probabilistic reasoning as in Exercise 12 to find the matrix $\boldsymbol{P}^n$. What must the whole trajectory look like, if we start at -1, make n steps, and finish at -1 again?

Exercise 18. Consider a series of Bernoulli trails with the probability of success $p \in (0, 1)$. Denote by X_n the length of a series of successes preceding the n-th trial (if the n-th trial is not a success then we set $X_n = 0$). Show that $X = (X_n, n \in \mathbb{N})$ is a homogeneous discrete-time Markov chain, find its transition probability matrix $\boldsymbol{P}$ and its n^{th} power $\boldsymbol{P}^n$ and classify its states.

If we want to find the whole matrix $\boldsymbol{P}^n$ we have more options. One is, of course, the Jordan decomposition. Using the Jordan decomposition we obtain matrices $\boldsymbol{S}$ and $\boldsymbol{J}$ such that $\boldsymbol{P} = \boldsymbol{S}\boldsymbol{J}\boldsymbol{S}^{-1}$. Then

$$\boldsymbol{P}^n = \underbrace{(\boldsymbol{S}\boldsymbol{J}\boldsymbol{S}^{-1}) \cdot (\boldsymbol{S}\boldsymbol{J}\boldsymbol{S}^{-1}) \cdot \ldots \cdot (\boldsymbol{S}\boldsymbol{J}\boldsymbol{S}^{-1})}_{n \text{ times}} = \boldsymbol{S}\boldsymbol{J}^n\boldsymbol{S}^{-1}$$

and finding the matrix $\boldsymbol{J}^n$ is easy due to its canonical form. Another way how to find $\boldsymbol{P}^n$ is via the Perron's formula that is given in the following theorem.

Theorem 2.2 (Perron's formula for matrix powers)**.** Let $\boldsymbol{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_k$ with multiplicities $m_1, \cdots, m_k$. Then it holds that

$$\boldsymbol{A}^n = \sum_{j=1}^{k} \frac{1}{(m_j - 1)!} \frac{d^{m_j - 1}}{d\lambda^{m_j - 1}} \left[\frac{\lambda^n \mathbf{Adj}\,(\boldsymbol{\lambda I} - \boldsymbol{A})}{\psi_j(\lambda)} \right]_{\lambda = \lambda_j}$$

where

$$\psi_j(\lambda) = \frac{\det(\boldsymbol{\lambda I} - \boldsymbol{A})}{(\lambda - \lambda_j)^{m_j}}.$$

Exercise 19. Consider a homogeneous discrete-time Markov chain X with the state space $S = \{0, 1\}$ and the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

where $a, b \in (0, 1)$. Find $\boldsymbol{P}^n$ and classify the states of X.

Solution to Exercise 19. We will use the Jordan decomposition to find the matrix $\boldsymbol{P}^n$, then we will take its diagonal elements and use the cookbook above to classify the states 0 and 1. The key ingredients for Jordan decomposition are the eigenvalues and eigenvectors. The eigenvalues are the roots of the characteristic polynomial $\psi(\lambda)$. We thus have to solve

$$\psi(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{P}) = \begin{pmatrix} \lambda - 1 + a & -a \\ -b & \lambda - 1 + b \end{pmatrix} = \lambda^2 - (2 - (a+b))\lambda + 1 - (a+b) = 0.$$

The corresponding eigenvectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are found as solutions to $\boldsymbol{P}\boldsymbol{v_i} = \lambda_i \boldsymbol{v_i}$, $i = 1, 2$. We arrive at

$$\begin{aligned} \lambda_1 &= 1 & \ldots \quad \boldsymbol{v_1}^T &= (1, 1) \\ \lambda_2 &= 1 - (a+b) & \ldots \quad \boldsymbol{v_2}^T &= \left(-\tfrac{a}{b}, 1\right) \end{aligned}$$

The Jordan form is then

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -\frac{a}{b} \\ 1 & 1 \end{pmatrix}}_{=:\boldsymbol{S}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1-(a+b) \end{pmatrix}}_{=:\boldsymbol{J}} \underbrace{\begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ -\frac{b}{a+b} & \frac{b}{a+b} \end{pmatrix}}_{=\boldsymbol{S}^{-1}}.$$

Now it is easy to find the power $\boldsymbol{J}^n$. We have that

$$\boldsymbol{J}^n = \begin{pmatrix} 1 & 0 \\ 0 & (1-(a+b))^n \end{pmatrix}$$

so that

$$\begin{aligned} \boldsymbol{P}^n &= \boldsymbol{S}\boldsymbol{J}^n\boldsymbol{S}^{-1} \\ &= \frac{1}{a+b} \begin{pmatrix} 1 & -\frac{a}{b} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1-(a+b))^n \end{pmatrix} \begin{pmatrix} b & a \\ -b & b \end{pmatrix} \\ &= \frac{1}{a+b} \begin{pmatrix} b + a(1-(a+b))^n & a - a(1-(a+b))^n \\ b - b(1-(a+b))^n & a + b(1-(a+b))^n \end{pmatrix}. \end{aligned}$$

Clearly,

$$p_{ii}^{(n)} = \begin{cases} b + a(1-(a+b))^n, & i = 0, \\ a + b(1-(a+b))^n, & i = 1, \end{cases} \quad \xrightarrow[n\to\infty]{} \quad \begin{cases} b, & i = 0, \\ a, & i = 1, \end{cases} \neq 0.$$

Hence, the sum $\sum_{n=1}^{\infty} p_{ii}^{(n)}$ cannot converge and because of this, the Markov chain under consideration has positive recurrent states. Since also $p_{ii} > 0$ for both $i = 0, 1$, the states are also aperiodic. △

Exercise 20 (Random walk on a triangle)**.** Consider a homogeneous discrete-time Markov chain with the state space $S = \{0, 1, 2\}$ and the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

Find $\boldsymbol{P}^n$ and classify the states of X.

Solution to Exercise 20. In order to successfully apply the Perron's formula we need the following ingredients: the eigenvalues $\lambda_1, \ldots, \lambda_k$ of $\boldsymbol{P}$, their multiplicities $m_1, \ldots, m_k$, the corresponding polynomials $\psi_j(\lambda)$ and the adjoint matrix of $\lambda \boldsymbol{I} - \boldsymbol{P}$. The we plug everything into the formula to obtain $\boldsymbol{P}^n$. As in the previous exercise, we have to find the roots of characteristic polynomial

$$\psi(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{P}) = \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = \lambda^3 - \frac{3}{4}\lambda - \frac{1}{4} = (\lambda - 1)\left(\lambda + \frac{1}{2}\right)^2 = 0$$

which yields

$$\begin{array}{lllll} \lambda_1 = 1 & \ldots & m_1 = 1 & \ldots & \psi_1(\lambda) = \left(\lambda + \frac{1}{2}\right)^2 \\ \lambda_2 = -\frac{1}{2} & \ldots & m_2 = 2 & \ldots & \psi_2(\lambda) = (\lambda - 1) \end{array}$$

The adjoint matrix can be obtained as follows. Given a square $n \times n$ matrix $\boldsymbol{A}$, the adjoint

$$\textbf{Adj}\,\boldsymbol{A} = \begin{pmatrix} +\det M_{11} & -\det M_{12} & \cdots & (-1)^{n+1}\det M_{1n} \\ -\det M_{21} & +\det M_{22} & \cdots & (-1)^{n+2}\det M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}\det M_{n1} & (-1)^{n+1}\det M_{n1} & \cdots & (-1)^{n+n}\det M_{n1} \end{pmatrix}^T$$

where M_{ij} is the (i,j) minor (i.e. the determinant of the $(n-1)\times(n-1)$ matrix which is obtained by deleting row i and column j of $\boldsymbol{A}$). We obtain

$$\textbf{Adj}\,(\boldsymbol{\lambda I} - \boldsymbol{P}) = \textbf{Adj} \begin{pmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 - \frac{1}{2} & \frac{1}{2}\lambda + \frac{1}{4} & \frac{1}{2}\lambda + \frac{1}{4} \\ \frac{1}{2}\lambda + \frac{1}{4} & \lambda^2 - \frac{1}{4} & \frac{1}{2}\lambda + \frac{1}{4} \\ \frac{1}{2}\lambda + \frac{1}{4} & \frac{1}{2}\lambda + \frac{1}{4} & \lambda^2 - \frac{1}{4} \end{pmatrix}.$$

Plugging everything into the formula, we obtain

$$\begin{aligned} \boldsymbol{P}^n &= \frac{1}{(1-1)!}\frac{\mathrm{d}^0}{\mathrm{d}\lambda^0}\left(\frac{\lambda^n \textbf{Adj}\,(\boldsymbol{\lambda I} - \boldsymbol{P})}{\left(\lambda + \frac{1}{2}\right)^2}\right)\Bigg|_{\lambda=1} + \frac{1}{(2-1)!}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\lambda^n \textbf{Adj}\,(\boldsymbol{\lambda I} - \boldsymbol{P})}{\lambda - 1}\right)\Bigg|_{\lambda=-\frac{1}{2}} \\ &= \frac{4}{9}\cdot\frac{3}{4}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{\lambda^n}{\lambda - 1}\begin{pmatrix} 2\lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 2\lambda \end{pmatrix}\Bigg|_{\lambda=-\frac{1}{2}} \\ &\quad + \frac{n\lambda^{n-1}(\lambda-1) - \lambda^n}{(\lambda-1)^2}\begin{pmatrix} \lambda^2 - \frac{1}{2} & \frac{1}{2}\lambda + \frac{1}{4} & \frac{1}{2}\lambda + \frac{1}{4} \\ \frac{1}{2}\lambda + \frac{1}{4} & \lambda^2 - \frac{1}{4} & \frac{1}{2}\lambda + \frac{1}{4} \\ \frac{1}{2}\lambda + \frac{1}{4} & \frac{1}{2}\lambda + \frac{1}{4} & \lambda^2 - \frac{1}{4} \end{pmatrix}\Bigg|_{\lambda=-\frac{1}{2}} \\ &= \begin{pmatrix} \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n \\ \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n \\ \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n \end{pmatrix}. \end{aligned}$$

Clearly for all $i \in S$ we have that

$$p_{ii}^{(n)} = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n$$

and hence, the sum

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \sum_{n=1}^{\infty}\frac{1}{3} + \frac{2}{3}\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = \infty$$

which implies that all the states are recurrent and, moreover, $p_{ii}^{(n)} \to \frac{1}{3} \neq 0$ which implies that they are positive recurrent. Further,

$$\{n \in \mathbb{N}_0 : p_{ii}^{(n)} > 0\} = \mathbb{N}_0$$

and hence, the states are all aperiodic since the greatest common divisor of all natural numbers is 1. Compare this periodicity to the periodicity of the random walk on the integers. △

Looking back at the solution to Exercise 20, there are two points worth noting. First, when computing the characteristic polynomial of $\boldsymbol{P}$, one has to solve a polynomial equation of higher order. It helps to know that when $\boldsymbol{P}$ is a stochastic matrix, one of its eigenvalues is always 1 (hence, we can divide the polynomial $\psi(\lambda)$ by $\lambda - 1$ to reduce its order and find the other eigenvalues quickly). The second is that one should not forget to take the transpose of the matrix when computing the adjoint $\mathbf{Adj}(\boldsymbol{\lambda I} - \boldsymbol{P})$.

Theorem 2.1 gives a characterisation of recurrence of a state in terms of limiting behaviour of $p_{ii}^{(n)}$ regardless of whether the state space is finite or infinite. The drawback is that it might be quite tedious to apply in practice. It turns out, however, that in many cases, we can only classify some of the states and all the states that communicate with these states will be of the same type. More precisely there are the following results.

Definition 2.5. Let $X = (X_n, n \in \mathbb{N}_0)$ be a homogeneous discrete-time Markov chain with a state space S and let $i, j \in S$ be two of its states. We say, that j is accessible from i (and write $i \to j$) if there is $m \in \mathbb{N}_0$ such that $p_{ij}^{(m)} > 0$. If the states i, j are mutually accessible, then we say that they communicate (and write $i \leftrightarrow j$). X is called irreducible if all its states communicate.

Theorem 2.3. Let $X = (X_n, n \in \mathbb{N}_0)$ be a homogeneous discrete-time Markov chain with a state space S. The following claims hold:

1. If two states communicate, they are of the same type.
2. If a state $j \in S$ is accessible from a recurrent state $i \in S$, then j is also recurrent and i and j communicate.

Some remarks are in place.

- When we say that two states are of the same type, we mean that both states are either null-recurrent, positive recurrent or transient and if one is periodic, the other is also periodic with the same period.
- Theorem 2.3 holds for both finite and countable state space S. Hence, if the Markov chain is irreducible (all its states communicate), then it suffices to classify only one state - the rest will be of the same type.
- If for a state $i \in S$ there is $j \in S$ such that $i \to j \not\to i$, then i must be transient.
- Clearly, the relation $i \leftrightarrow j$ is an equivalence relation on the state space and thus, we may factorize S into equivalence classes. If i is a recurrent state, then class of equivalence $[i]$ simply consists of all the states which are accessible from i.

In the case when the state space S is finite, the situation becomes very easy as stated in the next theorem. The case of Markov chains with an infinite state space will be discussed later.

Theorem 2.4. Let X be an irreducible homogeneous discrete-time Markov chain with a finite state space. Then all its states are positive recurrent.

Exercise 21. Suppose we have an unlimited source of balls and k drawers. In every step we pick one drawer at random (uniformly chosen) and we put one ball into this drawer. Let X_n be a number of drawers with a ball at the time n. Show that $X = (X_n, n \in \mathbb{N})$ is a homogeneous discrete-time Markov chain, find its transition probability matrix and classify its states.

Exercise 22 (Gambler's ruin)**.** Suppose a gambler goes to a casino to repeatedly play a game which ends by either the gambler losing the round or the gambler winning the round. Suppose further that our gambler has initial fortune $\$\,a$ and the croupier has $\$\,(s-a)$ with $\$\,s$ being the total amount of \$ in the game. Each turn they play and the winner gets one dollar from his opponent. The game ends when either side has no money. Suppose that the probability that our gambler wins is $0 < p < 1$. Model the evolution of capital of the gambler by a homogeneous discrete-time Markov chain (i.e. say what is X_n and prove that it is a homogeneous discrete-time Markov chain), find its transition matrix and classify its states.

Solution to Exercise 22. This famous and old problem is first mentioned in a letter from Blaise Pascal to Pierre de Fermat in 1656 and a year later published by Christiaan Huygens in [9].

Forget for a second, that we are given an exercise to solve and imagine yourself watching two your friends play at dice. Naturally, you ask yourself: Who is going to win? How long will they play? And so on. In order to answer these questions, you have to build a model of the situation. There are various ways how to do that and we will use a probabilistic model.

