Invariance Principles

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Contents

Remark from author

The text is under developing and is continuously improving. Its aim is to give a comprehensive and self-confident description of subject "Convergence of Random Processes". Basis of Topology, Product Topology, Topology with Randomness are starting points of our text. Metric spaces are remembered as a particular case of topological spaces. After this introduction, several spaces of real functions are introduced. Particularly, space of all real functions, space of bounded real functions, space of continuous real functions, Skorokhod space of càdlàg functions are treated. Talk finishes with Skorokhod imbedding and Strong Invariance principle.

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Used notation

$\mathcal{P}(A)$ .... the set of all subsets of A

$\mathsf{Fin}\,(\Lambda)$ .... the set of all nonempty finite subsets of Λ

$\mathbb{R}$ .... real numbers

$\mathbb{Z}$ .... integer numbers

$\mathbb{N}$ .... natural numbers

$\mathbb{N}_0$ .... natural numbers with zero

$\mathbb{Q}$ .... racional numbers

$\mathbb{C}$ .... complex numbers

$\mathbb{R}_+$ .... positive real numbers

$\mathbb{R}_{+,0}$ .... nonnegative real numbers

Chapter 1

Measures and Topology

1.1 Measures

The text is devoted to real random processes and their convergence in distribution. It means we have to deal with probability measures on some convenient topological spaces of real functions and their weak convergence. Let us begin with recalling notions of σ-algebra and measure.

Definition 1.1 *We say $\Sigma = (\Sigma, X)$ is a σ-algebra (cz. σ-algebra), if*

- $X \neq \emptyset$.
- $\Sigma \subset \mathcal{P}(X)$.
- $\emptyset, X \in \Sigma$.
- $X \setminus A \in \Sigma$ *for each set* $A \in \Sigma$.
- $\bigcup_{i=1}^{+\infty} A_i \in \Sigma$ *for each sequence of sets* $A_i \in \Sigma$, $i \in \mathbb{N}$.

Definition 1.2 *We say $\mu = (\mu, X, \Sigma)$ is a measure (cz. míra), if*

- $X \neq \emptyset$.
- $\Sigma = (\Sigma, X)$ *is a* σ*-algebra.*
- $\mu : \Sigma \to \mathbb{R}^*_{+,0}$.
- $\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} \mu(A_i)$ *for each sequence of pairwise disjoint sets* $A_i \in \Sigma$, $i \in \mathbb{N}$.

We say $\mu = (\mu, X, \Sigma)$ is a sign measure (cz. znaménková míra), if

- $X \neq \emptyset$.
- $\Sigma = (\Sigma, X)$ *is a* σ*-algebra.*
- $\mu : \Sigma \to \mathbb{R}^*$.
- $\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} \mu(A_i)$ *for each sequence of pairwise disjoint sets* $A_i \in \Sigma$, $i \in \mathbb{N}$.

Let us recall definitions of outer and inner measures.

Definition 1.3 *Let* $\mu = (\mu, X, \Sigma)$ *be a measure. Then, outer measure (cz. vnější míra) is defined as*

$$\mu^* : \mathcal{P}(X) \to \mathbb{R}^*_{+,0} : A \in \mathcal{P}(X) \mapsto \inf\{\mu(B) : A \subset B \in \Sigma\}$$

and inner measure (cz. vnitřní míra) is defined as

$$\mu_* : \mathcal{P}(X) \to \mathbb{R}^*_{+,0} : A \in \mathcal{P}(X) \mapsto \sup\{\mu(B) : A \supset B \in \Sigma\}.$$

Definition 1.4 *If* $\mu = (\mu, X, \Sigma)$ *is a measure, we denote* σ*-algebra of all* μ*-measurable sets (cz.* σ*-algebra všech* μ*-měřitelných množin) by*

$$\mathcal{MS}(\mu) = \{A \in \mathcal{P}(X) : \mu^*(A) = \mu_*(A)\}.$$

Lemma 1.5 $\mathcal{MS}(\mu)$ *is a* σ*-algebra.*

1.2 Topological spaces

Definition 1.6 *Space* $\mathcal{X} = (\mathcal{X}, \mathcal{G})$ *is called topological space (cz. topologický prostor), if* $\mathcal{X}$ *is a nonempty set and* $\mathcal{G} \subset \mathcal{P}(\mathcal{X})$ *fulfills*

1. $\emptyset, \mathcal{X} \in \mathcal{G}$.
2. *If* $G_1, G_2 \in \mathcal{G}$ *then* $G_1 \cap G_2 \in \mathcal{G}$.
3. *If* Λ *is a nonempty set and* $G_\lambda \in \mathcal{G}$ *for each* $\lambda \in \Lambda$ *then* $\bigcup_{\lambda \in \Lambda} G_\lambda \in \mathcal{G}$.

We will use the following standard notation and terminology.

Definition 1.7 *Let* $\mathcal{X} = (\mathcal{X}, \mathcal{G})$ *be a topological space. Then*

- *Any member of* $\mathcal{G}$ *is called an open set of the topological space* $\mathcal{X}$ *and* $\mathcal{G}$ *itself is called the set of all open sets of the topological space* $\mathcal{X}$ *and will be denoted by* $\mathcal{G}(\mathcal{X})$.

- $\mathcal{F}(\mathcal{X}) = \{\mathcal{X} \setminus G : G \in \mathcal{G}(\mathcal{X})\}$ *is called the set of all closed sets of the topological space* $\mathcal{X}$ *and its members are called closed sets of the topological space* $\mathcal{X}$.

- $\mathcal{B}(\mathcal{X}) = \sigma(\mathcal{G}(\mathcal{X}))$ *is called Borel* σ*-algebra of the topological space* $\mathcal{X}$ *and its members are called Borel sets of the topological space* $\mathcal{X}$.

  For real line, we will use a convention $\mathbb{B} = \mathcal{B}(\mathbb{R})$.

Open sets possesses a nice characterization.

Lemma 1.8 *Let* $\mathcal{X}$ *be a topological space and* $A \subset \mathcal{X}$. *Then*

$$A \in \mathcal{G}(\mathcal{X}) \Longleftrightarrow \forall\, a \in A\ \exists\, G \in \mathcal{G}(\mathcal{X}) \ \text{s.t.}\ a \in G \subset A.$$

Proof:

1. If $A \in \mathcal{G}(\mathcal{X})$ the property is trivial, since for $a \in A$ we can set $G = A$.
2. Let $\forall\, a \in A\ \exists\, G \in \mathcal{G}(\mathcal{X})$ s.t. $a \in G \subset A$.

   Choose $G_a \in \mathcal{G}(\mathcal{X})$ such that $a \in G_a \subset A$ for each $a \in A$.

   Hence, $A = \bigcup_{a \in A} G_a \in \mathcal{G}(\mathcal{X})$.

Q.E.D.

Borel sets can be constructed using δ, σ operations.

Definition 1.9 *Let* $\mathcal{X}$ *be a topological space. We define*

$$\begin{aligned}
\mathcal{G}_\delta(\mathcal{X}) &= \left\{ \bigcap_{i=1}^{+\infty} G_i : G_i \in \mathcal{G}(\mathcal{X}) \ \text{for all } i \in \mathbb{N} \right\}, \\
\mathcal{G}_{\delta\sigma}(\mathcal{X}) &= \left\{ \bigcup_{i=1}^{+\infty} G_i : G_i \in \mathcal{G}_\delta(\mathcal{X}) \ \text{for all } i \in \mathbb{N} \right\}, \\
\mathcal{F}_\sigma(\mathcal{X}) &= \left\{ \bigcup_{i=1}^{+\infty} F_i : F_i \in \mathcal{F}(\mathcal{X}) \ \text{for all } i \in \mathbb{N} \right\}, \\
\mathcal{F}_{\sigma\delta}(\mathcal{X}) &= \left\{ \bigcap_{i=1}^{+\infty} F_i : F_i \in \mathcal{F}_\sigma(\mathcal{X}) \ \text{for all } i \in \mathbb{N} \right\}.
\end{aligned}$$

Consequently, sets $\mathcal{G}_{\delta\sigma\delta}(\mathcal{X})$, $\mathcal{F}_{\sigma\delta\sigma}(\mathcal{X})$, $\mathcal{G}_{\delta\sigma\delta\sigma}(\mathcal{X})$, $\mathcal{F}_{\sigma\delta\sigma\delta}(\mathcal{X})$, *etc., can be defined.*

Definition 1.10 *Let $\mathcal{X}$ be a topological space. Then, for each set $A \in \mathcal{P}(\mathcal{X})$ we define its*

- *closure* $\mathsf{clo}\,(A\,;\,\mathcal{G}\,(\mathcal{X})) = \bigcap_{A\subset F\in\mathcal{F}(\mathcal{X})} F$;
- *interior* $\mathsf{int}\,(A\,;\,\mathcal{G}\,(\mathcal{X})) = \bigcup_{A\supset G\in\mathcal{G}(\mathcal{X})} G$;
- *boundary* $\partial\,(A\,;\,\mathcal{G}\,(\mathcal{X})) = \mathsf{clo}\,(A\,;\,\mathcal{G}\,(\mathcal{X})) \setminus \mathsf{int}\,(A\,;\,\mathcal{G}\,(\mathcal{X}))$.

One can check that the closure of a sets is a closed set and the interior of a sets is an open set.

Let us recall, that the notion of topology is introduced in [3] in more general way than we did. Setup is based on a closure operator $\vartheta : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$, The only requirements are monotonicity and closure of a union of finite number of sets must be union of their closures.

There are equivalent descriptions of closure, interior and boundary.

Lemma 1.11 *Let $\mathcal{X}$ be a topological space, $A \subset \mathcal{X}$ and $x \in \mathcal{X}$.*

i) $x \in \mathsf{clo}\,(A\,;\,\mathcal{G}\,(\mathcal{X})) \iff \forall G \in \mathcal{G}\,(\mathcal{X})\,,\ x \in G$ *we have* $A \cap G \neq \emptyset$.

ii) $x \notin \mathsf{clo}\,(A\,;\,\mathcal{G}\,(\mathcal{X})) \iff \exists G \in \mathcal{G}\,(\mathcal{X})$ *such that* $x \in G,\ A \cap G = \emptyset$.

iii) $x \in \mathsf{int}\,(A\,;\,\mathcal{G}\,(\mathcal{X})) \iff \exists G \in \mathcal{G}\,(\mathcal{X})$ *such that* $x \in G \subset A$.

iv) $x \notin \mathsf{int}\,(A\,;\,\mathcal{G}\,(\mathcal{X})) \iff \forall G \in \mathcal{G}\,(\mathcal{X})\,,\ x \in G$ *we have* $G \setminus A \neq \emptyset$.

v) $x \in \partial\,(A\,;\,\mathcal{G}\,(\mathcal{X})) \iff \forall G \in \mathcal{G}\,(\mathcal{X})\,,\ x \in G$ *we have* $A \cap G \neq \emptyset$ *and* $G \setminus \mathsf{clo}\,(A\,;\,\mathcal{G}\,(\mathcal{X})) \neq \emptyset$.

Definition 1.12 *Let $\mathcal{X}$ be a topological space and $H \subset \mathcal{X}$. We say H is dense in $\mathcal{X}$ whenever* $\mathrm{clo}\,(H) = \mathcal{X}$.

Compact sets are a specific class of sets in a topological space.

Definition 1.13 *Let $\mathcal{X}$ be a topological space. A set $K \subset \mathcal{X}$ is called a compact whenever the following properties are fulfilled:*

1. $K \in \mathcal{F}\,(\mathcal{X})$.
2. *If* $\Lambda \neq \emptyset$ *and* $G_\lambda \in \mathcal{G}\,(\mathcal{X})$, $\lambda \in \Lambda$ *are given such that* $K \subset \bigcup_{\lambda\in\Lambda} G_\lambda$ *then there is* $I \in \mathsf{Fin}\,(\Lambda)$ *with* $K \subset \bigcup_{i\in I} G_i$.