Analysis: What kind of model should we use? First notice, that dice are played in turns and that each time a specific amount of money travels from one player to his opponent. Further, there is a finite amount, say $\$\,s$, in the game so that it is enough to observe how much money one of the player has and the fortune of the other can be simply deduced from this. The observed player starts with a certain fortune, say $\$\,a$.

Building a mathematical model: Denote by X_n the fortune of the observed player after the n-th round. So far, X_n is just a symbol. We have to give a precise mathematical meaning to this symbol and it is very natural to assume that X_n is a random variable for each round n. The game ends when either side has no money, this means that all X_n's can take values in the set $S := \{0, 1, \dots, s\}$ with $X_0 = a$. Now, the i-th round can be naturally modelled by another random variable Y_i taking only two values - the observed player won or lost. This corresponds to the player either getting $\$\,1$ or losing it. Hence, each Y_i can take either the value -1 or $+1$ and we assume that the result of every round does not depend on the results of all the previous rounds (e.g. the player does not learn to throw the dice better, the die does not become damaged over time, etc.) - this allows us to assume that Y_i's are independent and identically distributed. Obviously, the X_n's and Y_i's are connected - X_n is the sum of X_0 and all Y_i for $i = 1, \dots, n$ unless X_n is either s or 0 in which case it does not change anymore. The formal model follows.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $i \in \mathbb{N}$, let

$$Y_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (\{-1, 1\}, \{\emptyset, \{-1\}, \{1\}, \{-1, 1\}\})$$

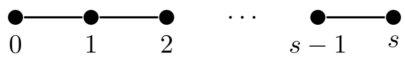
be a random variable and assume that $Y_1, Y_2, \dots$ are all mutually independent with the same probability distribution on $\{-1, 1\}$ given by

$$\mathbb{P}(Y_i = -1) = 1 - p, \quad \mathbb{P}(Y_i = 1) = p.$$

Define $X_0 := a$ and

$$X_n := X_0 + (X_{n-1} + Y_n)\mathbf{1}_{[X_{n-1} \notin \{0,s\}]} + X_{n-1}\mathbf{1}_{[X_{n-1} \in \{0,s\}]}, \quad n \in \mathbb{N}.$$

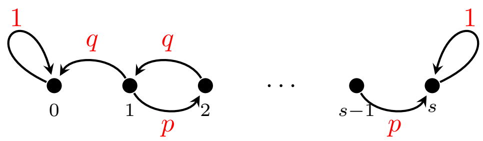
Then each X_n is a random variable taking the values in $S = \{0, 1, \dots, s\}$ (show this!) and, using Exercise 14, we can see that $X := (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete-time Markov chain with the state space S. Looking closely, we see that it is a random walk on the following graph



with absorption states at 0 and s. The transition matrix of X is built in the following way. Clearly, once we step into the state 0 or s, we will never leave it. Hence $p_{00} = p_{ss} = 1$ and $p_{0i} = p_{si} = 0$ for all $i \in \{1, \dots, s-1\}$. Furthermore, from the state j, $j \in \{1, \dots, s-1\}$, we can always reach only state $j-1$ (with probability $q = 1 - p$) and $j+1$ (with probability p), i.e. $p_{j,j-1} = q$ and $p_{j,j+1} = p$. We arrive at

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ s-1 \\ s \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & \dots & s-3 & s-2 & s-1 & s \end{array} \\ \left(\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

The transition diagram representing the Markov chain X is the following:



Classification of states: In order to classify the states, we make use of Theorem 2.3 and Theorem 2.4. We first notice that the chain is reducible. In particular, the state space can be written as

$$S = \{0\} \cup \{s\} \cup \{1, 2, \dots, s-1\}.$$

All the states in the set $\{1, \dots, s-1\}$ communicate and thus, they are of the same type. It suffices to analyse only one of them, say the state 1. Obviously,

$$\{n \in \mathbb{N}_0 : p_{11}^{(n)} > 0\} = \{2k, k \in \mathbb{N}_0\}$$

and hence, the state 1 is 2-periodic. Furthermore, $1 \to 0 \not\to 1$ and hence, 1 is transient. This means that all the states $\{1, \dots, s-1\}$ are 2-periodic and transient. Further, since $p_{00} > 0$ and $p_{ss} > 0$, both 0 and s are aperiodic. Since $p_{00} = \mathbb{P}_i(X_1 = 0) = 1$, we immediately have that 0 is a recurrent state and since $\tau_0(1) = 1, \mathbb{P}$-a.s., we also have that $\mathbb{E}_0\tau_0(1) = 1 < \infty$ which implies that 0 is positive recurrent. Alternatively, one can argue that we can define a (rather trivial) sub-chain of X which only has one state (i.e. 0) and which is therefore irreducible with a finite state space. Then we may appeal to Theorem 2.4 to infer that 0 is positive recurrent. The same holds for the state s. △

Exercise 23 (Heat transmission model, D. Bernoulli (1796))**.** The temperatures of two isolated bodies are represented by two containers with a number of balls in each. Altogether, we have $2l$ balls, numbered by $1, \dots, 2l$. The process is described as follows: in each turn, pick a number from $\{1, \dots, 2l\}$ uniformly at random and move the ball to the other container. Model the temperature in the first container by a homogeneous discrete-time Markov chain. Then find its transition matrix and classify its states.

Exercise 24 (Blending of two incompressible fluids, T. and P. Ehrenfest (1907))**.** Suppose we have two containers, each containing l balls (these are the molecule of our fluid). Altogether, we have l black balls and l white balls. The process of blending is described as follows: each turn, pick two balls, each from a different container, and switch them. This way the number of balls in each urn is constant in time. Model the process of blending by a homogeneous discrete-time Markov chain. Then find its transition matrix and classify its states.

2.3 Absorption probabilities

Recall the Gampler's ruin problem (Exercise 22). One may ask - what is the probability (which will clearly depend on the initial wealth a) that the whole game will eventually result in the ruin of the gambler? In general, what is the probability, that if a Markov chain starts at a transient state i, the first visited recurrent state is a given state j? These questions are discussed next.

Let X be a homogeneous discrete-time Markov chain with a state space S and denote by T the set of its transient states and by C the set of its recurrent states (so that $S = C \cup T$). Define

$$\tau := \min\{n \in \mathbb{N} : X_n \notin T\},$$

the time of leaving T and denote by X_τ the first visited recurrent state. Assume that $\mathbb{P}_i(\tau = \infty) = 0$ for all $i \in T$. Then we can define the absorption probabilities u_{ij} by

$$u_{ij} := \mathbb{P}_i(X_\tau = j), \quad i \in T, j \in C.$$

Theorem 2.5. The absorption probabilities satisfy the equation

$$u_{ij} = p_{ij} + \sum_{k \in T} p_{ik} u_{kj}, \quad i \in T, j \in C. \tag{2.1}$$

Theorem 2.5 can be easily applied in practice. Notice that if we relabel all the states in such a way that all the recurrent states have smaller index than all the transient states, we can write the (possibly infinite) transition matrix in the following way:

$$\boldsymbol{P} = \begin{array}{c} \\ C \\ T \end{array} \begin{array}{c} \begin{array}{cc} C & T \end{array} \\ \begin{pmatrix} \boldsymbol{P_C} & \boldsymbol{0} \\ \boldsymbol{Q} & \boldsymbol{R} \end{pmatrix} \end{array}.$$

Here $\boldsymbol{P_C} = (p_{ij})_{i,j \in C}$, $\boldsymbol{Q} = (p_{ij})_{i \in T, j \in C}$ and $\boldsymbol{R} = (p_{ij})_{i,j \in T}$. Formula (2.1) can then be written as

$$\boldsymbol{U} = \boldsymbol{Q} + \boldsymbol{R}\boldsymbol{U}$$

which has one solution in $[0,1]^{|T|\times|C|}$ if and only if $\mathbb{P}_i(\tau = \infty) = 0$ for every $i \in T$ (i.e. the chain leaves the set of transient states almost surely). The solution is, in general, written as a matrix geometric series

$$\hat{\boldsymbol{U}} = \sum_{n=0}^{\infty} \boldsymbol{R}^n \boldsymbol{Q}$$

and, in the case when the state space is finite it takes the form

$$\hat{\boldsymbol{U}} = (\boldsymbol{I}_T - \boldsymbol{R})^{-1}\boldsymbol{Q}. \tag{2.2}$$

Exercise 25. Classify the states of a Markov chain given by the transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

Solution to Exercise 25. Let us denote the Markov chain under consideration by X. Clearly, the state space of X is $S = \{0, 1, 2, 3\}$ and we have that $1 \to 2 \to 3 \to 1$ and $0 \to 1 \not\to 0$. Hence, by Theorem 2.3, we have that the states $1, 2, 3$ are positive recurrent (they define a finite irreducible sub-chain) and 0 is a transient state. All the states are aperiodic ($p_{00} > 0$ and $p_{22} > 0$).

Let us follow compute the absorption probabilities $\boldsymbol{U}$. The cookbook is as follows: first rearrange the states in such a way that all the recurrent states are in the upper left corner of $\boldsymbol{P}$, then compute $\hat{\boldsymbol{U}}$ from formula (2.2). We obtain

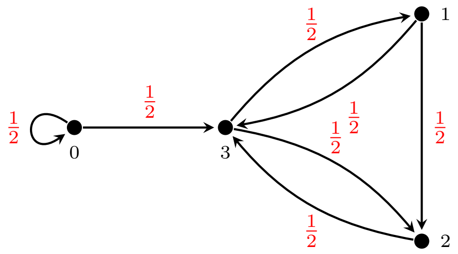
$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 0 \end{array} \\ \left(\begin{array}{ccc|c} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \hline 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right) \end{array}.$$

Now, we compute $\hat{\boldsymbol{U}}$ from formula (2.2):

$$\hat{\boldsymbol{U}} = (\boldsymbol{I}_1 - \boldsymbol{R})^{-1}\boldsymbol{Q} = \left(1 - \frac{1}{2}\right)^{-1} \begin{pmatrix} 0, & 0, & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0, & 0, & 1 \end{pmatrix} = \begin{pmatrix} u_{01}, & u_{02}, & u_{03} \end{pmatrix}.$$

The interpretation of the computed $\hat{\boldsymbol{U}}$ is that if X starts from 0, it jumps on spot for an unspecified amount of time but once it jumps to a different state then 0, it will be the state 3 almost surely.

Looking back at the solution: Being given a chain in Exercise 25 with this particular $\boldsymbol{P}$, we can immediately answer the question which of the recurrent states will be the first one X visits after leaving the set of transient states. Since $p_{03} > 0$ and $p_{01} = p_{02} = 0$, it must be that the first state which is visited after leaving 0 is 3. This becomes even clearer once we draw the transition diagram of X:



△

Exercise 26. Let $p \in [0,1]$. Classify the states of the Markov chain given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} p & 0 & 1-p & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \\ 0 & p & 0 & 1-p \end{pmatrix}$$

and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

Exercise 27. Classify the states of the Markov chain given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}$$

and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

Exercise 28. Classify the states of the Markov chain given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

Exercise 29. Classify the states of the Markov chain given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

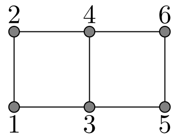
and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

Exercise 30. Classify the states of the Markov chain given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{8} & 0 & \frac{7}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

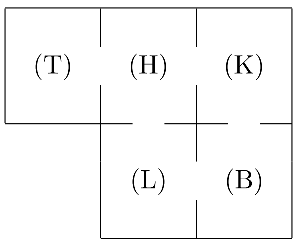
Exercise 31. A baby echidnea was born in the zoo. In order not to hurt itself, it is now only allowed to move in the following maze:



At each step, our baby echindea decides (uniformly randomly) upon one direction (west, east, north, or south) and then it goes in that direction until it hits a wall. If there is a wall in the chosen direction, the baby echidnea is confused and remains in the current spot. Denote by X_n its position at step $n \in \mathbb{N}_0$. Classify the states of the Markov chain $X = (X_n, n \in \mathbb{N}_0)$ and compute the matrix of absorption probabilities $\boldsymbol{U}$ into the set of recurrent states.

Exercise 32. Consider an urn and five balls. At each step, we shall add or remove the balls to/from the urn according to the following scheme. If the urn is empty, we will add all the five balls into it. If it is not empty, we remove either four or two balls or none at all, every case with probability 1/3. Should we remove more balls than the urn currently contains, we remove all the remaining balls. Denote by X_n the number of balls in the urn at the n-th step. Classify the states of the Markov chain $X = (X_n, n \in \mathbb{N}_0)$ and compute the matrix of absorption probabilities $\boldsymbol{U}$ into the set of recurrent states.

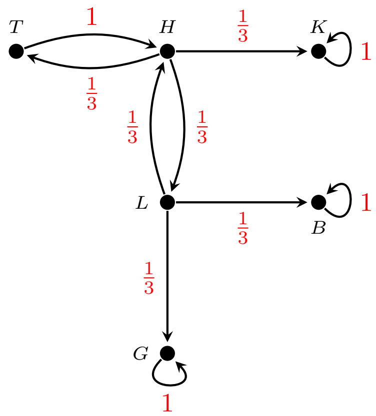
Exercise 33. There is a mouse in the house! The mouse moves in such a way that in each room, it chooses one of the adjacent rooms (each with the same probability) and runs there (the movements occur at times $n = 1, 2, \ldots$). In the schematic below, the rooms are the hall (H), kitchen (K), toilet (T), bedroom (B) and the living room (L). We can set two traps - one is mischievously installed in the bedroom and the other one in the kitchen since we really should not have a mouse there. As soon as the mouse enters a room with a trap, it is caught and it will never ever run into another room. Denote by X_n the position of the mouse at time n. Classify the states of the Markov chain $X = (X_n, n \in \mathbb{N}_0)$ and compute the matrix of absorption probabilities $\boldsymbol{U}$ into the set of recurrent states.



Exercise 34. Modify Exercise 33. Suppose now that we open the door from our living room (L) to the garden (G). Once the mouse leaves the flat and enters the garden, it will never come inside again. If the mouse starts at the toilet, what is the probability that it will escape the flat before it is caught in a trap?

Solution to Exercise 34. Building the formal model (i.e. defining X_n and proving that it is a homogeneous discrete-time Markov chain) is simple and could be done in a similar way as any random walk with absorbing states (see e.g. Exercise 22). We will focus on the task at hand: finding the probabilities u_{ij}. First, we shall find the transition probabilities $\boldsymbol{P}$ and the transition diagram that describes the Markov chain X. We have the following:

$$
\boldsymbol{P} = \begin{array}{c} \\ K \\ B \\ G \\ T \\ H \\ L \end{array} \begin{array}{c} \begin{array}{cccccc} K & B & G & T & H & L \end{array} \\ \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{array}\right) \end{array}
$$



Now we have two options. Either we compute the whole matrix $\hat{\boldsymbol{U}}$ or we compute u_{TG} directly.