The set of all compact sets of the topological space $\mathcal{X}$ is denoted by $\mathcal{K}\,(\mathcal{X})$.

Definition 1.14 *Let $\mathcal{X}$ be a topological space. A set $K \subset \mathcal{X}$ is called a relative compact (cz. relativní kompakt) whenever* $\mathrm{clo}(K) \in \mathcal{K}(\mathcal{X})$.

Definition 1.15 *We say $\mathcal{X}$ is a compact topological space if it is a topological space and $\mathcal{X} \in \mathcal{K}(\mathcal{X})$.*

We have a simple observation

Lemma 1.16 *Let $\mathcal{X}$ be a topological space. Then, always*

$$\mathcal{B}(\mathcal{X}) = \sigma(\mathcal{G}(\mathcal{X})) = \sigma(\mathcal{F}(\mathcal{X})) \supset \sigma(\mathcal{K}(\mathcal{X})). \tag{1.1}$$

Equality $\mathcal{B}(\mathcal{X}) = \sigma(\mathcal{K}(\mathcal{X}))$ is in power for $\mathcal{X} = \mathbb{R}^n$, for compact topological spaces, and, for some other particular cases.

Consider nice and helpful properties of compact sets.

Lemma 1.17 *If $K \in \mathcal{K}(\mathcal{X})$ and $F \in \mathcal{F}(\mathcal{X})$ then $K \cap F \in \mathcal{K}(\mathcal{X})$.*

Proof: Intersection of closed sets is a closed set, therefore, $K \cap F \in \mathcal{F}(\mathcal{X})$.
Take a system of open sets such that $K \cap F \subset \bigcup_{\lambda\in\Lambda} G_\lambda$.
Then $K \subset \bigcup_{\lambda\in\Lambda} G_\lambda \cup (\mathcal{X} \setminus F)$ is an open covering of compact K.
Therefore, there exists $I \in \mathsf{Fin}(\Lambda)$ such that $K \subset \bigcup_{i\in I} G_i \cup (\mathcal{X} \setminus F)$.
Thus, $K \cap F \subset \bigcup_{i\in I} G_i$ and, consequently, $K \cap F$ is a compact.

Q.E.D.

Proposition 1.18 *Let $K \in \mathcal{K}(\mathcal{X})$ and $F_\lambda \in \mathcal{F}(\mathcal{X})$ for each $\lambda \in \Lambda$.
Let $K \cap \bigcap_{\lambda\in\Lambda} F_\lambda = \emptyset$ then there exists $I \in \mathsf{Fin}(\Lambda)$ such that $K \cap \bigcap_{i\in I} F_i = \emptyset$.*

Proof: We have covering of the compact K by open sets $K \subset \bigcup_{\lambda\in\Lambda} (\mathcal{X} \setminus F_\lambda)$.
Thus, there exists $I \in \mathsf{Fin}(\Lambda)$ such that $K \subset \bigcup_{i\in I} (\mathcal{X} \setminus F_i)$.
In other words, $K \cap \bigcap_{i\in I} F_i = \emptyset$.

Q.E.D.

Proposition 1.18 is, actually, equivalent to the definition of compact set.
The following consequence of Proposition 1.18 possesses a great importance in measure theory.

Proposition 1.19 *Let $K_n \in \mathcal{K}(\mathcal{X})$, $n \in \mathbb{N}$ be such that $K_1 \supset K_2 \supset K_3 \supset \ldots$. Let $\bigcap_{n=1}^{+\infty} K_n = \emptyset$ then there exists $n_0 \in \mathbb{N}$ such that $K_{n_0} = \emptyset$.*

Proof: We have covering of a compact by open sets $K_1 \subset \bigcup_{n=1}^{+\infty} (\mathcal{X} \setminus K_n)$. Thus, there exists $n_0 \in \mathbb{N}$ such that $K_1 \subset \bigcup_{n=1}^{n_0} (\mathcal{X} \setminus K_n) = \mathcal{X} \setminus K_{n_0}$. Hence, $K_{n_0} = \emptyset$. since $K_{n_0} \subset K_1$.

Q.E.D.

According to the definition, topology is determined by a collection of open sets. Fortunately, there are smaller systems of sets fully describing the topology.

Definition 1.20 *Let $\mathcal{X}$ be a topological space and $\mathcal{G} \subset \mathcal{G}(\mathcal{X})$. Then,*

- *i) $\mathcal{G}$ is called a (topological) basis of $\mathcal{X}$, whenever, each open set can be written as a (possibly uncountable) union of sets from $\mathcal{G}$.*
- *ii) $\mathcal{G}$ is called a (topological) subbasis of $\mathcal{X}$, whenever, set of all intersections of finitely many sets from $\mathcal{G}$ forms a basis of $\mathcal{X}$.*

As an example, let us mention that $\{(a, b) : a, b \in \mathbb{R},\ a < b\}$ is a basis and $\{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, +\infty) : a \in \mathbb{R}\}$ is a subbasis of the natural topology on $\mathbb{R}$.

Let us introduce a construction of a topology. Assume Y is a nonempty set and $\mathcal{H} \subset \mathcal{P}(Y)$ then $\mathcal{H}$ uniquely determines a topology on Y. Topology is constructed in three steps:

1. $\mathcal{H}_0 = \mathcal{H} \cup \{\emptyset, Y\}$.
2. $\mathcal{H}_1 = \left\{ \bigcap_{i=1}^{k} H_i \ :\ H_1, H_2, \ldots, H_k \in \mathcal{H}_0,\ k \in \mathbb{N} \right\}$.
3. $\mathcal{H}_2 = \left\{ \bigcup_{\lambda \in \Lambda} H_\lambda \ :\ H_\lambda \in \mathcal{H}_1, \forall \lambda \in \Lambda,\ \Lambda \neq \emptyset \right\}$.

Proposition 1.21 *Let Y be a nonempty set and $\mathcal{H} \subset \mathcal{P}(Y)$ then $(Y, \mathcal{H}_2)$ is a topological space, $\mathcal{H}_1$ is its basis and $\mathcal{H}_0$ is its subbasis.*

We will denote this induced topology by $\tau(\mathcal{H}) := \mathcal{H}_2$.

Proof: We make some observations:

1. $\emptyset, Y \in \mathcal{H}_1$ and $\emptyset, Y \in \mathcal{H}_2$, since $\emptyset, Y \in \mathcal{H}_0$.
2. Let $H_1, H_2 \in \mathcal{H}_1$.

   Hence, $H_1 = \bigcap_{i=1}^{I_1} H_{1,i}$ and $H_2 = \bigcap_{j=1}^{I_2} H_{2,j}$ for some sets $H_{1,1}, \ldots, H_{1,I_1}, H_{2,1}, \ldots, H_{2,I_2} \in \mathcal{H}_0$.

   Then, $H_1 \cap H_2 = \bigcap_{k=1}^{2} \bigcap_{i=1}^{I_k} H_{k,i} \in \mathcal{H}_1$.

   Therefore, $\mathcal{H}_1$ is closed on intersection of finite number of sets.

3. Let $H_1, H_2 \in \mathcal{H}_2$.

   Hence, $H_1 = \bigcup_{\lambda\in\Lambda_1} H_{1,\lambda}$ and $H_2 = \bigcup_{\psi\in\Lambda_2} H_{2,\psi}$ for some sets $H_{1,\lambda}, H_{2,\psi} \in \mathcal{H}_1$, $\lambda \in \Lambda_1$, $\psi \in \Lambda_2$.

   Then, $H_1 \cap H_2 = \bigcup_{\lambda\in\Lambda_1} \bigcup_{\psi\in\Lambda_2} (H_{1,\lambda} \cap H_{2,\psi}) \in \mathcal{H}_2$, since $\mathcal{H}_1$ is closed on intersection of a finite number of sets.

   We have shown that $\mathcal{H}_2$ is also closed on intersection of a finite number of sets.

4. Let for each $\lambda \in \Lambda$ a set $H_\lambda \in \mathcal{H}_2$ be given.

   Then, $H_\lambda = \bigcup_{\psi\in\Psi_\lambda} H_{\lambda,\psi}$ for some sets $H_{\lambda,\psi} \in \mathcal{H}_1$.

   Hence, $\bigcup_{\lambda\in\Lambda} H_\lambda = \bigcup_{\lambda\in\Lambda} \bigcup_{\psi\in\Psi_\lambda} H_{\lambda,\psi} \in \mathcal{H}_2$.

   Thus, $\mathcal{H}_2$ is closed on union of an arbitrary number of sets.

We have verified that $\mathcal{H}_2$ is a system of open sets on Y, $\mathcal{H}_1$ is its basis and $\mathcal{H}_0$ is its subbasis.

Q.E.D.

The construction can be written as

$$\tau(\mathcal{H}) = \left\{ \bigcup_{\lambda\in\Lambda} \bigcap_{k=1}^{K_\lambda} H_{k,\lambda} \,:\, H_{k,\lambda} \in \mathcal{G}, K_\lambda \in \mathbb{N}, \Lambda \neq \emptyset \right\} \cup \{\emptyset\} \cup \{\mathcal{X}\}. \tag{1.2}$$

Examples of a topological basis and a subbasis:

- $\mathcal{X} = R$, $\mathcal{G}$:= open intervals of type (a, b), $a, b \in \mathbb{R}$, $a < b$;
- $\mathcal{X} = R$, $\mathcal{G}$:= closed intervals of type $[a, b]$, $a, b \in \mathbb{R}$, $a < b$;
- $\mathcal{X} = R$, $\mathcal{G}$:= half-closed intervals of type $[a, b)$, $a, b \in \mathbb{R}$, $a < b$;
- $\mathcal{X} = R$, $\mathcal{G}$:= open intervals of type $(a, +\infty)$, $a \in \mathbb{R}$;
- $\mathcal{X} = R$, $\mathcal{G}$:= closed intervals of type $[a, +\infty)$, $a \in \mathbb{R}$.

1.3 Topology - Local characterization

Topology can be fully described by local properties.

Definition 1.22 *Let $\mathcal{X}$ be a topological space and $\mathcal{U}_x \subset \mathcal{P}(\mathcal{X})$ be given for each $x \in \mathcal{X}$. If the following two properties are fulfilled for all $x \in \mathcal{X}$:*

i) For each $U \in \mathcal{U}_x$ there exists $G \in \mathcal{G}(\mathcal{X})$ such that $x \in G \subset U$.

ii) For each $G \in \mathcal{G}(\mathcal{X})$, $x \in G$ there exists $U \in \mathcal{U}_x$ such that $U \subset G$.

Then $\mathcal{U}_x$ is called (topological) (local) basis of neighborhoods at x (cz. baze okolí bodu x).

Let us emphasis that sets from $\mathcal{U}_x$ do not need to be open, even they can be non-Borel !

Basis of neighborhoods determine topology.

Lemma 1.23 *Let $\mathcal{X}$ be a topological space, $\mathcal{U}_x \subset \mathcal{P}(\mathcal{X})$ be basis of neighborhoods at x for each $x \in \mathcal{X}$ and $A \subset \mathcal{X}$. Then*

$$A \in \mathcal{G}(\mathcal{X}) \iff \forall a \in A \; \exists U \in \mathcal{U}_a \text{ s.t. } U \subset A.$$

Proof:

1. Let $A \in \mathcal{G}(\mathcal{X})$.

   Take $a \in A$. By definition 1.22, there is $U \in \mathcal{U}_a$ such that $U \subset A$.