Computing $\hat{\boldsymbol{U}}$*:* This can be done as in the previous exercises by using formula (2.2). We have

$$
\hat{\boldsymbol{U}} = (\boldsymbol{I}_3 - \boldsymbol{R})^{-1}\boldsymbol{Q} = \begin{pmatrix} 1 & -1 & 0 \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}.
$$

Of course, one can make many mistakes in computing the inverse of a matrix 3×3. you should always check that all the rows of your final matrix $\hat{\boldsymbol{U}}$ are probability distributions (i.e. that they sum up to 1).

Computing only u_{TG}*:* We can also appeal to the formula (2.1) and write down only those equations which interest us. We are interested in u_{LG} so let us rewrite (2.2) for $i = L$ and $j = G$. We obtain

$$
u_{TG} = p_{TG} + p_{TT}u_{TG} + p_{TH}u_{HG} + p_{TL}u_{LG} = u_{HG}
$$

Hence, we also need an equation for u_{HG}. We again apply formula (2.2) and look into $\boldsymbol{P}$ to obtain

$$
u_{HG} = p_{HG} + p_{HT}u_{TG} + p_{HH}u_{HG} + p_{HL}u_{LG} = \frac{1}{3}u_{TG} + \frac{1}{3}u_{LG}
$$

which means that we also need an equation for u_{LG}, namely,

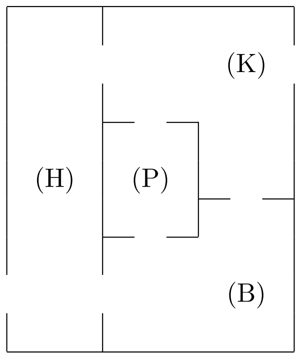
$$
u_{LG} = p_{LG} + p_{LT}u_{TG} + p_{LH}u_{HG} + p_{LL}u_{LG} = \frac{1}{3} + \frac{1}{3}u_{HG}.
$$

Finally, by choosing our favourite solution method for linear equations, we obtain $u_{TG} = \frac{1}{5}$. △

Exercise 35. There is a mouse in the house again! The rooms now are kitchen (K), hall (H), bedroom (B) and pantry (P) (see the schematic below). The mouse moves in such a way that in every room it chooses one of the doors (each with the same probability) and runs through those to a different room (or out of the flat). The movements occur at times $n = 1, 2, \ldots$. We have set one trap in the pantry. If the mouse enters a room with a trap, it is caught and it will never run again. If the mouse leaves the house, it forgets the way back and lives happily ever after (and it will never come back to our flat). Denote by X_n the position of the mouse at time n. Classify the states of the Markov chain $X = (X_n, n \in \mathbb{N}_0)$. What is the probability that if the mouse starts in the bedroom, it will escape the flat before it is caught in the trap?

Exercise 36. Recall the gambler's ruin problem (Exercise 22) and assume that the gambler starts with the initial wealth \$$a$ and that the total wealth in the game is \$$s$. Compute the probability that the game will eventually result in the ruin of the gambler.

Solution to Exercise 36. Recall that if X is the Markov chain modelling the gambler's wealth, then its state space is $S = \{0\} \cup \{s\} \cup \{1, 2, \ldots, s-1\}$ where 0 and s are absorbing states and $1, 2, \ldots, s-1$ are transient states. Let us assume that $a \in \{1, 2, \ldots, s-1\}$ (otherwise the game would be somewhat quick). Our task is to compute the probability of absorption from the transient state a to the absorbing state 0, i.e. we will compute u_{a0}. By rearranging the states, we obtain



$$\boldsymbol{P} = \begin{array}{c} \\ {\color{red}0} \\ {\color{red}s} \\ 1 \\ 2 \\ 3 \\ \vdots \\ s-2 \\ s-1 \end{array} \begin{array}{c} \begin{array}{cccccccc} {\color{red}0} & {\color{red}s} & 1 & 2 & 3 & \dots & s-2 & s-1 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \ddots & 0 & 0 \\ q & 0 & 0 & p & 0 & \ddots & 0 & 0 \\ 0 & 0 & q & 0 & p & \ddots & 0 & 0 \\ 0 & 0 & 0 & q & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & p \\ 0 & p & 0 & 0 & 0 & \cdots & q & 0 \end{pmatrix} \end{array}$$

With these ingredients in hand, we shall make use of the formula

$$\boldsymbol{U} = \boldsymbol{Q} + \boldsymbol{R}\boldsymbol{U}.$$

Now, we see that

$$\begin{aligned} u_1 - q - pu_2 &= 0 \\ u_i - qu_{i-1} - pu_{i+1} &= 0, \quad i = 2, 3, \dots, s-2 \\ u_{s-1} - qu_{s-2} &= 0 \end{aligned}$$

Let us write $u_i := u_{i0}$ for $i = 1, 2, \dots, s-1$ and define $u_0 := 1$ and $u_s := 0$. We obtain the following difference equation with boundary conditions

$$\begin{aligned} u_i - qu_{i-1} - pu_{i+1} &= 0, \quad i = 1, 2, \dots, s-1 \\ u_0 &= 1 \\ u_s &= 0 \end{aligned}$$

Its characteristic polynomial is

$$\chi(\lambda) = -p\lambda^2 + \lambda - q$$

and its roots are $\lambda_1 = 1$ and $\lambda_2 = q/p$.

The case $p \neq q$: In this case, we have two different roots λ_1 and λ_2, the fundamental solution to the difference equation is then $\{1, q/p\}$ and the general solution to the homogeneous equation is

$$u_i = A \cdot 1^i + B \cdot \left(\frac{q}{p}\right)^i, \quad i = 1, 2, \dots, s-1.$$

where A and B are constants which will be determined from the boundary conditions. Namely, we have

$$\begin{aligned} 1 &= A + B \\ 0 &= A + B\left(\frac{q}{p}\right)^s \end{aligned}$$

which gives

$$A = -\frac{\left(\frac{q}{p}\right)^s}{1-\left(\frac{q}{p}\right)^s}, \qquad B = \frac{1}{1-\left(\frac{q}{p}\right)^s}.$$

Finally, we arrive at

$$u_i = \frac{\left(\frac{q}{p}\right)^i - \left(\frac{q}{p}\right)^s}{1-\left(\frac{q}{p}\right)^s}, \quad i = 0, 1, \ldots, s.$$

The case $p = q$: In this case, $\chi(\lambda)$ only has one root and the general solution is given by

$$u_i = A + Bi, \quad i = 1, 2, \ldots, s-1,$$

where the constants A and B satisfy the equations

$$\begin{aligned} 1 &= A \\ 0 &= A + Bs. \end{aligned}$$

Hence, the sought absorption probability is

$$u_i = 1 - \frac{i}{s}, \quad i = 0, 1, \ldots, s.$$

△

2.4 Steady-state and large-time behaviour

Whenever discussing models of real world phenomena there is always an important question of stability. However, what precisely is meant by stability always depends on the particular situation. In the case of discrete-time Markov chains, stability means the existence of a probability distribution that would assure that if the Markov chain starts with this distribution, then its distribution is time invariant, i.e. it does not change in time. The existence of such a distribution, the so-called stationary distribution, is extremely important.

Definition 2.6. Let X be a homogeneous discrete-time Markov chain with a state space S and transition matrix $\boldsymbol{P}$. We say that a probability distribution $\boldsymbol{\pi} = (\pi_i)_{i \in S}$ is a stationary distribution for the Markov chain X if $\boldsymbol{\pi}$ satisfies the following equation:

$$\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \boldsymbol{P}. \tag{2.3}$$

Let us briefly comment on the above definition. Recall that if $(X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete-time Markov chain, we have that its distribution at a given time n can be computed as

$$\boldsymbol{p(n)}^T = \boldsymbol{p(0)}^T \boldsymbol{P}^n$$

where $\boldsymbol{p(0)}$ is the initial distribution of the Markov chain X. Let us assume that there exists a stationary distribution $\boldsymbol{\pi}$ for the chain X and that this stationary distribution is also its initial distribution. Then we have

$$\boldsymbol{p(n)}^T = \boldsymbol{p(0)}^T \boldsymbol{P}^T = \boldsymbol{\pi}^T \boldsymbol{P}^n = \boldsymbol{\pi}^T \boldsymbol{P} \cdot \boldsymbol{P}^{n-1} = \boldsymbol{\pi}^T \boldsymbol{P}^{n-1} = \ldots = \boldsymbol{\pi}^T.$$

Hence, we see that the stationary distribution does exactly what it was supposed to - if the Markov chain starts in its stationary distribution, then its distribution does not change in time (the right-hand side of the above chain of equalities does not depend on n).

Clearly, a stationary distribution must solve equation (2.3) but at the same time, it has to be a probability distribution on the state space. This may be true only if

$$\sum_{i \in S} \pi_i = 1. \tag{2.4}$$

Typically, when solving the system (2.3), one fixes an arbitrary π_0, finds a general solution in terms of π_0 and then tries to choose π_0 in such a way that the normalizing condition (2.4) holds.

Before we focus on the stabilizing properties of the stationary distribution, let us go back to the classification of states. We have already seen that if a homogeneous discrete-time Markov chain is irreducible and its state space is finite, then all its states are positive recurrent (Theorem 2.4). However, for Markov chains with an infinite state space this may not be true and we still need to use Theorem 2.1 (possibly in combination with Theorem 2.3) to determine whether the states are (positive or null) recurrent or transient. It turns out that the stationary distribution can be of great help in the state classification as stated in the following theorem.

Theorem 2.6. Let X be an irreducible homogeneous discrete-time Markov chain. Then X admits a stationary distribution if and only if all its states are positive recurrent.

Obviously, if the stationary distribution does not exist, then we are left with the question to determine if the states are all null-recurrent or transient. This can be decided using the following result.

Theorem 2.7. Let X be an irreducible homogeneous discrete-time Markov chain with the state space

$$S = \{0, 1, 2, \ldots\}$$

and the transition matrix $\boldsymbol{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$. Then all its states are recurrent if and only if the only solution of the system of equations

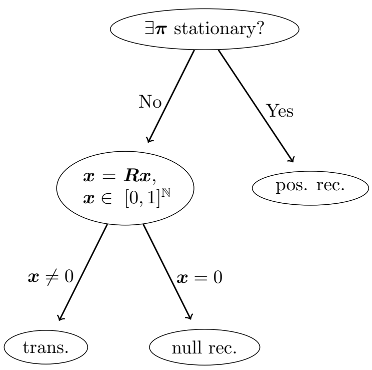
$$x_i = \sum_{j=1}^{\infty} p_{ij} x_j, \quad i = 1, 2, \ldots, \tag{2.5}$$

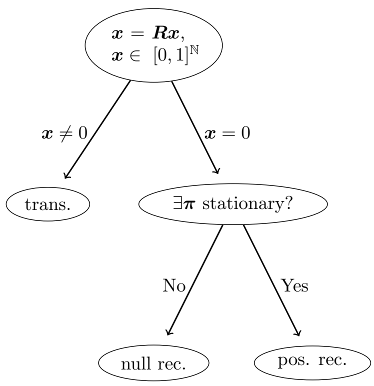
in the interval $[0, 1]$ is the trivial solution, i.e. $x_j = 0$ for all $j \in \mathbb{N}$.

If we adopt the notation $\boldsymbol{x} = (x_i)_{i\in\mathbb{N}}$ and $\boldsymbol{R} = (p_{ij})_{i,j\in\mathbb{N}}$ (notice that i, j are not zero - we simply forget the first column and row), the system (2.5) can be written as

$$\boldsymbol{x} = \boldsymbol{R}\boldsymbol{x}$$

which is to be solved for $\boldsymbol{x} \in [0, 1]^{\mathbb{N}}$. Theorems 2.6 and 2.7 together suggest two possible courses of action which can be taken. We either first try to find the stationary distribution and if that fails (i.e. the states can be either null-recurrent or transient), we can look at the reduced system (2.5) (which only tells us if all the states are recurrent or transient) or we can start by solving the reduced system (2.5) to see if the states are transient or recurrent and if we show that they are recurrent we can continue and look for a stationary distribution to see if the states are null or positive recurrent. This reasoning is summarized in the following cookbook:





Exercise 37. Classify the states of the homogeneous discrete-time Markov chain $X = (X_n, n \in \mathbb{N}_0)$ that

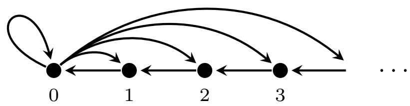
has the state space $S = \mathbb{N}_0$ and that is given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2^2} & \frac{1}{2^3} & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Solution to Exercise 37. We will follow the first guideline. First we should notice that the state space is countably infinite. When we look at the transition matrix $\boldsymbol{P}$, we see that $p_{0i} > 0$ for all $i \in \mathbb{N}_0$ and also $p_{i,i-1} > 0$ for $i \in \mathbb{N}$. Hence,

$$\begin{aligned} 0 &\to i, \quad i \in \mathbb{N}_0, \\ i &\to i-1, \quad i \in \mathbb{N}, \end{aligned}$$

and thus $i \leftrightarrow j$ for all $i, j \in S$. For a better visualisation we can draw the transition diagram of X:



This means that all the states of X are of the same type (Theorem 2.3). At this point, we are already able to determine the periodicity of all states. Since $p_{00} > 0$, the state 0 is aperiodic and thus, all the other states are aperiodic. In order to say more, we employ our cookbook according to which we have to decide if X admits a stationary distribution. We have to solve the equation $\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \boldsymbol{P}$ for an unknown stochastic vector $\boldsymbol{\pi}^T = (\pi_0, \pi_1, \pi_2, \ldots)$. The system reads as

$$\begin{aligned} \pi_0 &= \frac{1}{2}\pi_0 + \pi_1 \\ \pi_1 &= \frac{1}{2^2}\pi_0 + \pi_2 \\ \pi_2 &= \frac{1}{2^3}\pi_0 + \pi_3 \\ &\vdots \\ \pi_k &= \frac{1}{2^{k+1}}\pi_0 + \pi_{k+1}, \quad k = 0, 1, 2, \ldots \end{aligned} \tag{2.6}$$

which translates as

$$\begin{aligned} \pi_1 &= \pi_0 - \frac{1}{2}\pi_0 = \left(1 - \frac{1}{2}\right)\pi_0 \\ \pi_2 &= \pi_1 - \frac{1}{2^2}\pi_0 = \left(1 - \frac{1}{2} - \frac{1}{2^2}\right)\pi_0 \\ &\vdots \\ \pi_{k+1} &= \left(1 - \frac{1}{2} - \frac{1}{2^2} - \ldots - \frac{1}{2^{k+1}}\right)\pi_0, \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{2.7}$$

for a fixed π_0 whose value will be determined later. Formula (2.7) is easily proved from the recurrence (2.6) by induction (do not forget this step!). By solving for a general $k \in \mathbb{N}_0$, we obtain

$$\pi_{k+1} = \left(1 - \sum_{j=1}^{k+1} \frac{1}{2^j}\right)\pi_0 = \left(1 - \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{k+1}}}{1 - \frac{1}{2}}\right)\pi_0 = \frac{1}{2^{k+1}}\pi_0, \quad k = 0, 1, 2, \ldots$$

So far, we know that every vector $\boldsymbol{\pi}^T = (\pi_0, \frac{1}{2}\pi_0, \frac{1}{2^2}\pi_0, \ldots)$ solves the equation $\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \boldsymbol{P}$. Now we further require that the vector $\boldsymbol{\pi}$ is also a probability distribution, i.e. all its elements are non-negative and they sum up to 1.

$$1 \stackrel{!}{=} \sum_{k=0}^{\infty} \pi_k = \pi_0 + \sum_{k=1}^{\infty} \frac{1}{2^k}\pi_0 = 2\pi_0.$$

This means that the only π_0 for which $\boldsymbol{\pi}$ is a probability distribution on $\mathbb{N}_0$ is $\pi_0 = \frac{1}{2}$. Hence, we arrive at

$$\pi_k = \frac{1}{2^{k+1}}, \quad k = 0, 1, 2, \ldots$$

which is the sought stationary distribution of the Markov chain X. Hence, by Theorem 2.6, we know that the chain has only positive recurrent states. △

Exercise 38. Classify the states of the homogeneous discrete-time Markov chain $X = (X_n, n \in \mathbb{N}_0)$ whose state space is $S = \mathbb{N}_0$ and that is given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2\cdot 1} & \frac{1}{3\cdot 2} & \frac{1}{4\cdot 3} & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Solution to Exercise 38. Clearly the chain X has infinitely many states and in the same manner as in the previous exercise, all its states communicate and are aperiodic. Now we try to find the stationary distribution. The equation $\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \boldsymbol{P}$ reads as

$$\begin{aligned} \pi_0 &= \frac{1}{2\cdot 1}\pi_0 + \pi_1 \\ \pi_1 &= \frac{1}{3\cdot 2}\pi_0 + \pi_2 \\ &\vdots \\ \pi_k &= \frac{1}{(k+1)(k+2)}\pi_0 + \pi_{k+1}, \quad k = 0, 1, 2, \ldots, \end{aligned}$$

which translates as

$$\begin{aligned} \pi_1 &= \frac{1}{2}\pi_0 \\ \pi_2 &= \frac{1}{3}\pi_1 \\ \pi_3 &= \frac{1}{4}\pi_2 \\ &\vdots \end{aligned}$$

From the above system, we can guess that π_k satisfies

$$\pi_k = \frac{1}{k+1}\pi_0, \quad k = 1, 2, \ldots \tag{2.8}$$

This has to be proven by induction as follows. For $k = 1$, we immediately obtain that $\pi_1 = \frac{1}{2}\pi_0$ which is correct. Now we assume that formula (2.8) holds for some $k \in \mathbb{N}$ and we wish to show that it also holds for $k + 1$. We compute

$$\pi_{k+1} = \pi_k - \frac{1}{(k+1)(k+2)}\pi_0 = \frac{1}{k+1}\pi_0 - \frac{1}{(k+1)(k+2)}\pi_0 = \frac{1}{k+2}\pi_0$$

and hence, formula (2.8) is satisfied for π_{k+1}. This means that each vector $\boldsymbol{\pi}^T = (\pi_0, \frac{1}{2}\pi_0, \frac{1}{3}\pi_0, \ldots)$ is a good candidate for a stationary distribution. Now we have to choose π_0 in such a way that we have $\sum_{k=0}^{\infty} \pi_k = 1$. This is, however, impossible as the following computation shows.

$$1 \stackrel{!}{=} \sum_{k=0}^{\infty} \pi_k = \pi_0 + \sum_{k=1}^{\infty} \frac{1}{k+1}\pi_0 = \infty.$$

We thus see, that X does not admit a stationary distribution. Hence, the states cannot be positive recurrent and we need to decide whether they are null recurrent or transient. We define

$$\boldsymbol{R} := \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \ddots \\ 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

the $\mathbb{N} \times \mathbb{N}$ matrix which is obtained by crossing out the first row and column of $\boldsymbol{P}$. We need to solve $\boldsymbol{x} = \boldsymbol{R}\boldsymbol{x}$ for an unknown vector $\boldsymbol{x}^T = (x_1, x_2, \dots)$. We obtain

$$\begin{aligned} x_1 &= 0 \\ x_2 &= x_1 \\ &\vdots \\ x_{k+1} &= x_k, \quad k = 1, 2, \dots \end{aligned}$$

which implies that $0 = x_1 = x_2 = \dots$ and thus, this equation has only the trivial solution (particularly in the interval $[0,1]$). This means that all the states are recurrent and since they cannot be positive recurrent, they must be null-recurrent. △

When looking back at the previous solution, it helps to realize that the equation $\boldsymbol{x} = \boldsymbol{R}\boldsymbol{x}$ always admits the trivial solution. The question is whether this is the only solution which lives in the interval $[0,1]$ (more precisely, $\boldsymbol{x} \in [0,1]^{\mathbb{N}}$ or, $x_j \in [0,1]$ for all $j \in \mathbb{N}$). In Exercise 38, the interval $[0,1]$ was not important. However, sometimes the interval $[0,1]$ becomes important - it is in the situation when there is another solution to $\boldsymbol{x} = \boldsymbol{R}\boldsymbol{x}$ such that $x_j \to c$ as $j \to \infty$ where $c \in (1, \infty]$ (see hint to Exercise 40).

Exercise 39. Classify the states of the homogeneous discrete-time Markov chain $X = (X_n, n \in \mathbb{N}_0)$ whose state space is $S = \mathbb{N}_0$ and that is given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $0 < p_i < 1$ for all $i \in \mathbb{N}_0$ such that $\sum_{i=0}^{\infty} p_i = 1$.

Hint: Convince yourself that

$$\sum_{k=1}^{\infty} \left(1 - \sum_{j=0}^{k-1} p_j \right) = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} p_j = \sum_{k=1}^{\infty} k p_k.$$

Exercise 40. Classify the states of the homogeneous discrete-time Markov chain $X = (X_n, n \in \mathbb{N}_0)$ whose state space is $S = \mathbb{N}_0$ and that is given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & \cdots \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hint: Show that there is no stationary distribution and that the reduced system $\boldsymbol{x} = \boldsymbol{R}\boldsymbol{x}$ is solved by $x_k = \frac{k+1}{2} x_1$ for a fixed x_1. Now, if $x_1 \neq 0$, then $x_j \to \infty$ as $j \to \infty$ and hence, there must be an index j^* such that $x_{j^*} > 1$. Hence, one cannot choose $x_1 \neq 0$ in such a way that all x_k's are in the interval $[0,1]$ and we are left only with $x_1 = 0$ for which all x_k's belong to $[0,1]$.

Exercise 41. Classify the states of the homogeneous discrete-time Markov chain $X = (X_n, n \in \mathbb{N}_0)$ whose state space is $S = \mathbb{N}_0$ and that is given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & \cdots \\ \frac{2}{9} & \frac{2}{3} & \frac{1}{9} & 0 & 0 & \ddots \\ \frac{2}{27} & 0 & \frac{8}{9} & \frac{1}{27} & 0 & \ddots \\ \frac{2}{81} & 0 & 0 & \frac{26}{27} & \frac{1}{81} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Exercise 42. Classify the states of the homogeneous discrete-time Markov chain $X = (X_n, n \in \mathbb{N}_0)$ whose state space is $S = \mathbb{N}_0$ and that is given by the transition matrix

$$\mathbf{P} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & \cdots \\ \frac{3}{16} & \frac{3}{4} & \frac{1}{16} & 0 & 0 & \ddots \\ \frac{3}{64} & 0 & \frac{15}{16} & \frac{1}{64} & 0 & \ddots \\ \frac{3}{256} & 0 & 0 & \frac{63}{64} & \frac{1}{256} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Exercise 43. Classify the states of the homogeneous discrete-time Markov chain $X = (X_n, n \in \mathbb{N}_0)$ whose state space is $S = \mathbb{N}_0$ and that is given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $0 < p_i < 1$ for all $i \in \mathbb{N}_0$ such that $\sum_{i=0}^{\infty} p_i = 1$.

Exercise 44. Classify the states of the homogeneous discrete-time Markov chain $X = (X_n, n \in \mathbb{N}_0)$ whose state space is $S = \mathbb{N}_0$ and that is given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} q_0 & p_0 & 0 & 0 & \cdots \\ q_1 & 0 & p_1 & 0 & \cdots \\ q_2 & 0 & 0 & p_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $0 < p_i < 1$ for all $i \in \mathbb{N}_0$ and $q_i = 1 - p_i$.

Exercise 45. Classify the states of the homogeneous discrete-time Markov chain $X = (X_n, n \in \mathbb{N}_0)$ whose state space is $S = \mathbb{N}_0$ and that is given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} q_0 & 0 & p_0 & 0 & 0 & \cdots \\ q_1 & 0 & 0 & p_1 & 0 & \cdots \\ q_2 & 0 & 0 & 0 & p_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $0 < p_i < 1$ for all $i \in \mathbb{N}_0$ and $q_i = 1 - p_i$.

Hint: In this exercise, one could think that the state 1 (i.e. the one corresponding to the column of zeroes) is not reachable and should not belong to S. This depends on the initial distribution $\boldsymbol{p(0)}^T = (p(0)_0, p(0)_1, p(0)_2, \ldots)$. If $p(0)_1 = 0$, then we may cross out the state 1, relabel the states and this exercise becomes Exercise 44. Assume then that $p(0)_1 \neq 0$.

Exercise 46. Suppose a snail climbs an infinitely high tree. Each hour, the snail moves one centimetre up with probability 1/3 and it moves one centimetre down with probability 2/3 (it does not really want to fight gravity). If the snail reaches ground level, it moves one centimetre up in the next hour. Formulate the model for the position of the snail so that you obtain a homogeneous discrete-time Markov chain, find its transition matrix, and classify its states.

Exercise 47. Modify Exercise 46 in such a way that the probability of going up is $p \in (0, 1)$ and the probability of going down is $q := 1 - p$. Classify the states of the resulting Markov chain in terms of p.

Apart from being the steady-state distribution, the stationary distribution has a nice interpretation in terms of the mean return time to a given state. More precisely, there is the following result.

Theorem 2.8. Let X be an irreducible homogeneous discrete-time Markov chain with the state space S that has only positive recurrent states. Then its stationary distribution $\boldsymbol{\pi} = (\pi_i)_{i\in S}$ is unique and it satisfies the equality

$$\pi_i = \frac{1}{m_i}, \quad i \in S,$$

where $m_i := \mathbb{E}_i\tau_i(1)$ is the mean of the first return time to the state i (see Definition 2.4).

Theorem 2.8 provides a very efficient way how to compute the mean of the first return time m_i - we simply have to find the stationary distribution.

Exercise 48. Consider the standard chessboard with 8×8 squares and a knight figurine that moves around the board in such a way that in every step, it chooses one of the possible new positions that it can reach by a single jump uniformly at random and jumps there. Assume that the knight starts in one of the corners of the chessboard. What is the average number of steps it takes before it returns to its starting position?

Hint: Represent each square of the chessboard as a vertex of a graph (V, E) where two vertices are connected by an edge if and only if the knight can move from one square to the other by a single move. For $i, j \in V$ define

$$p_{ij} := \begin{cases} \frac{1}{v_i}, & (i,j) \in E, \\ 0, & \text{otherwise} \end{cases}$$

where v_i is the degree of the vertex i, i.e. the number of edges that lead to i. Show that $\boldsymbol{P} = (p_{ij})_{i,j\in V}$ defines a transition matrix of an irreducible homogeneous discrete-time Markov chain. Thus all its states are positive recurrent and there is a unique stationary distribution $\boldsymbol{\pi} = (\pi_i)_{i\in V}$. Show that

$$\pi_i = \frac{v_i}{\sum_{j\in V} v_j}, \quad i \in V$$

is the sought stationary distribution and compute $m_1 = \frac{1}{\pi_1}$ (we assume that the corner that the knight starts in is labelled by 1).

Definition 2.7. Let X be a homogeneous discrete-time Markov chain with the state space S. Provided it exits, the limit distribution for the Markov chain X is a probability distribution $\boldsymbol{\mu} = (\mu_i)_{i\in S}$ for which the convergence

$$\mu_j = \lim_{n\to\infty} p_{ij}^{(n)}$$

is satisfied for every $i, j \in S$.

Note that by the above definition, the limit distribution (if it exists) does not depend on the initial state $i \in S$. Of course, the limit distribution captures what the Markov chain looks like in the long run. It turns out that (again, if it exists) the limit distribution can be found quite easily.

Theorem 2.9. If a limit distribution for a Markov chain exists, then its stationary distribution also exists and the two distributions are equal.

In Example 19, we saw a Markov chain for which there is a limit distribution. However, we also have the following example that shows that sometimes, the limit distribution does not exist.

Exercise 49. Consider a homogeneous discrete-time Markov chain with the state space $S = \{0, 1\}$ defined by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that there is no limit distribution for this Markov chain.

Theorem 2.10. Assume that X is an irreducible aperiodic homogeneous discrete-time Markov chain that only has positive recurrent states. Then the limit distribution for the Markov chain X exists.

The limit distribution can be viewed as the long-run proportion of time that a Markov chain spends in a given state.

Exercise 50. Consider the mouse moving in our flat from Exercise 33. Assume that there are no traps and the doors to the garden are closed. We wish to analyse the behavioural patterns of our mouse. The mouse moves in the same way as before - in each room, it chooses (uniformly randomly) one of the adjacent rooms and moves there. What is the long-run proportion of time the mouse spends in each of the rooms?

2.5 Answers to Exercises

Answer to Exercise 12. The symmetric random walk is a homogeneous discrete-time Markov chain.

Answer to Exercise 13. The running maximum is a homogeneous discrete-time Markov chain.

Answer to Exercise 14. The process X is a homogeneous discrete-time Markov chain. What is more, the converse is also true as shown for example in [10, Proposition 7.6].