2. Let $\forall a \in A \; \exists U \in \mathcal{U}_a$ s.t. $U \subset A$.

   Take $a \in A$. Then there is $U \in \mathcal{U}_a$ such that $U \subset A$.

   By definition 1.22, there is $G \in \mathcal{G}(\mathcal{X})$ such that $a \in G \subset U \subset A$.

   According to Lemma 1.8, $A \in \mathcal{G}(\mathcal{X})$.

Q.E.D.

Now assume, for each $x \in \mathcal{X}$ systems of sets $\mathcal{S}_x \subset \mathcal{P}(\mathcal{X})$ are given. Consider following four properties:

(Neighbor-0) Far all $x \in \mathcal{X}$ we have $\mathcal{S}_x \neq \emptyset$.

(Neighbor-1) Far all $x \in \mathcal{X}$, $U \in \mathcal{S}_x$ we have $x \in U$.

(Neighbor-2) Far all $x \in \mathcal{X}$, $U, V \in \mathcal{S}_x$ there exists $Z \in \mathcal{S}_x$ such that $Z \subset U \cap V$.

(Neighbor-3) Far all $x \in \mathcal{X}$, $U \in \mathcal{S}_x$ there exists $Z \in \mathcal{S}_x$ and $U_y \in \mathcal{S}_y$, $y \in Z$ such that $U_y \subset U$ for each $y \in Z$.

These given systems build up a topology

$$\mathbf{t}(\mathcal{S}_x, x \in \mathcal{X}) = \{G \in \mathcal{P}(\mathcal{X}) : \forall y \in G \; \exists U \in \mathcal{S}_y, \; U \subset G\} . \qquad (1.3)$$

Proposition 1.24 *If properties (Neighbor-0), (Neighbor-2) are fulfilled then* $\mathbf{t}(\mathcal{S}_x, x \in \mathcal{X})$ *forms open sets in* $\mathcal{X}$.

Proof: Step by step we check properties of open sets.

1. From definition (1.3) we see $\emptyset \in \mathbf{t}(\mathcal{S}_x, x \in \mathcal{X})$.

2. Property (Neighbor-0) is giving $\mathcal{X} \in \mathbf{t}(\mathcal{S}_x, x \in \mathcal{X})$.

3. Let $G, H \in \mathbf{t}(\mathcal{S}_x, x \in \mathcal{X})$.

   Take $x \in G \cap H$.

   Then, there are $U, V \in \mathcal{S}_x$ such that $U \subset G$ and $V \subset H$. Accordingly to property (Neighbor-2), there exists $W \in \mathcal{S}_x$ such that $W \subset G \cap H$.

   We have checked $G \cap H \in \mathbf{t}(\mathcal{S}_x, x \in \mathcal{X})$.

4. Let $G_\lambda \in \mathbf{t}(\mathcal{S}_x, x \in \mathcal{X})$ for each $\lambda \in \Lambda$.

   Take $x \in \bigcup_{\lambda \in \Lambda} G_\lambda$.

   Then, there exists $\psi \in \Lambda$ and $U \in \mathcal{S}_x$ such that $U \subset G_\psi$.

   We have checked $\bigcup_{\lambda \in \Lambda} G_\lambda \in \mathbf{t}(\mathcal{S}_x, x \in \mathcal{X})$.

Q.E.D.

Proposition 1.25 *Requiring (Neighbor-0), (Neighbor-1), (Neighbor-2) and (Neighbor-3), then* $\mathbf{t}(\mathcal{S}_x, x \in \mathcal{X})$ *forms open sets in* $\mathcal{X}$ *and for each* $x \in \mathcal{X}$ *the system* $\mathcal{S}_x$ *is a basis of neighborhoods at point* x.

Proof: According to Proposition 1.24, we know $\mathbf{t}(\mathcal{S}_x, x \in \mathcal{X})$ are open sets in $\mathcal{X}$. It remains to verify $\mathcal{S}_x$ is a basis of neighborhoods at point x.

1. Take $G \in \mathbf{t}(\mathcal{S}_x, x \in \mathcal{X})$ and $x \in G$.

   Directly from Definition (1.3) there exists $U \in \mathcal{S}_x$ such that $U \subset G$.

2. Take $x \in \mathcal{X}$ and $U \in \mathcal{S}_x$.

   Denote $G = \{y \in U \ : \exists V \in \mathcal{S}_y \text{ such that } V \subset U\}$.

   Let $y \in G$ then there exists $V \in \mathcal{S}_y$ such that $V \subset U$.

   From property (Neighbor-3), there exists $W \in \mathcal{S}_y$ such that for each $w \in W$ there exists $Z_w \in \mathcal{S}_w$ and $Z_w \subset V$.

   Therefore, $W \subset G$.

   We have checked $G \in \mathbf{t}(\mathcal{S}_x, x \in \mathcal{X})$.

   Moreover, $G \subset U$ and from property (Neighbor-1) we have $x \in G$.

Q.E.D.

1.4 Classification of topological spaces

Definition 1.26 *Topological space $\mathcal{X}$ is called*

i) *T_0, if for each $x, y \in \mathcal{X}$, $x \neq y$ there exists $G \in \mathcal{G}(\mathcal{X})$ such that either $x \in G$, $y \notin G$ or $x \notin G$, $y \in G$.*

ii) *T_1, if for each $x, y \in \mathcal{X}$, $x \neq y$ there exists $G \in \mathcal{G}(\mathcal{X})$ such that $x \in G$, $y \notin G$.*

iii) *Hausdorff, if for each $x, y \in \mathcal{X}$, $x \neq y$ there are $G, Q \in \mathcal{G}(\mathcal{X})$ such that $x \in G$, $y \in Q$ and $G \cap Q = \emptyset$.*

iv) *regular, if for each $x \in \mathcal{X}$, $F \in \mathcal{F}(\mathcal{X})$, $F \neq \emptyset$, $x \notin F$ there are $G, Q \in \mathcal{G}(\mathcal{X})$ such that $x \in G$, $F \subset Q$ and $G \cap Q = \emptyset$.*

v) *normal, if for each $F, H \in \mathcal{F}(\mathcal{X})$, $F \neq \emptyset$, $H \neq \emptyset$, $F \cap H = \emptyset$ there are $G, Q \in \mathcal{G}(\mathcal{X})$ such that $F \subset G$, $H \subset Q$ and $G \cap Q = \emptyset$.*

vi) *locally compact, if for each $x \in \mathcal{X}$ there exists $G \in \mathcal{G}(\mathcal{X})$ such that $x \in G$ and $\mathsf{clo}(G\,;\,\mathcal{G}(\mathcal{X})) \in \mathcal{K}(\mathcal{X})$.*

vii) *fulfilling I. axiom of countability, if for each $x \in \mathcal{X}$ there exists countable basis of neighborhoods at point x.*

viii) *separable (fulfilling II. axiom of countability),*
whenever it possesses a countable basis.

Some observations and notes to the classification.

Lemma 1.27 *Let $\mathcal{X}$ be a topological space and $\sim$ be a relation defined on $\mathcal{X}$*

$$x \sim y \iff \forall\, G \in \mathcal{G}\left(\mathcal{X}\right) \text{ it is fulfilled } \left(x \in G \iff y \in G\right). \tag{1.4}$$

Then, $\sim$ is an equivalence on $\mathcal{X}$ and $\mathcal{X}\,|_\sim$ is T_0.

Topology of $\mathcal{X}\,|_\sim$ is determined as follows. For $x \in \mathcal{X}$ we denote $[x] = \{y \in \mathcal{X} : y \sim x\}$ and for $A \subset \mathcal{X}$ we set $[A] = \{[y] : y \in A\}$. Then, $\mathcal{X}\,|_\sim = \{[x] : x \in \mathcal{X}\}$ and $\mathcal{G}\left(\mathcal{X}\,|_\sim\right) = \{[G] : G \in \mathcal{G}\left(\mathcal{X}\right)\}$.

Proof: Evidently, the relation $\sim$ is an equivalence on $\mathcal{X}$.
Take $v, w \in \mathcal{X}\,|_\sim$ and $v \neq w$. Then, there are $x, y \in \mathcal{X}$, $v = [x]$, $w = [y]$ and there exists $G \in \mathcal{G}\left(\mathcal{X}\right)$ such that either $x \in G$, $y \notin G$ or $x \notin G$, $y \in G$.
Therefore either $v \in [G]$, $w \notin [G]$ or $v \notin [G]$, $w \in [G]$.
We have checked $\mathcal{X}\,|_\sim$ is T_0.

Q.E.D.

Space T_1 possesses an equivalent characterization.

Lemma 1.28 *Let $\mathcal{X}$ be a topological space. Then, $\mathcal{X}$ is T_1 if and only if $\{x\} \in \mathcal{F}\left(\mathcal{X}\right)$ for each $x \in \mathcal{X}$.*

Proof:

1. Let $\mathcal{X}$ be T_1 and $x \in \mathcal{X}$.

   Then, for each $y \in \mathcal{X}$, $y \neq x$ there exists $G_y \in \mathcal{G}\left(\mathcal{X}\right)$ such that $y \in G_y$ and $x \notin G_y$.

   Then,, $\{x\} = \mathcal{X} \setminus \bigcup_{y \neq x} G_y \in \mathcal{F}\left(\mathcal{X}\right)$.

2. Let $\{x\} \in \mathcal{F}\left(\mathcal{X}\right)$ for each $x \in \mathcal{X}$.

   Each $y \in \mathcal{X}$, $y \neq x$ fulfills $y \in \mathcal{X} \setminus \{x\} \in \mathcal{G}\left(\mathcal{X}\right)$.

   Therefore $\mathcal{X}$ is T_1.

Q.E.D.

Lemma 1.29 *Classification of topological spaces fulfills*

$$T_0 \Longleftarrow T_1 \Longleftarrow \text{Hausdorff} \Longleftarrow \text{regular} \wedge T_1 \Longleftarrow \text{normal} \wedge T_1.$$

Proof: According to Proposition 1.28, we know that T_1 means that points are closed sets. Then, all implication follows directly Definition 1.26.

Q.E.D.

Also, regular space possesses an equivalent characterization by means of neighborhoods at points.

Lemma 1.30 *Let $\mathcal{X}$ be a topological space. Then, the following properties are equivalent:*

i) Space $\mathcal{X}$ is regular.

ii) For all $x \in \mathcal{X}$ we have implication if $\mathcal{U}_x$ is a basis of neighborhoods at x then $\{\mathsf{clo}(U\,;\mathcal{G}(\mathcal{X})) : U \in \mathcal{U}_x\}$ is also basis of neighborhoods at x.

iii) For all $x \in \mathcal{X}$ there exists $\mathcal{U}_x \subset \mathcal{F}(\mathcal{X})$ basis of neighborhoods at x.

Proof:

1. Assume, $\mathcal{X}$ is a regular space and $\mathcal{U}_x$ is a basis of neighborhoods at point $x \in \mathcal{X}$.

   Denote $\mathcal{S}_x = \{\mathsf{clo}(U\,;\mathcal{G}(\mathcal{X})) : U \in \mathcal{U}_x\}$.

   (a) System of sets $\mathcal{S}_x$ fulfills condition (i) of Definition 1.22, since always $U \subset \mathsf{clo}(U\,;\mathcal{G}(\mathcal{X}))$.