Answer to Exercise 17. We have that $p^{(n)}_{-1,-1} = \left(\frac{1}{3}\right)^n$, $p^{(n)}_{0,0} = \left(\frac{2}{3}\right)^n$ and $p^{(n)}_{1,1} = 1$. Hence, $-1, 0$ are aperiodic, transient states and 1 is an aperiodic, positive recurrent state (the so-called absorbing state).

Answer to Exercise 18. The transition matrix is

$$\boldsymbol{P} = \begin{pmatrix} q & p & 0 & 0 & \cdots \\ q & 0 & p & 0 & \cdots \\ q & 0 & 0 & p & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \boldsymbol{P}^n = \begin{pmatrix} q & qp & qp^2 & \cdots & qp^{n-1} & p^n & 0 & 0 & \cdots \\ q & qp & qp^2 & \cdots & qp^{n-1} & 0 & p^n & 0 & \cdots \\ q & qp & qp^2 & \cdots & qp^{n-1} & 0 & 0 & p^n & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

All the states are aperiodic, positive recurrent.

Answer to Exercise 21. X is a homogeneous discrete-time Markov chain with the state space $S = \{0, 1, \dots, k\}$ and the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{k} & 1-\frac{1}{k} & 0 & \ddots & 0 & 0 \\ 0 & 0 & \frac{2}{k} & 1-\frac{2}{k} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & \frac{k-1}{k} & 1-\frac{k-1}{k} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The states $0, \dots, k-1$ are transient and aperiodic, the state k is absorbing (i.e. positive recurrent and aperiodic).

Answer to Exercise 23. Denote X_n the number of balls in the first container at time n. Then $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete-time Markov chain whose state space is $S = \{0, 1, \dots, 2l\}$ and transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2l} & 0 & 1-\frac{1}{2l} & 0 & \cdots & 0 & 0 \\ 0 & \frac{2}{2l} & 0 & 1-\frac{2}{2l} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2l}{2l} & 0 \end{pmatrix}$$

(i.e. $p_{01} = p_{2l-1,2l} = 1$ and $p_{k,k-1} = \frac{k}{2l}, p_{k,k+1} = 1 - \frac{k}{2l}$ for $k = 1, \dots, 2l-1$ and $p_{ij} = 0$ otherwise). All the states are positive recurrent and 2-periodic.

Answer to Exercise 24. Denote by X_n the number of white balls in the first container. Then $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete-time Markov chain whose state space is $S = \{0, 1, \dots, l\}$ and transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \left(\frac{1}{l}\right)^2 & \frac{2(l-1)}{l^2} & \left(\frac{l-1}{l}\right)^2 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{2}{l}\right)^2 & \frac{4(l-2)}{l^2} & \left(\frac{l-2}{l}\right)^2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \left(\frac{l}{l}\right)^2 & 0 \end{pmatrix}$$

(i.e. $p_{0,1} = p_{l-1,l} = 1$ and $p_{k,k-1} = \left(\frac{k}{l}\right)^2, p_{k,k} = \frac{2k(l-k)}{l^2}, p_{k,k+1} = \left(\frac{l-k}{l}\right)^2$ for $k = 1, \dots, l-1$ and $p_{ij} = 0$ otherwise). All the states are positive recurrent and aperiodic.

Answer to Exercise 26. If $p \in (0,1)$, then the chain is irreducible and all the states are aperiodic, positive recurrent. If $p = 1$, then the chain is not irreducible, the state space can be written as $S = \{0\} \cup \{1\} \cup \{2,3\}$ where 0 and 1 are absorbing states and 2, 3 transient states and the absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{20} & u_{21} \\ u_{30} & u_{31} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

If $p = 0$, then the chain is not irreducible, the state space can be written as $S = \{0,1\} \cup \{2\} \cup \{3\}$ where 0, 1 are transient states, 2 and 3 are absorbing states and the absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{02} & u_{03} \\ u_{12} & u_{13} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Answer to Exercise 27. The chain is not irreducible, its state space can be written as $S = \{0,3\} \cup \{1,2,4\}$ where 0, 3 are aperiodic transient states and 1, 2, 4 are aperiodic positive recurrent states. The absorption probabilities are

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{01} & u_{02} & u_{04} \\ u_{31} & u_{32} & u_{34} \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{2}{21} & \frac{16}{21} & \frac{3}{21} \end{pmatrix}.$$

Answer to Exercise 28. The chain is not irreducible, its state space can be written as $S = \{1,3\} \cup \{0,2,4\}$ where 1, 3 are 2-periodic, transient and 0, 2, 4 are aperiodic, positive recurrent. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{10} & u_{12} & u_{14} \\ u_{30} & u_{32} & u_{34} \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 4 & 3 & 2 \\ 2 & 3 & 4 \end{pmatrix}.$$

Answer to Exercise 29. The chain is not irreducible, its state space can be written as $S = \{3,4\} \cup \{1,2,5,6\}$ where 3, 4 are aperiodic transient states and 1, 2, 5, 6 are aperiodic positive recurrent states. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{31} & u_{32} & u_{35} & u_{36} \\ u_{41} & u_{42} & u_{45} & u_{46} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 \end{pmatrix}.$$

Answer to Exercise 30. The chain is not irreducible, its state space can be written as $S = \{0\} \cup \{1,2\} \cup \{4\} \cup \{3,5,6\}$ where 0 is a aperiodic transient state, 1, 2 are 2-periodic transient states, 4 is an absorbing state and 3, 5, 6 are 3-periodic positive recurrent states. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{03} & u_{04} & u_{05} & u_{06} \\ u_{13} & u_{14} & u_{15} & u_{16} \\ u_{23} & u_{24} & u_{25} & u_{26} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{7}{9} & \frac{1}{9} & \frac{1}{9} & 0 \\ \frac{8}{9} & \frac{1}{18} & \frac{1}{18} & 0 \end{pmatrix}.$$

Answer to Exercise 31. This is the same Markov chain as in Exercise 29.

Answer to Exercise 32. The state space of X is $S = \{0,1,2,3,4,5\}$ and its transition matrix is given by

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

S can be written as $S = \{2,4\} \cup \{0,1,3,5\}$ where $2,4$ are aperiodic, transient states and $0,1,3,5$ are aperiodic positive recurrent states. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{20} & u_{21} & u_{23} & u_{25} \\ u_{40} & u_{41} & u_{43} & u_{45} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This is not surprising at all when you look at the transition diagram for X (draw this yourself!).

Answer to Exercise 33. The states T, H, L are 2-periodic, transient and K and B are both absorbing. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{TK} & u_{TB} \\ u_{HK} & u_{HB} \\ u_{LK} & u_{LB} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Answer to Exercise 35. Denote by (O) the state "outside". Then the states K, B and H are 2-periodic, transient and the states P and O are absorbing. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{HP} & u_{HO} \\ u_{KP} & u_{KO} \\ u_{BP} & u_{BO} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 & 5 \\ 4 & 4 \\ 5 & 3 \end{pmatrix}$$

and, in particular, $u_{BO} = 3/8$.

Answer to Exercise 39. The stationary distribution exists if and only if the sum $\sum_{k=1}^{\infty} kp_k < \infty$. If this is so, the states are all positive recurrent and aperiodic. If $\sum_{k=1}^{\infty} kp_k = \infty$, then there is no stationary distribution and all the states are null-recurrent and aperiodic.

Answer to Exercise 40. There is no stationary distribution. All the states are aperiodic and null-recurrent.

Answer to Exercise 41. There is no stationary distribution. All the states are aperiodic and null-recurrent.

Answer to Exercise 42. There is no stationary distribution. All the states are aperiodic and null-recurrent.

Answer to Exercise 43. There is a stationary distribution $\boldsymbol{\pi}^T = (p_k)_{k=0}^{\infty}$ where $p_0 = \frac{1}{2-p_0}$ and $p_k = \frac{p_k}{1-p_0}$ for $k \in \mathbb{N}$. All the states are aperiodic and positive recurrent.

Answer to Exercise 44. The stationary distribution exists if and only if $\sum_{k=1}^{\infty} \prod_{j=0}^{k-1} p_j < \infty$. If this is so, all the states are positive recurrent and aperiodic. If $\sum_{k=1}^{\infty} \prod_{j=0}^{k-1} p_j = \infty$, then all the states are null-recurrent and aperiodic.

Answer to Exercise 45. All the states are aperiodic. The state 1 is transient. If $\sum_{k=1}^{\infty}\prod_{j=0}^{k-1} p_j < \infty$, the states $0, 2, 3, 4, \ldots$ are positive recurrent. If $\sum_{k=1}^{\infty}\prod_{j=0}^{k-1} p_j = \infty$, then the state $0, 2, 3, 4, \ldots$ are null-recurrent.

Answer to Exercise 46. Let $Y = (Y_i, i \in \mathbb{N})$ be a sequence of independent, identically distributed random variables with values in $\{-1, 1\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let their common distribution be

$$\mathbb{P}(Y_1 = -1) = \frac{2}{3}, \quad \mathbb{P}(Y_1 = 1) = \frac{1}{3}.$$

Define $X_0 := 0$ and

$$X_{n+1} := \mathbf{1}_{[X_n=0]} + \mathbf{1}_{[X_n \neq 0]}(X_n + Y_{n+1}), \quad n = 0, 1, 2, \ldots$$

X_n represents the height in which the snail is at time n. Then $X = (X_n, n \in \mathbb{N})$ is a homogeneous discrete-time Markov chain with the state space $S = \mathbb{N}_0$. Its transition matrix is

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & \ddots \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

X admits a stationary distribution $\pi_0 = \frac{1}{4}$ and $\pi_k = \frac{3}{4}\left(\frac{1}{2}\right)^k$ for $k = 1, 2, \ldots$ All its states are 2-periodic and positive recurrent.

Answer to Exercise 47. Regardless of the value of the parameter p, X defined similarly as in Exercise 46 has 2-periodic states. If $p \leq q$, then all the states are null-recurrent. If $p > q$, then all the states are transient.

Answer to Exercise 48. The average number of step before the knight returns to its original position is $m_1 = 168$.

Answer to Exercise 50. In the long-run, the mouse spends 1/10 of time at the toilet, 3/10 of time in the hall, 2/10 of time in the kitchen, in the bedroom, and in the living room.

3 Continuous-time Markov chains

We start by recalling the notion of continuous-time Markov chains.

Definition 3.1. A $\mathbb{Z}$-valued random process $X = (X_t, t \geq 0)$ is called a continuous-time Markov chain with the state space S if

1. $S = \{i \in \mathbb{Z} : \exists\, t \geq 0 \text{ such that } \mathbb{P}(X_t = i) > 0\}$,
2. and the equality
$$\mathbb{P}(X_t = j | X_s = i, X_{s_n} = i_n, \ldots, X_{s_1} = i_1) = \mathbb{P}(X_t = j | X_s = i),$$
holds for all $n \in \mathbb{N}_0$ and all states $i, j, i_n, \ldots, i_1 \in S$ and times $0 \leq s_1 < s_2 < \ldots < s_n < s < t$ that satisfy the following condition:
$$\mathbb{P}(X_s = i, X_{s_n} = i_n, \ldots, X_{s_1} = i_1) > 0.$$

The first condition means that we only consider the effective states - those states we are able to reach at some time t and the second condition is called the Markov property. Again, roughly speaking, Markov property says that to determine the probability distribution of the process at a future time, we only need to know the current state and not the whole preceding history.

Similarly as in the previous section, for a continuous-time Markov chain X, we define the transition probability from the state i at the time s to the state j at the time $s + h$ by

$$p_{ij}(s, s+h) := \mathbb{P}(X_{s+h} = j | X_s = i)$$

where $i, j \in S$ and $s, h \geq 0$. With these, we can form the stochastic matrix

$$\boldsymbol{P(s, s+h)} = (p_{ij}(s, s+h))_{i,j \in S}$$

which is called the transition matrix of X from the time s to the time $s + h$.

Definition 3.2. If for every $h \geq 0$, there is a stochastic matrix $\boldsymbol{P(h)}$ such that the equality

$$\boldsymbol{P(s, s+h)} = \boldsymbol{P(h)}$$

is satisfied for all $s \geq 0$, then X is called a (time) homogeneous continuous-time Markov chain.

In the case of continuous-time Markov chains, there is no natural notion of a "step". Because of this, we cannot expect to obtain a formula analogous to $\boldsymbol{P(k)} = \boldsymbol{P}^k$ which we obtained for discrete-time chains. However, it turns out that the fundamental property of transition matrices of a homogeneous continuous-time Markov chain is the Chapman-Kolmogorov equality that reads as

$$\boldsymbol{P(s+t)} = \boldsymbol{P(s)P(t)} \tag{3.1}$$

and that holds for every $s, t \geq 0$ with $\boldsymbol{P(0)} := \boldsymbol{I}_S$. If we are interested in the probability distribution of the random variable X_t, denoted by $\boldsymbol{p(t)}$, we can compute it by using the formula

$$\boldsymbol{p(t)}^T = \boldsymbol{p(0)}^T \boldsymbol{P(t)}, \quad t \geq 0,$$

where $\boldsymbol{p(0)}$ denotes the initial distribution of the Markov chain X, i.e. the vector which contains the probabilities $\mathbb{P}(X_0 = k) > 0$ for $k \in S$.

3.1 Generator

So far, the definitions and properties were analogous to the discrete-time case. The major exception is that for a homogeneous chain, we do not have only one matrix $\boldsymbol{P}$ but a whole system of matrices $(\boldsymbol{P(t)}, t \geq 0)$ that is sometimes called the the transition semigroup of X. In order to describe the process

X, we need to know the matrix $\boldsymbol{P(t)}$ for every $t \geq 0$. When we had discrete-time Markov chains, we could generate the system $(\boldsymbol{P(k)}, k \in \mathbb{N}_0)$ simply by multiplying the matrix $\boldsymbol{P}$. Naturally, the question is whether it is possible to find one object that can be used to generate the continuous system $(\boldsymbol{P(t)}, t \geq 0)$? As it turns out, this really is the case.

Theorem 3.1. Assume that the transition probabilities of a homogeneous continuous-time Markov chain whose state state is S are right continuous functions at zero[a]. Then for every state $i \in S$ there exists the (finite or infinite) limit

$$\lim_{h \to 0+} \frac{1 - p_{ii}(h)}{h} =: q_i =: -q_{ii}$$

and for all $i, j \in S$ there exists the (finite) limit

$$\lim_{h \to 0+} \frac{p_{ij}(h)}{h} =: q_{ij}.$$

[a]That is, the convergence $\lim_{t \to 0+} p_{ij}(t) = \delta_{ij}$ holds for every $i, j \in S$.

In Theorem 3.1, the numbers q_{ij} for $i \neq j$ are called transition intensities or transition rates from $i \in S$ to $j \in S$ and the numbers q_i are called exit intensities/rates from the state $i \in S$. Now, if X is a homogeneous continuous-time Markov chain with the state space S for which it holds that $\sum_{j \in S} q_{ij} = 0$ for all $i \in S$, then it follows from Theorem 3.1 that we can define the matrix

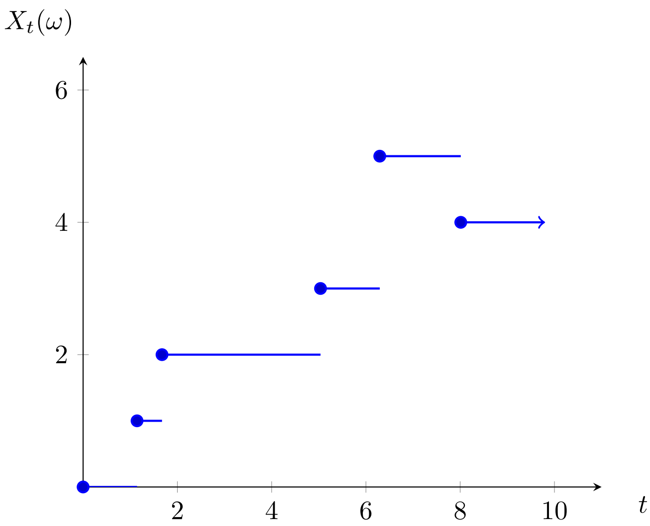
$$\boldsymbol{Q} := \lim_{h \to 0+} \frac{\boldsymbol{P(h)} - \boldsymbol{I}_S}{h}$$

which is called the intensity matrix, transition rate matrix or generator of the Markov chain X. As we will see in the sequel, the matrix $\boldsymbol{Q}$ can be used to generate the whole system $(\boldsymbol{P(t)}, t \geq 0)$.

For Theorem 3.1 and other technical reasons, we will always assume that every continuous-time Markov chain under consideration satisfies the following assumptions:

- X is homogeneous,
- transition probabilities are right continuous functions at zero,
- X has right continuous sample paths,
- $\sum_{j \in S} q_{ij} = 0$ for all $i \in S$.

A typical sample path $t \mapsto X_t(\omega)$ of a continuous-time Markov chain X is the following:



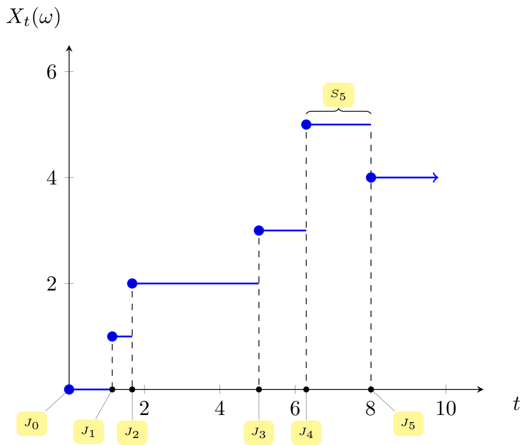
The picture suggests it might be reasonable to consider the following random variables. We will define

$$J_0 := 0$$
$$J_{n+1} := \inf(t > 0 : X_t \neq X_{J_n}), \quad n \in \mathbb{N}_0,$$

with the convention that $\inf \emptyset := \infty$. A careful reader should stop here. In general, there is no guarantee that X_{J_n} are random variables since J_n are themselves random variables. However, it can be shown that J_n are indeed measurable. The random variables J_n are called the jump times of the Markov chain X. Furthermore, we can define for $n \in \mathbb{N}_0$ the following random variables:

$$S_{n+1} := \begin{cases} J_{n+1} - J_n, & \text{if } J_n < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

The random variables S_n are called the holding times of the Markov chain X.

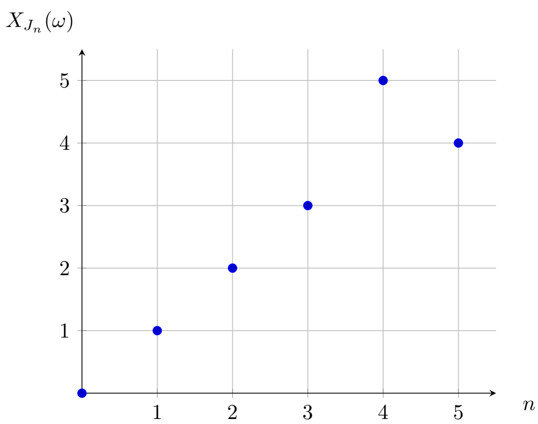


Finally, let us define the matrix

$$\boldsymbol{Q}^* = (q^*_{ij})_{i,j \in S}, \qquad \text{where} \qquad q^*_{ij} := \begin{cases} \frac{q_{ij}}{q_i} \mathbf{1}_{[i \neq j]}, & \text{if } q_i \neq 0 \\ \mathbf{1}_{[i=j]}, & \text{otherwise.} \end{cases}$$

Theorem 3.2. The stochastic process $Y = (Y_n, n \in \mathbb{N}_0)$ where $Y_n := X_{J_n}$ for $n \in \mathbb{N}_0$ is a homogeneous discrete-time Markov chain whose transition matrix is $\boldsymbol{Q}^*$.

The discrete-time Markov chain Y from Theorem 3.2 is called the embedded chain of the continuous-time Markov chain X. Theorem 3.2 says in particular that if X is in the state i (and stays there for some time), then the probability that the state which X will reach when it jumps next will be a given state j is $\frac{q_{ij}}{q_i}$ (we assume that $i \neq j$ and $q_i \neq 0$). A sample path of the embedded chain corresponding to our example is as follows:



The next theorem answers the question how long the Markov chain X will stay in a given state.

Theorem 3.3. Let $(S_n)_{n\in\mathbb{N}}$ be the holding times of a homogeneous continuous-time Markov chain. Then

$$S_{n+1}|Y_0,\dots,Y_n;S_1,\dots,S_n;[J_n<\infty] \quad \sim \quad \text{Exp}\,(q_{Y_n}), \quad n\in\mathbb{N}.$$

In particular, Theorem 3.3 says that the holding times are all independent and exponentially distributed random variables. So, if you ever want to simulate paths of continuous-time Markov chains, it suffices to generate independent exponentially distributed holding times (with appropriate parameters) and its embedded chain.

The analysis of continuous-time Markov chains is done in more steps. First, we need to built a model. This is typically done in such a way that we define the approximate behaviour of the process on a small time interval, i.e. we are given the Taylor expansion of $p_{ij}(h)$ up to the first order (or we have to construct it). This will be called the infinitesimal definition here. In the next step, we will find the generator $\boldsymbol{Q}$; and once we have $\boldsymbol{Q}$, we are able to obtain the whole system $(\boldsymbol{P(t)}, t\geq 0)$ which fully describes the time evolution of the process.

$$\boldsymbol{P(h)} \text{ for small } h \longrightarrow \boldsymbol{Q} \longrightarrow \boldsymbol{P(t)} \text{ for every } t\geq 0$$

Hence, we need to be able to find the generator $\boldsymbol{Q}$. This is practised in the following exercises.

Exercise 51 (Poisson process)**.** Consider certain events which repeatedly occur randomly in continuous-time (e.g. incoming calls, arrivals of customers). We assume that the numbers of events which occur in disjoint time intervals are independent random variables whose probability distributions only depend on the length of these time intervals. Let $\lambda>0$ and consider a small time interval $(t,t+h]$. We assume that there is exactly one event in the interval $(t,t+h]$ with probability $\lambda h+o(h)$, $h\to 0+$; two or more events with probability $o(h)$, $h\to 0+$; and no event with probability $1-\lambda h+o(h)$, $h\to 0+$. Let $N_0:=0$ and let N_t denote the number of events which occurred in the interval $(0,t]$. Find the generator $\boldsymbol{Q}$ of the homogeneous continuous-time Markov chain $(N_t,t\geq 0)$ and find the transition matrix $\boldsymbol{Q^*}$ of its embedded chain. More rigorously, the chain is defined as follows:

Definition 3.3. Let $\lambda>0$. The stochastic process $N=(N_t,t\geq 0)$ of count random variables is called the Poisson process with the parameter λ if

1. $N_0=0$ is satisfied almost surely,
2. the process N has independent and stationary increments,
3. $\mathbb{P}(N_{t+h}-N_t=1)=\lambda h+o(h)$, $\mathbb{P}(N_{t+h}-N_t\geq 2)=o(h)$ for $h\to 0+$ for all $t\geq 0$.

Exercise 52. Show that Definition 3.3 is equivallent to the following definition that justifies its name.

Definition 3.4. Let $\lambda>0$. The stochastic process $N=(N_t,t\geq 0)$ of count random variables is called the Poisson process with parameter λ if

1. $N_0=0$,
2. the process N has independent increments,
3. $N_{t+s}-N_s\sim \text{Po}\,(\lambda s)$ for all times $s,t\geq 0$.

Exercise 53 (Linear birth-death process)**.** Consider a population of identical reproducing and dying organisms. Assume that each organism dies within the time interval $(t,t+h]$ with probability $\mu h+o(h)$, $h\to 0+$. Similarly, in $(t,t+h]$, each organism gives birth to exactly one organism with probability $\lambda h+o(h)$, $\to 0+$, to two or more organisms with probability $o(h)$, $h\to 0+$ or it will not reproduce with probability $1-\lambda h+o(h)$, $h\to 0+$. The fates of individual organisms are independent and $\mu>0,\lambda>0$. Let X_t be the size of the population at time $t\geq 0$. Find the generator $\boldsymbol{Q}$ of the homogeneous continuous-time Markov chain $(X_t,t\geq 0)$ and the transition matrix of its embedded chain $\boldsymbol{Q^*}$.

Exercise 54 (Telephone switchboard)**.** Consider a telephone switchboard with N lines. Assume that there will be exactly one incoming phone call within the time interval $(t, t+h]$ with probability $\lambda h + o(h)$, $h \to 0+$ and $\lambda > 0$. The probability is the same for all $t \geq 0$. The probability that two or more calls will come within the same interval is $o(h)$, $h \to 0+$. No phone call will come with probability $1 - \lambda h + o(h)$, $h \to 0+$. All the incoming calls are independent of each other and if all the N lines are engaged, then the incoming phone call is lost. Assume that the length of a phone call has exponential distribution with mean $1/\mu$, $\mu > 0$.

1. Find the probability that a phone call ends within the time interval $(t, t+h]$ if we know that it has not ended by the time t.
2. Denote by X_t the number of engaged lines at time t. Find the generator of the homogeneous continuous-time Markov chain $X = (X_t, t \geq 0)$.
3. Find the transition matrix $\boldsymbol{Q}^*$ of the embedded chain.

Exercise 55 (Factory machine)**.** A factory machine is either operational or not. The time for which it does work is exponentially distributed with mean $1/\lambda$, $\lambda > 0$. Then it breaks and has to be repaired. The time it takes to repair the machine is exponentially distributed with mean $1/\mu$, $\mu > 0$. Model the state of the machine (works/does not work) by a homogeneous continuous-time Markov chain, find its generator $\boldsymbol{Q}$ and the transition matrix $\boldsymbol{Q}^*$ of its embedded chain.

Exercise 56 (Unimolecular reaction)**.** Consider a chemical reaction during which the molecules of the compound A change irreversibly to the molecules of the compound B. Suppose the initial concentration of A is N molecules. If there are j molecules of A at time t, then each of these molecules changes to a B molecule within the time interval $(t, t+h]$ with probability $qh + o(h)$, $h \to 0+$ where $q > 0$. Model the number of A molecules by a homogeneous continuous-time Markov chain, find its generator $\boldsymbol{Q}$ and the transition matrix $\boldsymbol{Q}^*$ of its embedded chain.

Exercise 57 (Bimolecular reaction)**.** Consider a liquid solution of two chemical compounds A and B. The chemical reaction is described as follows:

$$\mathrm{A} + \mathrm{B} \to \mathrm{C}$$

i.e. one A molecule reacts with one B molecule and they produce one C molecule. Let the initial concentrations of the compounds A, B, and C be a, b, and 0 molecules, respectively, and set $N := \min\{a, b\}$. If there are j molecules of C at time t, then the reaction will produce exactly one molecule of the compound C within the time interval $(t, t+h]$ with probability

$$q(a-j)(b-j)h + o(h), \quad h \to 0+$$

for $j = 0, 1, \ldots, N$. Model the concentration of C at time t by a homogeneous continuous-time Markov chain $X = (X_t, t \geq 0)$, find its generator $\boldsymbol{Q}$ and the transition matrix $\boldsymbol{Q}^*$ of its embedded chain.

3.2 Classification of states

In this section, we briefly revisit the concept of irreducibility and transience/recurrence of a Markov chain from the perspective of continuous-time Markov chains.

Definition 3.5. Let X be a homogeneous continuous-time Markov chain with the state space S. We say that the state $j \in S$ is accessible from the state $i \in S$ (and write $i \to j$) if there is $t > 0$ such that

$$\mathbb{P}_i(X_t = j) = p_{ij}(t) > 0.$$

If the states i, j are mutually accessible, then we say that they communicate (and write $i \leftrightarrow j$). X is called irreducible if all its states communicate..

It can be shown that $i \to j$ in the continuous-time Markov chains X if and only if $i \to j$ in its embedded discrete-time Markov chain Y.

Definition 3.6. Let X be a homogeneous continuous-time Markov chain with the state space S. Denote by

$$T_i(1) := \inf(t \geq J_1 : X_t = i)$$

the first return time to the state $i \in S$. The state i is called

- recurrent if either $(q_i = 0)$ or $(q_i > 0$ and $\mathbb{P}_i(T_i(1) < \infty) = 1)$,
- transient if $(q_i > 0$ and $\mathbb{P}_i(T_i(1) = \infty) > 0)$.

If $i \in S$ is recurrent, we call it

- positive if either $(q_i = 0)$ or $(q_i > 0$ and $\mathbb{E}_i T_i(1) < \infty)$,
- null if $(q_i > 0$ and $\mathbb{E}_i T_i(1) = \infty)$.

The intuition behind these definitions is the same as in the discrete-time case. If a state is recurrent, the chain will revisit it in a finite time almost surely. If it is transient, it might happen (with positive probability), that the state will never be visited again.

In fact, if we consider a chain X which describes a moving particle (on, say, $\mathbb{N}_0$), then if the chain turns out to have only transient states, then the particle drifts to infinity almost surely. In this case, the state space should be enriched with the state ∞, an absorbing state, which may be reached in finite or infinite time. More precisely, for any irreducible homogeneous continuous-time Markov chain X on $\mathbb{N}_0$ with transient states, it holds for every $i \in S$ that

$$\mathbb{P}_i(\lim_{t\to\infty} X_t = \infty) = 1.$$

Recurrent states, on the other hand, are visited infinitely many times. However, the mean time it may take the process to revisit a given state can be either finite or infinite. These two options correspond to either positive recurrence or null recurrence of the chain.