   (b) Take $G \in \mathcal{G}(\mathcal{X})$, $x \in G$. Then, $\mathcal{X} \setminus G \in \mathcal{F}(\mathcal{X})$ and $x \notin \mathcal{X} \setminus G$.
   From regularity of $\mathcal{X}$, there are $W, Q \in \mathcal{G}(\mathcal{X})$ such that $x \in W$, $\mathcal{X} \setminus G \subset Q$ and $W \cap Q = \emptyset$.
   Then, there exists $U \in \mathcal{U}_x$ such that $U \subset W$.
   Hence, $\mathsf{clo}(U\,;\mathcal{G}(\mathcal{X})) \subset \mathcal{X} \setminus Q \subset G$.
   Thus, system of sets $\mathcal{S}_x$ fulfills condition (ii) of Definition 1.22.

   Therefore, $\mathcal{S}_x$ is a basis of neighborhoods at point x.

2. Property (ii) implies (iii).

3. Assume property (iii) and take $x \in \mathcal{X}$, $F \in \mathcal{F}(\mathcal{X})$, $x \notin F$.

   Then, there exists $\mathcal{U}_x \subset \mathcal{F}(\mathcal{X})$ basis of neighborhoods at point x.

   Then, $x \in \mathcal{X} \setminus F \in \mathcal{G}(\mathcal{X})$ and there exists $U \in \mathcal{U}_x$ such that $U \subset \mathcal{X} \setminus F$.

   Moreover, there exists $G \in \mathcal{G}(\mathcal{X})$ such that $x \in G \subset U$.

   Finally, $x \in G$, $F \subset \mathcal{X} \setminus U \in \mathcal{G}(\mathcal{X})$ a $G \cap \mathcal{X} \setminus U = \emptyset$.

   We have shown, the space $\mathcal{X}$ is regular.

Q.E.D.

Lemma 1.31 *Let $\mathcal{X}$ be a topological space. Then, $\mathcal{X}$ is separable if and only if $\mathcal{X}$ fulfills I. axiom of separability and there is a countable set $H \subset \mathcal{X}$ which is dense in $\mathcal{X}$.*

Proof:

1. Let $\mathcal{G}$ be a countable basis of $\mathcal{X}$ and $x \in \mathcal{X}$.

   Then, $\mathcal{U}_x = \{G \in \mathcal{G} : x \in G\}$ is a basis of neighborhoods at x.

   Moreover, $\mathcal{U}_x$ is countable, therefore, $\mathcal{X}$ fulfills I. axiom of separability.

   Select for each nonempty $G \in \mathcal{G}$ a point $\xi_G \in G$.

   Hence, $H = \{\xi_G : G \in \mathcal{G}, G \neq \emptyset\}$ is countable and dense in $\mathcal{X}$, since otherwise $\mathcal{X} \setminus \mathrm{clo}(H) \in \mathcal{G}(\mathcal{X})$ is nonempty, therefore, containing a point from H.

2. Let for each $x \in \mathcal{X}$, $\mathcal{U}_x$ is a countable basis of neighborhoods at x and H is a countable dense subset of $\mathcal{X}$.

   Hence, $\mathcal{G} = \bigcup_{x \in H} \mathcal{U}_x$ is a countable basis of $\mathcal{X}$.

Q.E.D.

Definition of compact set become more simple in regular T_1 topological spaces.

Lemma 1.32 *Let $\mathcal{X}$ be a regular T_1 topological space and $A \subset \mathcal{X}$. Then, the following is equivalent:*

i) $A \in \mathcal{K}(\mathcal{X})$.

ii) Let $G_\lambda \in \mathcal{G}(\mathcal{X})$, $\lambda \in \Lambda$ and $A \subset \bigcup_{\lambda\in\Lambda} G_\lambda$, then one is able to select $I \in \mathsf{Fin}(\Lambda)$ such that $A \subset \bigcup_{\lambda\in I} G_\lambda$.

Proof: Compact sets fulfill condition (ii). To show reverse implication, we have only to show that (ii) implies compactness.
Assume, $A \in \mathcal{P}(\mathcal{X})$ fulfilling condition (ii) and $A \notin \mathcal{F}(\mathcal{X})$.
Then, there exists $x \in \mathcal{X} \setminus A$ such that $G \cap A \neq \emptyset$ for each $G \in \mathcal{U}_x := \{U \in \mathcal{G}(\mathcal{X}) \,:\, x \in U\}$.
Since $\mathcal{X}$ is regular, for each $U \in \mathcal{U}_x$ there are $G_U, H_U \in \mathcal{G}(\mathcal{X})$, $x \in H_U$, $G_U \supset \mathcal{X} \setminus U$ and $G_U \cap H_U = \emptyset$.
Then, $\bigcup_{U\in\mathcal{U}_x} G_U \supset A$ since $\mathcal{X}$ is T_1.
According to property (ii), there exists $I \in \mathsf{Fin}(\mathcal{U}_x)$ such that $\bigcup_{U\in I} G_U \supset A$.
That is a contradiction since $\bigcup_{U\in I} G_U \cap \bigcap_{U\in I} H_U = \emptyset$ and in the same time $\bigcap_{U\in I} H_U \cap A \neq \emptyset$, since $\bigcap_{U\in I} H_U \in \mathcal{U}_x$.
Therefore $A \in \mathcal{F}(\mathcal{X})$ and then also $A \in \mathcal{K}(\mathcal{X})$.

Q.E.D.

Lemma 1.33 *Let $\mathcal{X}$ be a topological space. Then, $\mathcal{X} \setminus \mathcal{K}(\mathcal{X})$ is a basis $\mathcal{X}$ if and only if $\mathcal{F}(\mathcal{X}) \setminus \{\mathcal{X}\} \subset \mathcal{K}(\mathcal{X})$.*

Proof:

1. Let $\mathcal{X} \setminus \mathcal{K}(\mathcal{X})$ be basis $\mathcal{X}$ and $F \in \mathcal{F}(\mathcal{X}) \setminus \{\mathcal{X}\}$.

   Then, there exists $x \notin F$ and $\mathcal{X} \setminus F \in \mathcal{G}(\mathcal{X})$.

   One can find $K \in \mathcal{K}(\mathcal{X})$ such that $\mathcal{X} \setminus K \subset \mathcal{X} \setminus F$.

   Consequently, $F \subset K$ and then $F \in \mathcal{K}(\mathcal{X})$.

2. Let $\mathcal{F}(\mathcal{X}) \setminus \{\mathcal{X}\} \subset \mathcal{K}(\mathcal{X})$, then $\mathcal{X} \setminus \mathcal{K}(\mathcal{X})$ is a basis $\mathcal{X}$.

Q.E.D.

Particularly, if $\mathcal{X}$ is compact space then $\mathcal{X} \setminus \mathcal{K}(\mathcal{X})$ is a basis $\mathcal{X}$.
Let us present some examples.

Example 1.34: Consider $\mathcal{X} = \{1, 2\}$ equipped with topology $\mathcal{G}(\mathcal{X}) = \{\emptyset, \{1\}, \{1, 2\}\}$. Then, $\mathcal{X}$ is T_0 and is not T_1.

△

Example 1.35: Consider $\mathcal{X} = \mathbb{N}$ equipped with topology

$$\mathcal{G}(\mathcal{X}) = \{A \subset \mathbb{N} : \mathbb{N} \setminus A \in \mathsf{Fin}(\mathbb{N})\} \cup \{\emptyset\}.$$

Then, $\mathcal{X}$ is T_1 and is not Hausdorff.

Proof is simple.

1. Let $x, y \in \mathbb{N}$, $x \neq y$ then $G = \mathbb{N} \setminus \{y\}$ fulfills $G \in \mathcal{G}(\mathcal{X})$, $x \in G$ and $y \notin G$. Therefore $\mathcal{X}$ is T_1.
2. Let $G, Q \in \mathcal{G}(\mathcal{X})$, $G \neq \emptyset$, $Q \neq \emptyset$ then $G \cap Q \neq \emptyset$.

   Therefore $\mathcal{X}$ cannot be Hausdorff.

△

Example 1.36: Consider $\mathcal{X}$ equipped with topology

$$\mathcal{G}(\mathcal{X}) = \{A \subset \mathcal{X} : \mathcal{X} \setminus A \in \mathsf{Fin}(\mathcal{X})\} \cup \{\emptyset\}.$$

Then, $\mathcal{X}$ is T_1 Hausdorff if and only if $\mathcal{X}$ is a finite set.

Proof is simple.

1. Let $x, y \in \mathcal{X}$, $x \neq y$ then $G = \mathcal{X} \setminus \{y\}$ fulfills $G \in \mathcal{G}(\mathcal{X})$, $x \in G$ and $y \notin G$. Therefore $\mathcal{X}$ is T_1.
2. Let $\mathcal{X}$ be a finite set.

   Let $x \in \mathcal{X}$ then $\{x\} = \mathcal{X} \setminus (\mathcal{X} \setminus \{x\}) \in \mathcal{G}(\mathcal{X})$, since $\mathcal{X} \setminus \{x\}$ is finite.

   Therefore, if $x, y \in \mathcal{X}$, $x \neq y$, then $\{x\}, \{y\} \in \mathcal{G}(\mathcal{X})$.
3. Let $\mathcal{X}$ be at least countable.

   Let $G, Q \in \mathcal{G}(\mathcal{X})$, $G \neq \emptyset$, $Q \neq \emptyset$ then $G \cap Q \neq \emptyset$. Therefore $\mathcal{X}$ cannot be Hausdorff.

△

Example 1.37: Consider topological space $(\mathcal{X}, \tau)$ which is T_1. We introduce another topology denoted by σ

$$\mathcal{G}(\mathcal{X}; \sigma) = \{\mathcal{X} \setminus K : K \in \mathcal{K}(\mathcal{X}; \tau)\} \cup \{\emptyset\}.$$

Then, $(\mathcal{X}, \sigma)$ is T_1.

- If $\mathcal{X} \notin \mathcal{K}(\mathcal{X};\tau)$ then $(\mathcal{X};\sigma)$ cannot be Hausdorff.
- If $\mathcal{X} \in \mathcal{K}(\mathcal{X};\tau)$ then $(\mathcal{X};\tau)$, $(\mathcal{X};\sigma)$ coincide.

Here is a proof:

1. Space $(\mathcal{X};\tau)$ is T_1, therefore, its points are compacts.
   Take $x, y \in \mathcal{X}$, $x \neq y$.
   Then $G = \mathcal{X} \setminus \{y\} \in \mathcal{G}(\mathcal{X};\sigma)$, $x \in G$ and $y \notin G$.
   Therefore $(\mathcal{X};\sigma)$ is T_1.
2. Let $x, y \in \mathcal{X}$, $x \neq y$, $G, Q \in \mathcal{G}(\mathcal{X};\sigma)$, $x \in G$, $y \in Q$ and $G \cap Q = \emptyset$.
   Then, there are $K, L \in \mathcal{K}(\mathcal{X};\tau)$ such that $G = \mathcal{X} \setminus K$, $Q = \mathcal{X} \setminus L$.
   Consequently, $\mathcal{X} = K \cup L \in \mathcal{K}(\mathcal{X};\tau)$.
   We have derived if $\mathcal{X} \notin \mathcal{K}(\mathcal{X};\tau)$ then $(\mathcal{X};\sigma)$ is not Hausdorff.
3. Let $\mathcal{X} \in \mathcal{K}(\mathcal{X};\tau)$.
   Then, $\mathcal{F}(\mathcal{X};\tau) = \mathcal{K}(\mathcal{X};\tau) = \mathcal{F}(\mathcal{X};\sigma)$.
   Consequently, $(\mathcal{X};\tau)$, $(\mathcal{X};\sigma)$ coincide.