Definition 3.7. Let X be a continuous-time Markov chain with the state space S. Let $i \in S$.

- If $q_i = 0$, then we call the state i absorbing.
- If $q_i \in (0, \infty)$, then we call the state i stable.
- If $q_i = \infty$, then we call the state i unstable.

Clearly, if $q_i = 0$, then from the definition of the embedded chain we can see, that $q_{ii}^* = 1$ which means that i is absorbing (in the embedded chain). The interpretation of q_i is the rate at which the chain jumps to a different state. The higher q_i is, the less likely the chain is to stay in i.

Theorem 3.4. Let X be a homogeneous continuous-time Markov chain with the state space S and let Y be its embedded chain. Then a state $i \in S$ is recurrent in X if and only if i is recurrent in Y.

Exercise 58 (General birth-death process)**.** Let $X = (X_t, t \geq 0)$ be a homogeneous continuous-time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & 0 & 0 & \ddots \\ 0 & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 & 0 & \ddots \\ 0 & 0 & \mu_3 & -(\mu_3+\lambda_3) & \lambda_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{3.2}$$

where $\mu_i > 0$ for all $i \in \mathbb{N}$ and $\lambda_i > 0$ for all $i \in \mathbb{N}_0$. Show that all the states of the Markov chain X are recurrent if and only if

$$\sum_{n=1}^{\infty} \frac{\mu_1 \mu_2 \cdot \ldots \cdot \mu_n}{\lambda_1 \lambda_2 \cdot \ldots \cdot \lambda_n} = \infty. \tag{3.3}$$

Hint: Find the transition matrix $\boldsymbol{Q}^*$ of the embedded chain and use Theorem 2.7.

3.3 Kolmogorov differential equations

In this section, we show how to obtain the system $(\boldsymbol{P(t)}, t \geq 0)$ of a given homogeneous continuous-time Markov chain from its generator $\boldsymbol{Q}$. The cookbook is provided by the following theorem.

Theorem 3.5 (Kolmogorov differential equations (KDE))**.** Let $X = (X_t, t \geq 0)$ be a homogeneous continuous-time Markov chain with a state space S and the generator $\boldsymbol{Q} = (q_{ij})_{i,j \in S}$. Assume that the finiteness condition $q_i < \infty$ is satisfied for every $i \in S$. Then the following claims are true:

1. The transition probabilities $p_{ij}(\cdot)$ of X are differentiable in $(0, \infty)$ and it holds that

$$\boldsymbol{P'(t)} = \boldsymbol{Q}\boldsymbol{P(t)}, \quad t > 0. \tag{3.4}$$

2. If, moreover, there is the convergence[a]

$$\lim_{h \to 0+} \sup_{j \in S} \left| \frac{p_{ij}(h)}{h} - q_{ij} \right| = 0, \quad j \in S, \tag{3.5}$$

   then it also holds that

$$\boldsymbol{P'(t)} = \boldsymbol{P(t)}\boldsymbol{Q}, \quad t > 0. \tag{3.6}$$

[a] That is, for every $j \in S$, $p_{ij}(h)/h$ converges to q_{ij} as $h \to 0+$ uniformly in $i \in S$.

The first differential equation, (3.4), always holds for chains with stable states and it is called the Kolmogorov backward equation. The second one, called the Kolmogorov forward equation, holds, for example, for stable chains with finite state space (in which case there is the uniform convergence (3.5)).

Theorem 3.6 (Solution to KDE)**.** Let X be a homogeneous continuous-time Markov chain whose state space S is finite and that has only stable states. Then both equations (3.4) and (3.6) admit a unique solution that satisfies the initial condition $\boldsymbol{P(0)} = \boldsymbol{I}_S$. The solution is the same for both equations and it can be written as the matrix exponential

$$\boldsymbol{P(t)} = \mathrm{e}^{\boldsymbol{Q}t} = \sum_{k=0}^{\infty} \frac{t^k \boldsymbol{Q}^k}{k!}, \quad t \geq 0.$$

If we want to use Theorem 3.6, we have two options. We can either use the definition of matrix exponential (as an infinite sum) for which we will need to know $\boldsymbol{Q}^k$ for all $k \in \mathbb{N}_0$ or we can use the fact that the solution is matrix exponential directly via the following Perron's formula for holomorphic functions.

Theorem 3.7 (Perron's formula for holomorphic functions)**.** Let $f : U(0, R) \to \mathbb{C}$ be a holomorphic function on some R-neighbourhood of 0, $0 < R \leq \infty$ and let $\boldsymbol{A}$ be a square matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ whose multiplicities are $m_1, m_2, \ldots, m_k$. If $|\lambda_j| < R$ for all $j = 1, 2, \ldots, k$, then

$$f(\boldsymbol{A}) = \sum_{j=1}^{k} \frac{1}{(m_j - 1)!} \frac{\mathrm{d}^{m_j - 1}}{\mathrm{d}\lambda^{m_j - 1}} \left[f(\lambda) \frac{\mathbf{Adj}(\boldsymbol{\lambda I} - \boldsymbol{A})}{\psi_j(\lambda)} \right]_{\lambda = \lambda_j}$$

where

$$\psi_j(\lambda) = \frac{\det(\boldsymbol{\lambda I} - \boldsymbol{A})}{(\lambda - \lambda_j)^{m_j}}.$$

Exercise 59. Let X be the homogeneous continuous-time Markov chain with the state space $S = \{0, 1\}$ and the generator $\boldsymbol{Q}$ that is given by

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

where $\lambda > 0$ and $\mu > 0$. Compute the transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ of X and the distribution of X_t for all $t > 0$ if the initial distribution of X on S is $\mathbf{p(0)}^T = (p, q)$ where $0 < 1 - q = p < 1$.

Exercise 60. Let X be the homogeneous continuous-time Markov chain with the state space $S = \{0, 1, 2\}$ and the generator $\boldsymbol{Q}$ that is given by

$$\boldsymbol{Q} = \begin{pmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}.$$

Compute the transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ of X and the distribution of X_t for all $t > 0$ if the initial distribution of X is uniform on S.

Exercise 61. Let X be the homogeneous continuous-time Markov chain with the state space $S = \{0, 1, 2\}$ and the generator $\boldsymbol{Q}$ that is given by

$$\boldsymbol{Q} = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{pmatrix}.$$

Compute the transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ of X and the distribution of X_t for all $t > 0$ if the initial distribution of X is uniform on S.

3.4 Explosive and non-explosive chains

In this section, we will take a closer look at the number of jumps that can occur within a bounded time interval. Two cases are distinguished. Either the number of jumps is finite or it is countably infinite.

Definition 3.8 (Regular chains)**.** Let X be a homogeneous continuous-time Markov chain with the state space S and assume that all the states of the Markov chain X are stable. Denote

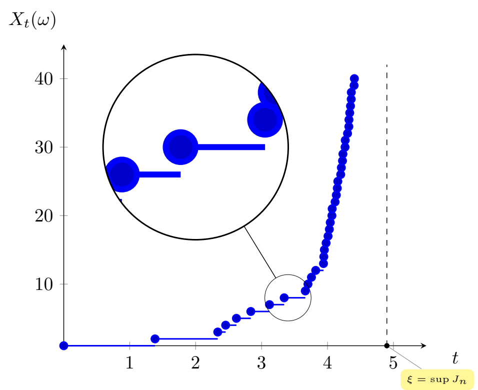
$$\xi := \sup J_n = \sum_{k=1}^{\infty} S_k.$$

The Markov chain X is called regular or non-explosive if the condition

$$\mathbb{P}_i(\xi = \infty) = 1$$

is satisfied for every $i \in S$. The $N_0 \cup \{\infty\}$-valued random variable ξ is called the time to explosion.

If a chain X is regular, then within a bounded time interval only a finite number of jumps can occur. In the following picture, there is a sample path of an explosive Markov chain.



It is a natural question whether non-explosive homogeneous continuous-time Markov chains can be characterized. Such characterization is given in the following theorem.

Theorem 3.8 (Characterization of non-explosive chains)**.** Let X be a homogeneous continuous-time Markov chain with the state space $S = \mathbb{N}_0$, the generator $\boldsymbol{Q}$, and with the embedded chain Y. Then X is non-explosive if and only if the condition

$$\mathbb{P}_i\left(\sum_{k=0}^{\infty} \frac{1}{q_{Y_k}} = \infty\right) = 1$$

is satisfied for every state $i \in \mathbb{N}_0$.

Exercise 62. Show that if the exit rates $\{q_i\}_{i=0}^{\infty}$ of a homogeneous continuous-time Markov chain X with state space $S = \mathbb{N}_0$ are uniformly bounded (i.e. there exists a finite positive constant C such that the inequality $\sup_{i\in\mathbb{N}_0} q_i \leq C$ is satisfied), then the Markov chain X is non-explosive.

Exercise 63. Show that if a homogeneous continuous-time Markov chain with the state space $S = \mathbb{N}_0$ is irreducible and all its states are recurrent, then the Markov chain X is non-explosive.

Exercise 64. Decide whether the Poisson process is non-explosive or explosive.

Exercise 65. Let X be the homogeneous continuous-time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -4 & 4 & 0 & 0 & \ddots \\ 0 & 0 & -9 & 9 & 0 & \ddots \\ 0 & 0 & 0 & -16 & 16 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Decide whether the Markov chain X is explosive or non-explosive.

3.5 Steady-state and large-time behaviour

In this section, stability of continuous-time Markov chains will be discussed.

Definition 3.9. Let $X = (X_t, t \geq 0)$ be a homogeneous continuous-time Markov chain with a state space S and a transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$.

- A vector $\boldsymbol{\eta}^T = (\eta_i)_{i\in S}$ with non-negative entries is called an invariant measure of the Markov chain X if the equation
$$\boldsymbol{\eta}^T = \boldsymbol{\eta}^T \boldsymbol{P(t)} \tag{3.7}$$
is satisfied for every time $t \geq 0$.
- An invariant measure $\boldsymbol{\eta}$ of X is called the stationary distribution of X if it is also a probability distribution on its state space S, i.e. if η_i it holds that $\sum_{i\in S} \eta_i = 1$.

Note that as opposed to the discrete-time case, the situation in the continuous-time case is more complicated. Specifically, in the discrete-time case, it was sufficient to find a probability vector $\boldsymbol{\pi}^T$ that satisfies the equality $\boldsymbol{\pi}^T = \boldsymbol{\pi}^T\boldsymbol{P}$. Then, by induction, that the equality

$$\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \boldsymbol{P(n)}$$

is satisfied for every step $n \in \mathbb{N}_0$. As it turns out however, even in the continuous-time case, the balance equation (3.7) can be reduced to a single equation.

Theorem 3.9 (Existence of invariant measure)**.** Let X be a homogeneous continuous-time Markov chain with the generator $\boldsymbol{Q}$ and the embedded chain Y. Assume that the chain Y is irreducible and all its states are recurrent. Then X has an invariant measure $\boldsymbol{\eta} = (\eta_i)_{i\in S}$ which is the unique (up to a multiplicative constant) positive ($\eta_i > 0$ for every $i \in S$) solution to the equation

$$\boldsymbol{\eta}^T \boldsymbol{Q} = \boldsymbol{0}^T. \tag{3.8}$$

Furthermore, if

$$\sum_{i\in S} \eta_i < \infty,$$

then the chain X has positive recurrent states and $\boldsymbol{\pi} = (\pi_i)_{i\in S}$ where

$$\pi_i := \frac{\eta_i}{\sum_{k\in S} \eta_k}, \quad i \in S,$$

is its stationary distribution. If

$$\sum_{i\in S} \eta_i = \infty,$$

then X has null recurrent states and it does not admit a stationary distribution.

Exercise 66. Consider the Markov chain X with state space $S = \{0, 1\}$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Show that the stationary distribution of X exists and find it.

Exercise 67. There are N workers in a factory, each of whom uses a certain electrical device. A worker that is not using the device at time t will start using it within the time interval $(t, t+h]$ with probability $\lambda h + o(h)$, $h \to 0+$, independently of the other workers. A worker who is using the device at time t will stop using it within the interval $(t, t+h]$ with probability $\mu h + o(h)$, $h \to 0+$. Here, $\lambda > 0$ and $\mu > 0$ are fixed parameters. Model the number of workers using the device at time t by a continuous-time Markov chain. Find its generator $\boldsymbol{Q}$, show that its stationary distribution exists, and find it.

Exercise 68. Consider the telephone switchboard model from Exercise 54. Decide whether there is a steady-state regime for the system and if there is, find it.

Exercise 69. Let $0 < p = 1 - q < 1$. Let X be the homogeneous continuous-time Markov chain with the state space $S = \mathbb{N}_0$ and the generator $\boldsymbol{Q}$ that is given by

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots \\ pq & -p & p^2 & 0 & \cdots \\ p^2q & 0 & -p^2 & p^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Decide, whether the stationary distribution of X exists or not; and if it does exist, find it.

Exercise 70. Let X be the continuous-time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -3 & 2 & 0 & 0 & \cdots \\ 1 & 0 & -4 & 3 & 0 & \cdots \\ 1 & 0 & 0 & -5 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Decide, whether the stationary distribution of X exists or not; and if it does exist, find it.

Exercise 71 ($M|M|3$)**.** Assume that customers arriving to pay for their grocery are governed by the Poisson process, i.e. the times between two consecutive arrivals are independent exponentially distributed random variables with mean 10 seconds. The customers are served at three cash desks and if these are full, the customers wait in a queue that is common for both cash desks and that can be (potentially) infinitely long. The mean time it takes for a customer to pay is 20 seconds and it is assumed that the times needed for different customers to pay are independent. Model the number of customers in the system (both at the cash desks and in the queue) by a continuous-time Markov chain X.

1. Prove that there is a steady-state regime (i.e. stationary distribution) for the system.
2. What is the mean number of customers being served in the steady-state regime?
3. What is the mean number of customers waiting in the queue in the steady-state regime?
4. What is the probability that an arriving customer will not need to wait in the queue?

Exercise 72. Let X be a recurrent birth-death process, i.e. X is the homogeneous continuous-time Markov chain with the state space $S = \mathbb{N}_0$ and the generator $\boldsymbol{Q}$ that is given by formula (3.2) such that the condition (3.3) is satisfied. Show that the Markov chain X has positive recurrent states if and only if

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty. \tag{3.9}$$

Show also that if the finiteness condition (3.9) is satisfied, then the chain X admits a stationary distribution that is given by

$$\pi_j := \frac{\rho_j}{\sum_{n=0}^{\infty} \rho_n}, \quad j \in \mathbb{N}_0, \tag{3.10}$$

where $\rho_0 := 1$ and $\rho_n := \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$ for $n \in \mathbb{N}$.

The following example shows that a homogeneous continuous-time Markov chain can have positive recurrent states (in the sense of Definition 3.6) while its embedded chain has null recurrent states (in the sense of Definition 2.4).

Exercise 73. Let X be the continuous-time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \ddots \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & \ddots \\ 0 & 0 & \frac{1}{9} & -\frac{2}{9} & \frac{1}{9} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Show that X has only positive recurrent states but its embedded chain has null recurrent states.

Solution to Exercise 73. The process X is a birth-death process with $\lambda_0 = 1$ and $\mu_n = \frac{1}{n^2} = \lambda_n$ for $n \in \mathbb{N}$. We have that

$$\sum_{n=1}^{\infty} \frac{\mu_1 \cdot \ldots \cdot \mu_n}{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_n} = \infty$$

so that the chain X has recurrent states by Exercise 58. Moreover, we have that

$$\sum_{n=1}^{\infty} \rho_n = \sum_{n=1}^{\infty} \frac{\lambda_0 \cdot \ldots \cdot \lambda_{n-1}}{\mu_1 \cdot \ldots \cdot \mu_n} = \sum_{n=1}^{\infty} \frac{\lambda_0}{\mu_n} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

so that the states of X are positive recurrent by Exercise 72. On the other hand, the transition matrix of the embedded chain of the Markov chain X is given by

$$\boldsymbol{Q}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \ddots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

which is the transition matrix of a symmetric random walk on $\mathbb{N}_0$ with a reflecting boundary at 0 which can be shown to have only null-recurrent states similarly as in section 2. △

The following theorem says that the stationary distribution is closely connected to a limit distribution of a Markov chain which, as in the discrete-time case, can be viewed as the long-run proportion of time that the Markov chain spends in a given state.

Theorem 3.10 (Existence of limit measure)**.** Let X be a homogeneous continuous-time Markov chain with the generator $\boldsymbol{Q}$ and the embedded chain Y. Assume that the chain Y is irreducible and all its states are recurrent. If $\boldsymbol{\pi} = (\pi_i)_{i\in S}$ is a solution to equation (3.7) such that $\pi_i > 0$ and $\sum_{i\in S} \pi_i = 1$, then it holds for the transition probabilities $\boldsymbol{P}(t) = (p_{ij}(t))_{i,j\in S}$ of the chain X that

$$\lim_{t\to\infty} p_{ij}(t) = \pi_j, \quad i, j \in S,$$

and for the distribution $\boldsymbol{p(t)} = (p_j(t))_{j\in S}$ of the chain X that

$$\lim_{t\to\infty} p_j(t) = \pi_j, \quad j \in S.$$

Exercise 74 ($M|M|\infty$)**.** Assume that a web page is being visited according to the Poisson process, i.e. the times between two consecutive displays of the web page are independent random variables with the exponential distribution with mean $\frac{1}{\lambda}$ for some parameter $\lambda > 0$. Assume that the time each visitor spends viewing the web page is also exponentially distributed and has mean $\frac{1}{\mu}$ for some parameter $\mu > 0$; and assume also that the viewing times of the visitors are mutually independent. Finally, we assume that there is no limit to how many visitors can view the web page at the same time. What is the long-run proportion of time that the web page is viewed by $k \in \mathbb{N}_0$ visitors at the same time?

3.6 Answers to exercises

Answer to Exercise 51. The state space of the Poisson proces N is $S = \mathbb{N}_0$, its generator $\boldsymbol{Q}$ and the transition matrix of its embedded chain $\boldsymbol{Q}^*$ are

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & \ddots \\ 0 & 0 & -\lambda & \lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \boldsymbol{Q}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \ddots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Answer to Exercise 53. The state space of the linear birth-death process is $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & \ddots \\ 0 & 2\mu & -2(\mu+\lambda) & 2\lambda & 0 & \ddots \\ 0 & 0 & 3\mu & -3(\mu+\lambda) & 3\lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Its embedded chain is an asymmetric random walk with the absorbing state 0 whose transition matrix is

$$\boldsymbol{Q}^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} & 0 & 0 & \ddots \\ 0 & \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} & 0 & \ddots \\ 0 & 0 & \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Answer to Exercise 54. The probability that one incoming call ends within the interval $(t, t+h]$ for small $h > 0$, is $1 - e^{-\mu h}$. The state space of X is $S = \{0, 1, \ldots, N\}$ and its generator is

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\mu & -(2\mu+\lambda) & \lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (N-1)\mu & -((N-1)\mu+\lambda) & \lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & N\mu & -N\mu \end{pmatrix}.$$

The embedded chain of X is a random walk on S with reflecting boundaries and its transition matrix is

$$\boldsymbol{Q}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{2\mu}{2\mu+\lambda} & 0 & \frac{\lambda}{2\mu+\lambda} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{(N-1)\mu}{(N-1)\mu+\lambda} & 0 & \frac{\lambda}{(N-1)\mu+\lambda} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Answer to Exercise 55. The state space of X is $S = \{0, 1\}$ where 0 represents "the machine does not work" and 1 represents "the machine works". The generator and the transition matrix of the embedded chain are

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}, \quad \boldsymbol{Q}^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Answer to Exercise 56. The state space of X is $S = \{0, 1, \ldots, N\}$ and its embedded chain is the left-shift process with absorbing state 0. The generator and transition matrix of the embedded chain are

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ q & -q & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2q & -2q & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3q & -3q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & Nq & -Nq \end{pmatrix}, \quad \boldsymbol{Q}^* = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Answer to Exercise 57. The state space of X is $S = \{0, 1, \ldots, N\}$ and its embedded chain is the right-shift with the absorbing state N. The generator of X is

$$\boldsymbol{Q} = \begin{pmatrix} -abq & abq & 0 & 0 & \cdots & 0 & 0 \\ 0 & -(a-1)(b-1)q & (a-1)(b-1)q & 0 & \cdots & 0 & 0 \\ 0 & 0 & -(a-2)(b-2)q & (a-2)(b-2)q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and the transition matrix of the embedded chain is

$$\boldsymbol{Q}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Answer to Exercise 59. The transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ is

$$\boldsymbol{P(t)} = \frac{1}{\lambda + \mu} \begin{pmatrix} \mu + \lambda e^{-(\lambda+\mu)t} & \lambda - \lambda e^{-(\lambda+\mu)t} \\ \mu - \mu e^{-(\lambda+\mu)t} & \lambda + \mu e^{-(\lambda+\mu)t} \end{pmatrix}.$$

The distribution of X_t is

$$\boldsymbol{p(t)}^T = \frac{1}{\lambda + \mu} \left(\mu + (p\lambda - q\mu)e^{-(\lambda+\mu)}, \lambda - (p\lambda - q\mu)e^{-(\lambda+\mu)} \right).$$

Answer to Exercise 60. The transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ is

$$\boldsymbol{P(t)} = \begin{pmatrix} e^{-3t} & 1 - e^{-3t} & 0 \\ 0 & 1 & 0 \\ e^{-2t} - e^{-3t} & 1 + e^{-3t} - 2e^{-2t} & e^{-2t} \end{pmatrix}.$$

The distribution of X_t is

$$\boldsymbol{p(t)}^T = \frac{1}{3} \left(e^{-2t}, 3 - 2e^{-2t}, e^{-2t} \right).$$

Answer to Exercise 61. The transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ is

$$\boldsymbol{P(t)} = \begin{pmatrix} \frac{1}{5} + \frac{2}{3}e^{-2t} + \frac{2}{15}e^{-5t} & \frac{2}{5} - \frac{2}{5}e^{-5t} & \frac{2}{5} - \frac{2}{3}e^{-2t} + \frac{4}{15}e^{-5t} \\ \frac{1}{5} - \frac{1}{5}e^{-5t} & \frac{2}{5} - \frac{2}{5}e^{-5t} & \frac{2}{5} - \frac{2}{5}e^{-5t} \\ \frac{1}{5} - \frac{1}{3}e^{-2t} + \frac{2}{15}e^{-5t} & \frac{2}{5} - \frac{2}{5}e^{-5t} & \frac{2}{5} + \frac{1}{3}e^{-2t} + \frac{4}{15}e^{-5t} \end{pmatrix}.$$

The distribution of X_t is

$$\boldsymbol{p(t)}^T = \frac{1}{3} \left(\frac{3}{5} + \frac{1}{3}e^{-2t} + \frac{1}{15}e^{-5t}, \frac{6}{5} - \frac{1}{5}e^{-5t}, \frac{6}{5} - \frac{1}{3}e^{-2t} + \frac{2}{15}e^{-5t} \right).$$

Answer to Exercise 64. The Poisson process is non-explosive.

Answer to Exercise 65. The Markov chain X is explosive.

Answer to Exercise 66. The stationary distribution exists because the chain X is finite and irreducible. The stationary distribution of X is given by $\boldsymbol{\pi}^T = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Answer to Exercise 67. The state space of X is $S = \{0, 1, \ldots, N\}$, the generator $\boldsymbol{Q}$ of X is given by

$$\boldsymbol{Q} = \begin{pmatrix} -N\lambda & N\lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu & -(\mu + (N-1)\lambda) & (N-1)\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\mu & -(2\mu + (N-2)\lambda) & (N-2)\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (N-1)\mu & -((N-1)\mu + \lambda) & \lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & N\mu & -N\mu \end{pmatrix},$$

its stationary distribution exists (finite, irreducible chain) and is given by

$$\pi_k = \binom{N}{k} \left(\frac{\lambda}{\mu}\right)^k \left(1+\frac{\lambda}{\mu}\right)^{-N}, \quad k = 0, 1, \ldots, N.$$

Answer to Exercise 68. The stationary distribution of X exists and if we denote by $\rho := \frac{\lambda}{\mu}$, it takes the form

$$\pi_k = \frac{\rho^k}{k!} \left(\sum_{j=0}^{N} \frac{\rho^j}{j!}\right)^{-1}, \quad k = 0, 1, \ldots, N.$$

Answer to Exercise 69. The chain X admits an invariant measure that is given by $\eta_i = \frac{1}{p}\eta_0$ for $i \in \mathbb{N}$. This invariant measure cannot be normalized to a stationary distribution.

Answer to Exercise 70. The stationary distribution of X exists and it takes the form

$$\pi_0 = \frac{1}{2} \quad \text{and} \quad \pi_k = \frac{1}{(k+1)(k+2)}, \quad k \in \mathbb{N}.$$

Answer to Exercise 71. The arrival rate is 6 customers per minute while the service rate is 3 customers per minute. Therefore, the system is modeled by the homogeneous continuous-time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -6 & 6 & 0 & 0 & 0 & 0 & \cdots \\ 3 & -9 & 6 & 0 & 0 & 0 & \ddots \\ 0 & 6 & -12 & 6 & 0 & 0 & \ddots \\ 0 & 0 & 9 & -15 & 6 & 0 & \ddots \\ 0 & 0 & 0 & 9 & -15 & 6 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

1. The stationary distribution of X exists and it is given by

$$\pi_k = \begin{cases} \frac{1}{9}\frac{2^j}{j!}, & j = 0, 1, 2, 3, \\ \frac{1}{3}\left(\frac{2}{3}\right)^{j-1}, & j = 4, 5, \ldots. \end{cases}$$

2. The mean number of customers in the system in the steady-state regime is

$$\sum_{j=1}^{\infty} j\pi_j = \frac{26}{9}.$$

3. The mean number of customers in the queue in the steady-state regime is

$$\sum_{j=1}^{\infty} j\pi_{j+3} = \frac{16}{9}.$$

4. The probability that an arriving customer will not need to wait in the queue is

$$\pi_0 + \pi_1 + \pi_2 = \frac{5}{9}.$$

Answer to Exercise 74. The number of visitors is modeled by the homogeneous continuous-time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \ddots \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 & \ddots \\ 0 & 0 & 3\mu & -(\lambda+3\mu) & \lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The chain admits a stationary distribution that is given by

$$\pi_k = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \mathrm{e}^{-\frac{\lambda}{\mu}}, \quad k \in \mathbb{N}_0$$

which is also the long-run proportion of time the chain spends in the state $k \in \mathbb{N}_0$.

References

[1] P. R. de Montmort, *Essay d'analyse sur les jeux de hazard*, Jacque Quillau, Paris, 2nd ed., 1713. Rêvue et augmentée de plusieurs Lettre.

[2] V. Dupač and J. Dupačová, *Markovovy procesy II.*, Univerzita Karlova, Praha, 1980.

[3] R. Durrett, *Essentials of Stochastic Processes*, Springer, Cham, 3rd ed., 2016.

[4] P. A. Gagniuc, *Markov Chains. From Theory to Implementation and Experimentation.*, John Wiley & Sons, Inc., 2017.

[5] F. Galton and H. W. Watson, *On the probability of the extinction of families*, Journal of the Royal Anthropological Institute, (1875), pp. 138–144.

[6] G. R. Grimmett and D. Stirzaker, *Probability and Random Processes*, Oxford University Press, New York, 2001.

[7] O. Häggström, *Finite Markov Chains and Algorithmic Applications*, Cambridge University Press, Cambridge, 2002.

[8] L. L. Helms, *Introduction to Probability Theory with Contemporary Applications*, Courier Corporation, 2012.

[9] C. Huygens, *De ratiociniis in ludo aleae*, Ex officinia J. Elsevirii, 1657.

[10] O. Kallenberg, *Foundations of Modern Probability*, Springer-Verlag New York, 1997.

[11] S. Karlin and H. E. Taylor, *A First Course in Stochastic Processes*, Academic Press, New York-London, 2nd ed., 1975.

[12] S. Karlin and H. M. Taylor, *A Second Course in Stochastic Processes*, Academinc Press, New York-London, 1981.

[13] J. Medhi, *Stochastic Processes*, John Wiley & Sons, Inc., New York, 2nd ed., 1994.

[14] J. R. Norris, *Markov Chains*, Cambridge University Press, Cambridge, 1998.

[15] Z. Prášková and P. Lachout, *Základy náhodných procesů I.*, Matfyzpress, 2nd ed., 2012.

[16] N. Privault, *Understanding Markov Chains*, Springer, Singapore, 2nd ed., 2018.

[17] S. I. Resnick, *Adventures in Stochastic Processes*, Birkhäuser Boston, 1992.

[18] D. Revuz, *Markov Chains*, North-Holland Publishing Co., Amsterdam, 2nd ed., 1984.

[19] S. M. Ross, *Stochastic processes*, John Wiley & Sons, Inc., New York, 2nd ed., 1996.