△

1.5 Relative and product topology

Definition 1.38 *Let $\mathcal{X}$ be a topological space and $Y \subset \mathcal{X}$, $Y \neq \emptyset$. Then, relative topology on Y induced by topology of $\mathcal{X}$ is a topology determined by open sets $\mathcal{G}(Y) = \mathcal{G}(\mathcal{X}) \cap Y = \{G \cap Y : G \in \mathcal{G}(\mathcal{X})\}$.*

Lemma 1.39 *Let $\mathcal{X}$ be a topological space and $Y \subset \mathcal{X}$, $Y \neq \emptyset$ be equipped with relative topology. Then, $\mathcal{F}(Y) = \mathcal{F}(\mathcal{X}) \cap Y = \{F \cap Y : F \in \mathcal{F}(\mathcal{X})\}$.*

Let $\mathcal{X}$ be T_1 regular topological space then $\mathcal{K}(Y) = \{K \in \mathcal{K}(\mathcal{X}) : K \subset Y\}$.

Proof:

1. Take $F \in \mathcal{F}(\mathcal{X})$.

   Then, $\mathcal{X} \setminus F \in \mathcal{G}(\mathcal{X})$. Therefore, $(\mathcal{X} \setminus F) \cap Y = Y \setminus F \cap Y \in \mathcal{G}(Y)$ and, consequently, $F \cap Y \in \mathcal{F}(Y)$.

2. Take $F \in \mathcal{F}(Y)$.
   Then, $Y \setminus F \in \mathcal{G}(Y)$ and there exists $G \in \mathcal{G}(\mathcal{X})$ such that $G \cap Y = Y \setminus F$.
   Hence, $F = (\mathcal{X} \setminus G) \cap Y$ and $(\mathcal{X} \setminus G) \in \mathcal{F}(\mathcal{X})$.

3. Let $K \in \mathcal{K}(\mathcal{X})$ and $K \subset Y$.
   Take $G_\lambda \in \mathcal{G}(Y)$, $\lambda \in \Lambda$ such that $K \subset \bigcup_{\lambda\in\Lambda} G_\lambda$.
   Then, there are $Q_\lambda \in \mathcal{G}(\mathcal{X})$ such that $G_\lambda = Q_\lambda \cap Y$ for each $\lambda \in \Lambda$.
   Hence, $K \subset \bigcup_{\lambda\in\Lambda} Q_\lambda$ and one can find $I \in \mathsf{Fin}(\Lambda)$ such that $K \subset \bigcup_{\lambda\in I} Q_\lambda$.
   Then also $K \subset \bigcup_{\lambda\in I} G_\lambda$ and $K \in \mathcal{K}(Y)$.

4. Let $K \in \mathcal{K}(Y)$.
   Then, immediately $K \subset Y$.
   Space $\mathcal{X}$ is regular T_1 then according to lemma 1.32 we have to verify, that from each open cover we are able to select a finite subcover.
   Take $G_\lambda \in \mathcal{G}(\mathcal{X})$, $\lambda \in \Lambda$ such that $K \subset \bigcup_{\lambda\in\Lambda} G_\lambda$.
   Then, $G_\lambda \cap Y \in \mathcal{G}(Y)$ for each $\lambda \in \Lambda$ a $K \subset \bigcup_{\lambda\in\Lambda} G_\lambda \cap Y$.
   Hence, there is $I \in \mathsf{Fin}(\Lambda)$ such that $K \subset \bigcup_{\lambda\in I} G_\lambda \cap Y \subset \bigcup_{\lambda\in I} G_\lambda$.
   We have shown $K \in \mathcal{K}(\mathcal{X})$.

Q.E.D.

Product of topological spaces is equipped with a product topology. We will use the following notation.

$$ {}_{\Psi}\Pi_{\Gamma} \quad \text{projection from } \mathop{\times}_{t\in\Psi} \mathcal{X}_t \text{ to } \mathop{\times}_{t\in\Gamma} \mathcal{X}_t, \text{ where } \emptyset \neq \Gamma \subset \Psi \subset T, \tag{1.5} $$

$$ {}_{\Psi}\Pi_{\Gamma}^{-1} \quad \text{inverse of projection.} \tag{1.6} $$

Definition 1.40 *Let $T \neq \emptyset$ and $\mathcal{X}_t$, $t \in T$ be topological spaces. Then, we define a product topological space*

$$ \bigotimes_{t\in T} \mathcal{X}_t = \bigotimes_{t\in T} (\mathcal{X}_t, \tau_t) = \left(\mathop{\times}_{t\in T} \mathcal{X}_t, \bigotimes_{t\in T} \tau_t \right), $$

where $\mathop{\times}_{t\in T} \mathcal{X}_t$ is Cartesian product and product topology $\bigotimes_{t\in T} \tau_t$ is determined by a subbasis

$$ \mathcal{G} = \left\{ {}_{T}\Pi_{\{t\}}^{-1}(G) \,:\, G \in \mathcal{G}(\mathcal{X}_t), t \in T \right\}. $$

A basis of neighborhoods at point $x \in \bigtimes_{t\in T} \mathcal{X}_t$ can be taken as

$$\mathcal{U}_x = \left\{ {}_T\Pi_I^{-1}\left(\bigtimes_{i\in I} H_i\right) : {}_T\Pi_I\, x \in \bigtimes_{i\in I} H_i,\ \forall i \in I\ H_i \in \mathcal{G}\left(\mathcal{X}_i\right), I \in \mathsf{Fin}\left(T\right)\right\}.$$

An important observation is, Cartesian product of compacts is a compact in product topology.

Theorem 1.41 (Tikhonov): *Let $T \neq \emptyset$ and for each $t \in T$ a topological space $\mathcal{X}_t$ be given. If $K_t \in \mathcal{K}\left(\mathcal{X}_t\right)$ for each $t \in T$ then*

$$\bigtimes_{t\in T} K_t \in \mathcal{K}\left(\bigotimes_{t\in T} \mathcal{X}_t\right).$$

Proof: A proof can be found in [3].

Q.E.D.

1.6 Topology and convergence

Topology is inducing a convergence.

Definition 1.42 *Set $\Lambda = (\Lambda, \leq)$ is called directed (or, directed preorder, filtered set) (cz. usměrněná množina), if*

1. $\Lambda \neq \emptyset$.
2. *(**reflexivity**) $\lambda \leq \lambda$ for each $\lambda \in \Lambda$.*
3. *(**transitivity**) For each $\lambda, \psi, \gamma \in \Lambda$, if $\lambda \leq \psi$ and $\psi \leq \gamma$ then $\lambda \leq \gamma$.*
4. *For each $\lambda, \psi \in \Lambda$, there exists $\omega \in \Lambda$ such that $\lambda \leq \omega$ and $\psi \leq \omega$.*

Recall, we speak about **preorder** if reflexivity and transitivity are fulfilled.

Definition 1.43 *Let Λ be directed set and A_λ, $\lambda \in \Lambda$ be logical expressions. Then, we say*

i) A_λ is true eventually, If there exists $\lambda_0 \in \Lambda$ such that A_λ is true for of all $\lambda \geq \lambda_0$, $\lambda \in \Lambda$.

ii) A_λ is true confinally, if for each $\lambda \in \Lambda$ there exists $\varphi \in \Lambda$, $\varphi \geq \lambda$ such that A_φ is true.

Definition 1.44 *Let Λ, Ψ be directed sets and $\omega : \Psi \to \Lambda$. We say that*

i) ω is monotone, if $\omega(\psi_1) \leq \omega(\psi_2)$ whenever $\psi_1, \psi_2 \in \Psi$, $\psi_1 \leq \psi_2$.

ii) ω is confinal, if for each $\lambda \in \Lambda$ there exists $\psi \in \Psi$ such that $\lambda \leq \omega(\psi)$.

iii) ω is eventual, If there exists $\lambda_0 \in \Lambda$ such that for each $\lambda \in \Lambda$, $\lambda \geq \lambda_0$ there exists $\psi \in \Psi$ such that $\lambda = \omega(\psi)$.

Definition 1.45 *Let A be a nonempty set, Λ be an directed set and $a_\lambda \in A$ be given for each $\lambda \in \Lambda$. Collection $< a_\lambda >_{\lambda\in\Lambda}$ is called net in A (or, generalized sequence) (cz. net, zobecněná sequence).*

Definition 1.46 *Let $< a_\lambda >_{\lambda\in\Lambda}$ and $< b_\psi >_{\psi\in\Psi}$ be nets in A. We say that $< b_\psi >_{\psi\in\Psi}$ is subnet of $< a_\lambda >_{\lambda\in\Lambda}$, whenever there exists monotone confinal mapping $\omega : \Psi \to \Lambda$ such that $b_\psi = a_{\omega(\psi)}$ for all $\psi \in \Psi$.*

Definition 1.47 *Let $\mathcal{X}$ be a topological space, $< a_\lambda >_{\lambda\in\Lambda}$ be a net in $\mathcal{X}$ and $a \in \mathcal{X}$. We say that $< a_\lambda >_{\lambda\in\Lambda}$ converges to a (cz. konverguje k), whenever for each $G \in \mathcal{G}(\mathcal{X})$, $a \in G$ there is $a_\lambda \in G$ eventually.*

Convergence will be denoted by $a_\lambda \xrightarrow[\lambda\in\Lambda]{} a$ in $\mathcal{X}$.

Limit of a net is not determined uniquely in general topological space. The set of all limit points of a net (cz. množina všech limitních bodů) will be denoted by $\mathsf{Li}_{\lambda\in\Lambda}(a_\lambda \| \mathcal{X})$.

Lemma 1.48 *Let $\mathcal{X}$ be a Hausdorff topological space then limit of a net is determined uniquely.*

Proof: Take $x, y \in \mathcal{X}$, $x \neq y$.
Since the space is Hausdorff, given points can be separated by open sets. Hence, if a net converges to one of these points, then it cannot converge to the second one.

Q.E.D.

Definition 1.49 *Let $\mathcal{X}$ be a topological space and $< a_\lambda >_{\lambda\in\Lambda}$ be a net in $\mathcal{X}$. We say that $x \in \mathcal{X}$ is a cluster point of $< a_\lambda >_{\lambda\in\Lambda}$ (cz. hromadný bod), if there exists its subnet which converges to x.*

The set of all cluster points of the net will be denoted by $\mathsf{Ls}_{\lambda\in\Lambda}(a_\lambda \| \mathcal{X})$.

Lemma 1.50 *Let $\mathcal{X}$ be a topological space and $< a_\lambda >_{\lambda\in\Lambda}$ be a net in $\mathcal{X}$. Then the set of all cluster points possesses a description*

$$\begin{aligned}&\mathsf{Ls}_{\lambda\in\Lambda}\left(a_\lambda\,||\mathcal{X}\right) = \\ &= \left\{x \in \mathcal{X} \,:\, \forall\, G \in \mathcal{G}\left(\mathcal{X}\right),\; x \in G \text{ we have } a_\lambda \in G \text{ confinally}\right\}.\end{aligned}$$

Proof:

1. Let $x \in \mathsf{Ls}_{\lambda\in\Lambda}\left(a_\lambda\,||\mathcal{X}\right)$.

   Then, there is $< b_\psi >_{\psi\in\Psi}$ subnet of $< a_\lambda >_{\lambda\in\Lambda}$ which converges to x.

   Hence, $b_\psi \in G$ eventually for each $G \in \mathcal{G}\left(\mathcal{X}\right)$, $x \in G$.

   Consequently, $a_\lambda \in G$ confinally for each $G \in \mathcal{G}\left(\mathcal{X}\right)$, $x \in G$.

2. Let $a_\lambda \in G$ confinally for each $G \in \mathcal{G}\left(\mathcal{X}\right)$, $x \in G$.

   Define an index set with a preordering

   $$\begin{aligned}&\Psi = \left\{(\lambda, G) \,:\, G \in \mathcal{G}\left(\mathcal{X}\right), x \in G,\; a_\lambda \in G\right\},\\ &(\lambda_1, G_1) \le (\lambda_2, G_2) \longleftrightarrow \lambda_1 \le \lambda_2,\; G_1 \supset G_2.\end{aligned}$$

   Take $(\lambda_1, G_1), (\lambda_2, G_2) \in \Psi$.

   Set $G = G_1 \cap G_2$, then, $G \in \mathcal{G}\left(\mathcal{X}\right), x \in G$.

   Since Λ is directed, there is $\gamma \in \Lambda$ such that $\gamma \ge \lambda_1$, $\gamma \ge \lambda_2$.

   We know $a_\lambda \in G$ confinally.

   Hence, there is $\psi \in \Lambda$ such that $\gamma \le \psi$ and $a_\psi \in G$.

   We have constructed $(\psi, G) \in \Psi$ with property $(\psi, G) \ge (\lambda_1, G_1)$, $(\psi, G) \ge (\lambda_2, G_2)$. We have checked, Ψ is a directed set.

   Setting $b_{(\lambda,G)} = a_\lambda$ for $(\lambda, G) \in \Psi$, we are receiving $< b_\psi >_{\psi\in\Psi}$ subnet of $< a_\lambda >_{\lambda\in\Lambda}$. Moreover, $< b_\psi >_{\psi\in\Psi}$ converges to x.

   Thus, $x \in \mathsf{Ls}_{\lambda\in\Lambda}\left(a_\lambda\,||\mathcal{X}\right)$.

Q.E.D.

Lemma 1.51 *Let $\mathcal{G}$ be a subbasis of a topological space $\mathcal{X}$, $< a_\lambda >_{\lambda\in\Lambda}$ be a net in $\mathcal{X}$ and $a \in \mathcal{X}$. Then, $< a_\lambda >_{\lambda\in\Lambda}$ converges to a if and only if $a_\lambda \in G$ eventually for each $G \in \mathcal{G}$, $a \in G$.*

Proof:

1. Let net $< a_\lambda >_{\lambda\in\Lambda}$ converge to a.

   Then, $a_\lambda \in G$ eventually for each $G \in \mathcal{G}$, $a \in G$, because $\mathcal{G} \subset \mathcal{G}(\mathcal{X})$.

2. Let $a_\lambda \in G$ eventually for each $G \in \mathcal{G}$, $a \in G$.

   Take $Q \in \mathcal{G}(\mathcal{X})$, $a \in Q$.

   Then, there exist $k \in \mathbb{N}$ and $G_1, G_2, \ldots, G_k \in \mathcal{G}$ such that $a \in G_1 \cap G_2 \cap \cdots \cap G_k \subset Q$.

   We know $a_\lambda \in G_1$ eventually, $a_\lambda \in G_2$ eventually, $\ldots$, $a_\lambda \in G_k$ eventually. Then, $a_\lambda \in G_1 \cap G_2 \cap \cdots \cap G_k$ eventually, and therefore, $a_\lambda \in Q$ eventually.

   We have checked $< a_\lambda >_{\lambda\in\Lambda}$ converges to a.

Q.E.D.

Lemma 1.52 *Let $\mathcal{X}$ be a topological space, $a \in \mathcal{X}$, $\mathcal{U}_a$ be a basis of neighborhoods at point a, $< a_\lambda >_{\lambda\in\Lambda}$ be a net in $\mathcal{X}$. Then, $< a_\lambda >_{\lambda\in\Lambda}$ converges to a if and only if $a_\lambda \in U$ eventually for each $U \in \mathcal{U}_a$.*

Proof:

1. Let $< a_\lambda >_{\lambda\in\Lambda}$ converge to a.

   Take $U \in \mathcal{U}_a$.

   Then, there is $G \in \mathcal{G}(\mathcal{X})$, $a \in G$ such that $G \subset U$.

   Then, $a_\lambda \in G$ eventually.

   Hence, $a_\lambda \in U$ eventually, since $G \subset U$.

2. Let $a_\lambda \in U$ eventually for each $U \in \mathcal{U}_a$.

   Take $Q \in \mathcal{G}(\mathcal{X})$, $a \in Q$.

   Then, there exists $U \in \mathcal{U}_a$ such that $U \subset Q$.

   Then, $a_\lambda \in U$ eventually.

   Finally, $a_\lambda \in Q$ eventually, since $U \subset Q$.

   We have checked $< a_\lambda >_{\lambda\in\Lambda}$ converges to a.

Q.E.D.

Lemma 1.53 *Let $\mathcal{X}$ be a topological space and $A \subset \mathcal{X}$. Then, $A \in \mathcal{F}(\mathcal{X})$ if and only if for each net $< a_\lambda >_{\lambda\in\Lambda}$ in A, $a_\lambda \xrightarrow[\lambda\in\Lambda]{} a \in \mathcal{X}$ we have $a \in A$.*

Proof:

1. Let $A \in \mathcal{F}(\mathcal{X})$, $< a_\lambda >_{\lambda\in\Lambda}$ be a net in A, $a_\lambda \xrightarrow[\lambda\in\Lambda]{} a \in \mathcal{X}$ and $a \notin A$.

   Then, $a \in \mathcal{X} \setminus A$.

   But, $\mathcal{X} \setminus A \in \mathcal{G}(\mathcal{X})$.

   Hence, $a_\lambda \in \mathcal{X} \setminus A$ eventually.

   That contradicts with assumption $a_\lambda \in A$ for each $\lambda \in \Lambda$.

2. Let for each net $< a_\lambda >_{\lambda\in\Lambda}$ in A, $a_\lambda \xrightarrow[\lambda\in\Lambda]{} a \in \mathcal{X}$ we have $a \in A$.

   Assume, $A \notin \mathcal{F}(\mathcal{X})$.

   Then, there exists $x \in \mathcal{X} \setminus A$ such that for each $Q \in \mathcal{G}(\mathcal{X})$, $x \in Q$ we have $Q \cap A \neq \emptyset$.

   Define an index set with an ordering

   $$\begin{aligned} \Psi &= \{Q \in \mathcal{G}(\mathcal{X}) \,:\, x \in Q\}, \\ Q_1 \leq Q_2 &\iff Q_1 \supset Q_2, \end{aligned}$$

   we are receiving a directed set.

   Far each $G \in \Psi$ we select $a_G \in G \cap A$, since we know $G \cap A \neq \emptyset$.

   Then, $< a_G >_{G\in\Psi}$ is a net in A and $a_G \xrightarrow[G\in\Psi]{} x \in \mathcal{X}$.

   According to our assumption $x \in A$. That is a contradiction, since point was chosen such that $x \notin A$.

Q.E.D.

Lemma 1.54 *Let $\mathcal{X}$ be a topological space, $A \subset \mathcal{X}$ and $< a_\lambda >_{\lambda\in\Lambda}$ be a net in A. Then $\mathsf{Ls}_{\lambda\in\Lambda}(a_\lambda \| \mathcal{X}) \subset \mathrm{clo}(A)$.*

Proof: The observation is a direct consequence of cluster points Definition 1.49 and previous Lemma 1.53.

Q.E.D.

Lemma 1.55 *Let $\mathcal{X}$ be a topological space and $< a_\lambda >_{\lambda\in\Lambda}$ be a net in $\mathcal{X}$. Then $\mathsf{Ls}_{\lambda\in\Lambda}\left(a_\lambda \,||\mathcal{X}\right) \in \mathcal{F}\left(\mathcal{X}\right)$.*

Proof: Take a net $< \xi_\gamma >_{\gamma\in\Gamma}$ in $\mathsf{Ls}_{\lambda\in\Lambda}\left(a_\lambda \,||\mathcal{X}\right)$ with $\xi_\gamma \xrightarrow[\gamma\in\Gamma]{} \eta \in \mathcal{X}$.
Consider $G \in \mathcal{G}\left(\mathcal{X}\right)$, $\eta \in G$.
Then, there is $\gamma \in \Gamma$ such that $\xi_\gamma \in G$.
Accordingly to Lemma 1.50, $a_\lambda \in G$ confinally, since $\xi_\gamma \in \mathsf{Ls}_{\lambda\in\Lambda}\left(a_\lambda \,||\mathcal{X}\right)$.
Hence, Lemma 1.50 says $\eta \in \mathsf{Ls}_{\lambda\in\Lambda}\left(a_\lambda \,||\mathcal{X}\right)$.
Finally according to Lemma 1.53, $\mathsf{Ls}_{\lambda\in\Lambda}\left(a_\lambda \,||\mathcal{X}\right) \in \mathcal{F}\left(\mathcal{X}\right)$.

Q.E.D.

Lemma 1.56 *Let $\mathcal{X}$ be a topological space and $F \in \mathcal{F}\left(\mathcal{X}\right)$. Then, $F \in \mathcal{K}\left(\mathcal{X}\right)$ if and only if $\mathsf{Ls}_{\lambda\in\Lambda}\left(a_\lambda \,||\mathcal{X}\right) \neq \emptyset$ for each net $< a_\lambda >_{\lambda\in\Lambda}$ in F.*

Proof:

1. Let $F \in \mathcal{K}\left(\mathcal{X}\right)$.

   Assume, $< f_\lambda >_{\lambda\in\Lambda}$ in F possesses no cluster point.

   According to Lemma 1.54, $\mathsf{Ls}_{\lambda\in\Lambda}\left(f_\lambda \,||\mathcal{X}\right) \subset F$, since $F \in \mathcal{F}\left(\mathcal{X}\right)$.

   Then, for each point $g \in F$ there exists $G_g \in \mathcal{G}\left(\mathcal{X}\right)$ and $\lambda_g \in \Lambda$ such that $g \in G_g$ and $f_\lambda \notin G_g$ for each $\lambda \geq \lambda_g$, $\lambda \in \Lambda$.

   Then, $F \subset \bigcup_{g\in F} G_g$.

   Since F is a compact, one can select $I \in \mathsf{Fin}\left(F\right)$ such that $F \subset \bigcup_{g\in I} G_g$.

   Then, for all $\lambda \in \Lambda$, fulfilling $\lambda \geq \lambda_g$ for of all $g \in I$, we have $f_\lambda \notin \bigcup_{g\in I} G_g$. Therefore $f_\lambda \notin F$, that is a contradiction.

2. Let each net $< f_\lambda >_{\lambda\in\Lambda}$ in F possess a cluster point.

   Assume, $F \notin \mathcal{K}\left(\mathcal{X}\right)$.

   Then, there exists index set Λ and $G_\lambda \in \mathcal{G}\left(\mathcal{X}\right)$ for $\lambda \in \Lambda$ such that $F \subset \bigcup_{\lambda\in\Lambda} G_\lambda$ and nobody is able to select any finite subcover.

   Index set $\Psi = \mathsf{Fin}\left(\Lambda\right)$ with ordering

   $$I_1 \leq I_2 \iff I_1 \subset I_2$$

   is a directed set.

   Since there exists no finite subcover, for each $I \in \Psi$ we can select a point $f_I \in F \setminus \bigcup_{\lambda\in I} G_\lambda$.

We are receiving a net $< f_I >_{I\in\Psi}$ in F. According to our assumption, there is at least one cluster point of the net, say h.

According to Lemma 1.54, $\mathsf{Ls}_{\lambda\in\Lambda}\left(f_\lambda \,||\mathcal{X}\right) \subset F$, since $F \in \mathcal{F}(\mathcal{X})$.

Therefore, $h \in F$.

Then, there exists $\varphi \in \Lambda$ such that $h \in G_\varphi$.

Then, for each $I \in \Psi$, $I \geq \{\varphi\}$, i.e. $\varphi \in I$, we have $f_I \notin G_\varphi$.

That is a contradiction because h is a cluster point of $< f_I >_{I\in\Psi}$.

Q.E.D.

Theorem 1.57: *Let $\mathcal{X}$ be a topological space, $< x_\lambda >_{\lambda\in\Lambda}$ be a net in $\mathcal{X}$ and $x \in \mathcal{X}$. Then,*

$$x_\lambda \xrightarrow[\lambda\in\Lambda]{} x \text{ in } \mathcal{X} \iff \begin{array}{l}\text{from each subnet } < y_\psi >_{\psi\in\Psi} \text{ of } < x_\lambda >_{\lambda\in\Lambda} \\ \text{one is able to select a subnet } < z_\phi >_{\phi\in\Phi} \\ \text{such that } z_\phi \xrightarrow[\phi\in\Phi]{} x \text{ in } \mathcal{X}.\end{array}$$

Proof:

1. If net converges to x then each its subnet converges to x.

2. Let $< x_\lambda >_{\lambda\in\Lambda}$ be a net in $\mathcal{X}$ such that from each its subnet one is able to select a subnet which converges to $x \in \mathcal{X}$.

   Assume, the net is not converging to x.

   Then, there exists $G \in \mathcal{G}(\mathcal{X})$, $x \in G$ such that $x_\lambda \notin G$ confinally.

   Define

   $$\Psi = \{\lambda \in \Lambda : x_\lambda \notin G\}.$$

   The set is directed, if ordering of Λ is considered.

   Define a net $< b_\lambda >_{\lambda\in\Psi}$ in $\mathcal{X}$ setting $b_\lambda = x_\lambda$ for all $\lambda \in \Psi$.

   Then, $< b_\lambda >_{\lambda\in\Psi}$ is a subnet of $< x_\lambda >_{\lambda\in\Lambda}$.

   According to our assumption, there is its subnet $< \xi_\gamma >_{\gamma\in\Gamma}$ such that $\xi_\gamma \xrightarrow[\gamma\in\Gamma]{} x$. That is a contradiction with selection of $< b_\lambda >_{\lambda\in\Psi}$.

Q.E.D.

This property is typical for convergence induced by a topology.

Convergence induced by product topology is a convergence via coordinates.

Theorem 1.58: *Let $T \neq \emptyset$ and for each $t \in T$ a topological space $\mathcal{X}_t$ be given. Let $< x_\psi >_{\psi \in \Psi}$ be a net in $\bigtimes_{t \in T} \mathcal{X}_t$ and $x \in \bigtimes_{t \in T} \mathcal{X}_t$. Then,*

$$x_\psi \xrightarrow[\psi \in \Psi]{} x \text{ in } \bigotimes_{t \in T} \mathcal{X}_t \iff \forall\, t \in T : \; x_{\psi,t} \xrightarrow[\psi \in \Psi]{} x_t \text{ in } \mathcal{X}_t .$$

Proof: Product topology is determined by a subbasis

$$\mathcal{G} = \left\{ {}_T\Pi_{\{t\}}^{-1}(G) \;:\; G \in \mathcal{G}(\mathcal{X}_t), t \in T \right\}.$$

1. Let $x_\psi \xrightarrow[\psi \in \Psi]{} x$ in $\bigotimes_{t \in T} \mathcal{X}_t$.

   Fix $t \in T$.

   Then for each $G \in \mathcal{G}(\mathcal{X}_t)$, $x_t \in G$ we have $x_\psi \in {}_T\Pi_{\{t\}}^{-1}(G)$ eventually, since $x \in {}_T\Pi_{\{t\}}^{-1}(G)$ and ${}_T\Pi_{\{t\}}^{-1}(G) \in \mathcal{G}$.

   That means $x_{\psi,t} \in G$ eventually.

   Thus, $\forall\, t \in T$ we have $x_{\psi,t} \xrightarrow[\psi \in \Psi]{} x_t$ in $\mathcal{X}_t$.

2. Let $\forall\, t \in T$ we have $x_{\psi,t} \xrightarrow[\psi \in \Psi]{} x_t$ in $\mathcal{X}_t$.

   Then for each $G \in \mathcal{G}(\mathcal{X}_t)$, $t \in T$, $x_t \in G$ we have $x_{\psi,t} \in G$ eventually.

   That means $x_\psi \in {}_T\Pi_{\{t\}}^{-1}(G)$ eventually.

   Thus, $x_\psi \xrightarrow[\psi \in \Psi]{} x$ in $\bigotimes_{t \in T} \mathcal{X}_t$.

Q.E.D.

If a topological space fulfills I.axiom of countability, topology is determined by convergence of sequences.

Lemma 1.59 *Let $\mathcal{X}$ be a topological space which fulfills I.axiom of countability, and $A \subset \mathcal{X}$. Then, $A \in \mathcal{F}(\mathcal{X})$ if and only if for each sequence $x_n \in A$, $n \in \mathbb{N}$, which converges to a point $x \in \mathcal{X}$, we have $x \in A$.*

Proof: Take net $< x_\lambda >_{\lambda\in\Lambda}$ in A which converges to $x \in \mathcal{X}$.
Space fulfills I.axiom of countability and, thus, there exists countable basis of neighborhoods at point x. Without any loss of generality we can assume a basis of neighborhoods with property $U_1 \supset U_2 \supset U_3 \supset U_4 \supset \dots$.
Then, there are indexes $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \dots$ such that for each $k \in \mathbb{N}$ and for all $\lambda \in \Lambda$, $\lambda \geq \lambda_k$ we have $x_\lambda \in U_k$.
Therefore $x_{\lambda_k} \xrightarrow[k\to+\infty]{} x$.
Consequently $x \in A$ and A is a closed set.

Q.E.D.

Let us mention, $< x_{\lambda_k} >_{k\in\mathbb{N}}$ does not have to be a subnet of $< x_\lambda >_{\lambda\in\Lambda}$.

Lemma 1.60 *Let T be uncountable and a topological space $\mathcal{X}_t$ is given for each $t \in T$. If for each $t \in T$ there is $G_t \in \mathcal{G}(\mathcal{X}_t)$ such that $G_t \neq \emptyset$ and $G_t \neq \mathcal{X}_t$ then product topological space $(\bigtimes_{t\in T} \mathcal{X}_t, \bigotimes_{t\in T} \tau(\mathcal{X}_t))$ does not fulfill the I. axiom of countability.*

Proof: Select for each $t \in T$ points $0_t \in G_t$, $1_t \notin G_t$ and denote $\mathbf{0} = \{0_t : t \in T\}$.
Assume $\mathcal{U}_\mathbf{0}$ is a countable basis of neighborhoods at $\mathbf{0}$.
Then, members of the basis can be numbered $\mathcal{U}_\mathbf{0} = \{U_i : i \in \mathbb{N}\}$.
Hence, for each $i \in \mathbb{N}$ there is $Q_i \in \mathcal{G}\left(\bigtimes_{t\in T} \mathcal{X}_t\right)$ such that $\mathbf{0} \in Q_i \subset U_i$.
Moreover, for each $i \in \mathbb{N}$ there is $I_i \in \mathsf{Fin}(T)$, $H_{i,t} \in \mathcal{G}(\mathcal{X}_t)$ for all $t \in I_i$ such that $\mathbf{0} \in {}_T\Pi_{I_i}^{-1}\left(\bigtimes_{t\in I_i} H_{i,t}\right) \subset Q_i \subset U_i$.
There is $\tau \in T \setminus \bigcup_{i\in\mathbb{N}} I_i$, since T is uncountable and $\bigcup_{i\in\mathbb{N}} I_i$ is at most countable.
Consider point $\delta \in \bigtimes_{t\in T} \mathcal{X}_t$, where $\delta_\tau = 1_\tau$ and $\delta_t = 0_t$ for all $t \in T$, $t \neq \tau$.
Then, $\mathbf{0} \in {}_T\Pi_{\{\tau\}}^{-1}(G_\tau) \in \mathcal{G}\left(\bigtimes_{t\in T} \mathcal{X}_t\right)$ and $\delta \notin {}_T\Pi_{\{\tau\}}^{-1}(G_\tau)$ but $\delta \in U_i$ for all $i \in \mathbb{N}$.
That is a contradiction, because $\mathcal{U}_\mathbf{0}$ is a basis of neighborhoods at $\mathbf{0}$.

Q.E.D.

There is a general theory on convergence, i.e. Convergence Spaces. Their theory lays outside of the concept of the lecture. Convergence almost surely is a straightforward example of convergence which is induced by no topology. Even for real random variables we have no topology inducing almost sure convergence. It is seen because statement of Theorem 1.57 is violated.

1.7 Continuity of functions

In this section, we introduce definitions and basic properties of continuous and semicontinuous functions.

Definition 1.61 *Let $\mathcal{X}$, $\mathcal{Y}$ be topological spaces and $f : \mathcal{X} \to \mathcal{Y}$ be a function. We say f is continuous (cz. spojitá) whenever for all $G \in \mathcal{G}(Y)$ we have $f^{-1}(G) \in \mathcal{G}(\mathcal{X})$.*

The set of all continuous functions from $\mathcal{X}$ to $\mathcal{Y}$ will be denoted by $\mathsf{C}(\mathcal{X}, \mathcal{Y})$. If $\mathcal{Y} = \mathbb{R}$ we abbreviate notation by $\mathsf{C}(\mathcal{X})$.

Definition 1.62 *Let $\mathcal{X}$ be a topological space and $f : \mathcal{X} \to \mathbb{R}$ be a function.*

- *We say that f is lower semicontinuous (cz. zdola polospojitá) whenever for all $r \in \mathbb{R}$ we have $f^{-1}((r, +\infty)) \in \mathcal{G}(\mathcal{X})$.*
- *We say that f is upper semicontinuous (cz. zhora polospojitá) whenever for all $r \in \mathbb{R}$ we have $f^{-1}((+\infty, r)) \in \mathcal{G}(\mathcal{X})$.*

These notions possess equivalent description by convergent nets. For that we need to explain a notation.

Definition 1.63 *Let $\mathcal{X}$ be a topological space, $A \subset \mathcal{X}$, $A \neq \emptyset$, $x \in \mathrm{clo}(A)$.*

- *Let $\mathcal{Y}$ be a topological space, $y \in \mathcal{Y}$ and $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. We say that $f(\xi)$ is tending to y while ξ is tending to x with respect to A if for each $G \in \mathcal{G}(Y)$, $y \in G$ there is $H \in \mathcal{G}(\mathcal{X})$, $x \in H$ such that for each $\xi \in H \cap A$, $\xi \neq x$ we have $f(\xi) \in G$. We denote the fact by the symbol*

$$\lim_{\substack{\xi \to x \\ \xi \in A}} f(\xi) = y.$$

- *Let $f : \mathcal{X} \to \mathbb{R}$ be a function and $y \in \mathbb{R}$. We say that limes inferior of $f(\xi)$ is y while ξ is tending to x with respect to A if*
  - *For each $r \in \mathbb{R}$, $r < y$ there is $H \in \mathcal{G}(\mathcal{X})$, $x \in H$ such that for each $\xi \in H \cap A$, $\xi \neq x$ we have $f(\xi) > r$.*
  - *For each $r \in \mathbb{R}$, $r > y$, $H \in \mathcal{G}(\mathcal{X})$, $x \in H$ there is $\xi \in H \cap A$, $\xi \neq x$ with $f(\xi) < r$.*

  We denote the fact by the symbol

$$\liminf_{\substack{\xi \to x \\ \xi \in A}} f(\xi) = y.$$

- *Let $f : \mathcal{X} \to \mathbb{R}$ be a function and $y \in \mathbb{R}$. We say that limes superior of $f(\xi)$ is y while ξ is tending to x with respect to A if*
  - *For each $r \in \mathbb{R}$, $r > y$ there is $H \in \mathcal{G}(\mathcal{X})$, $x \in H$ such that for each $\xi \in H \cap A$, $\xi \neq x$ we have $f(\xi) < r$.*
  - *For each $r \in \mathbb{R}$, $r < y$, $H \in \mathcal{G}(\mathcal{X})$, $x \in H$ there is $\xi \in H \cap A$, $\xi \neq x$ with $f(\xi) > r$.*

  We denote the fact by the symbol

$$\limsup_{\substack{\xi \to x \\ \xi \in A}} f(\xi) = y.$$

For $A = \mathcal{X}$ we simplify the notation

$$\lim_{\xi \to x} f(\xi), \ \liminf_{\xi \to x} f(\xi), \ \limsup_{\xi \to x} f(\xi).$$

These limits can be explained using nets.

Lemma 1.64 *Let $\mathcal{X}$ be a topological space, $A \subset \mathcal{X}$, $A \neq \emptyset$ and $x \in \mathrm{clo}\,(A)$.*

- *Let $\mathcal{Y}$ be a topological space, $y \in \mathcal{Y}$ and $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. Then*

$$\lim_{\substack{\xi \to x \\ \xi \in A}} f(\xi) = y$$

  iff

  for each net $< x_\lambda >_{\lambda \in \Lambda}$ in A with $x_\lambda \xrightarrow[\lambda \in \Lambda]{} x$ in $\mathcal{X}$ and $x_\lambda \neq x$ for all $\lambda \in \Lambda$ we have $f(x_\lambda) \xrightarrow[\lambda \in \Lambda]{} y$ in $\mathcal{Y}$.

- *Let $y \in \mathbb{R}$ and $f : \mathcal{X} \to \mathbb{R}$ be a mapping. Then*

$$\liminf_{\substack{\xi \to x \\ \xi \in A}} f(\xi) = y$$

  iff

  - *For each net $< x_\lambda >_{\lambda \in \Lambda}$ in A with $x_\lambda \xrightarrow[\lambda \in \Lambda]{} x$ in $\mathcal{X}$ and $x_\lambda \neq x$ for all $\lambda \in \Lambda$ we have $\eta \geq y$ for each $\eta \in \mathsf{Ls}_{\lambda \in \Lambda} f(x_\lambda)$.*
  - *There is a net $< \xi_\psi >_{\psi \in \Psi}$ in A with $\xi_\psi \xrightarrow[\psi \in \Psi]{} x$ in $\mathcal{X}$, $\xi_\psi \neq x$ for all $\psi \in \Psi$ and $f(\xi_\psi) \xrightarrow[\psi \in \Psi]{} y$ in $\mathbb{R}$.*

- *Let $y \in \mathbb{R}$ and $f : \mathcal{X} \to \mathbb{R}$ be a mapping. Then*

$$\limsup_{\substack{\xi \to x \\ \xi \in A}} f(\xi) = y$$

iff

  - *For each net $< x_\lambda >_{\lambda \in \Lambda}$ in A with $x_\lambda \xrightarrow[\lambda \in \Lambda]{} x$ in $\mathcal{X}$ and $x_\lambda \neq x$ for all $\lambda \in \Lambda$ we have $\eta \leq y$ for each $\eta \in \mathsf{Ls}_{\lambda \in \Lambda} f(x_\lambda)$.*
  - *There is a net $< \xi_\psi >_{\psi \in \Psi}$ in A with $\xi_\psi \xrightarrow[\psi \in \Psi]{} x$ in $\mathcal{X}$, $\xi_\psi \neq x$ for all $\psi \in \Psi$ and $f(\xi_\psi) \xrightarrow[\psi \in \Psi]{} y$ in $\mathbb{R}$.*

Lemma 1.65

- *Let $\mathcal{X}$, $\mathcal{Y}$ be topological spaces and $f : \mathcal{X} \to \mathcal{Y}$ be a function. The function f is continuous iff for all $x \in \mathcal{X}$*

$$\lim_{y \to x} f(y) = f(x).$$

- *Let $\mathcal{X}$ be a topological space and $f : \mathcal{X} \to \mathbb{R}$ be a function. The function f is lower semicontinuous iff for each $x \in \mathcal{X}$*

$$\liminf_{y \to x} f(y) \geq f(x).$$

- *Let $\mathcal{X}$ be a topological space and $f : \mathcal{X} \to \mathbb{R}$ be a function. The function f is upper semicontinuous iff for each $x \in \mathcal{X}$*

$$\limsup_{y \to x} f(y) \leq f(x).$$

Continuity of a function can be treated at a single point.

Definition 1.66 *Let $\mathcal{X}$, $\mathcal{Y}$ be topological spaces, $f : \mathcal{X} \to \mathcal{Y}$ be a function and $x \in \mathcal{X}$.*

- *We say that f is continuous at x, whenever*

$$\lim_{y \to x} f(y) = f(x).$$

- *Let $A \subset \mathcal{X}$. We say that f is continuous at x on A, whenever*

$$\lim_{\substack{y \to x \\ y \in A}} f(y) = f(x).$$

Continuous functions determine an important σ-algebra.

Definition 1.67 *Let $\mathcal{X}$ be a topological space. The smallest σ-algebra for which all continuous real functions $f : \mathcal{X} \to \mathbb{R}$ are measurable is called Baire σ-algebra. We will use notation $\mathsf{Baire}(\mathcal{X})$.*

Members of $\mathsf{Baire}(\mathcal{X})$ are called Baire sets of the space $\mathcal{X}$.

Evidently, $\mathsf{Baire}(\mathcal{X}) \subset \mathcal{B}(\mathcal{X})$.

1.8 Measures on topological spaces

Combining measures with topology presents a very powerful tool.

Definition 1.68 *Let $\mathcal{X}$ be a topological space. All measures defined on Borel σ-algebra $\mathcal{B}(\mathcal{X})$ are called Borel measures.*

We will consider Borel probability measures, or simply probabilities, in this text.

Definition 1.69 *Let $\mathcal{X}$ be a topological space. We denote by $\mathcal{M}_1(\mathcal{X})$ the set of all Borel probability measures, i.e. all probability measures defined on Borel σ-algebra $\mathcal{B}(\mathcal{X})$.*

Definition 1.70 *Let $\mathcal{X}$ be a topological space. A family of sets*

$$\mathcal{U}(\mathcal{X}) = \bigcap_{\mu \in \mathcal{M}_1(\mathcal{X})} \mathcal{MS}(\mu)$$

is called σ-algebra of universally measurable sets of $\mathcal{X}$ (cz. σ-algebra univerzálně měřitelných množin).

Direct consequence of the definition is a chain of inclusions
$\mathsf{Baire}(\mathcal{X}) \subset \mathcal{B}(\mathcal{X}) \subset \mathcal{U}(\mathcal{X})$.

By definition, probabilities are σ-additive. But they could possess better and more helpful properties.

Definition 1.71 *Let $\mathcal{X}$ be a topological space. We say $\mu \in \mathcal{M}_1(\mathcal{X})$ is regular (cz. regulární) Borel probability measure if for each $A \in \mathcal{B}(\mathcal{X})$ we have*

$$\mu(A) = \sup\{\mu(F) : F \subset A,\ F \in \mathcal{F}(\mathcal{X})\}.$$

Definition 1.72 *Let $\mathcal{X}$ be a topological space. We say $\mu \in \mathcal{M}_1(\mathcal{X})$ is Radon (or, tight) (cz. Radonova, těsná) probability measure if for each $A \in \mathcal{B}(\mathcal{X})$ we have*

$$\mu(A) = \sup\{\mu(K) : K \subset A,\ K \in \mathcal{K}(\mathcal{X})\}.$$

The set of all Radon probability measures on $\mathcal{X}$ will be denoted by $\mathcal{M}_{1,t}(\mathcal{X})$.

Definition 1.73 *Let A be a nonempty set. We say that $\mathcal{A} \subset \mathcal{P}(A)$ is a filter, (cz. filtr) if $\mathcal{A}$ is nonempty and for each $B, C \in \mathcal{A}$ there exists $\omega \in \mathcal{A}$ such that $B \supset \omega$ and $C \supset \omega$.*

Everybody can notice that $\mathcal{A}$ is a filter equivalently means $(\mathcal{A}, \leq)$ is directed with $B \leq C$ denoting $B \supset C$.

Definition 1.74 *Let $\mathcal{X}$ be a topological space. We say $\mu \in \mathcal{M}_1(\mathcal{X})$ is τ-additive (cz. τ-aditivní) Borel probability measure if for each filter $\mathcal{F} \subset \mathcal{F}(\mathcal{X})$ we have*

$$\mu\left(\bigcap_{F \in \mathcal{F}} F\right) = \inf\{\mu(F) : F \in \mathcal{F}\}.$$

These properties of Borel probability measures are related. Let us introduce some relations among them.

Lemma 1.75 *Let $\mathcal{X}$ be a topological space and $\mu \in \mathcal{M}_1(\mathcal{X})$. Then μ is regular iff for each $A \in \mathcal{B}(\mathcal{X})$ we have*

$$\mu(A) = \inf\{\mu(G) : G \supset A,\ G \in \mathcal{G}(\mathcal{X})\}.$$

Lemmas from [8], pp.64-65, 1.7.9, 1.7.10, 1.7.14.

Lemma 1.76 *Let $\mathcal{X}$ be a regular topological space. If $\mu \in \mathcal{M}_1(\mathcal{X})$ is τ-additive then μ is regular.*

Proof: Consider that for each $F \in \mathcal{F}(\mathcal{X})$ we have

$$\mu(F) = \inf\{\mu(\mathrm{int}(H)) : \mathrm{int}(H) \supset F,\ H \in \mathcal{F}(\mathcal{X})\}.$$

Q.E.D.

Lemma 1.77 *Let $\mathcal{X}$ be a topological space. If $\mu \in \mathcal{M}_1(\mathcal{X})$ is Radon then μ is τ-additive.*

Lemma 1.78 *Let $\mathcal{X}$ be a locally compact topological space. If $\mu \in \mathcal{M}_1(\mathcal{X})$ is τ-additive then μ is Radon.*

Theorem 1.79: *Let $\mathcal{X}$ be a compact topological space. If $\mu : \mathsf{Baire}(\mathcal{X}) \to [0,1]$ is a probability measure then there is uniquely defined $\tilde{\mu} \in \mathcal{M}_1(\mathcal{X})$ which is regular, τ-additive and enlarging μ, i.e. for all $B \in \mathsf{Baire}(\mathcal{X})$ we have $\tilde{\mu}(B) = \mu(B)$.*

Proof: See Theorem II.8.8, p.177 in [8].

Q.E.D.

1.9 Random maps

In this section, we consider a probability space $(\Omega, \mathcal{A}, \mathsf{P})$, a nonempty set H and maps from Ω to H.

1.9.1 General definitions

At first, let us recall definitions of outer and inner probabilities.

Definition 1.80 *Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space. Then, outer probability (cz. vnější pravděpodobnost) is defined as*

$$\mathsf{P}^* : \mathcal{P}(\Omega) \to [0,1] : A \in \mathcal{P}(\Omega) \mapsto \inf \{\mathsf{P}(B) \,:\, A \subset B \in \mathcal{A}\}$$

and inner probability (cz. vnitřní pravděpodobnost) is defined as

$$\mathsf{P}_* : \mathcal{P}(\Omega) \to [0,1] : A \in \mathcal{P}(\Omega) \mapsto \sup \{\mathsf{P}(B) \,:\, A \supset B \in \mathcal{A}\}.$$

Definition 1.81 *Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space, H be a nonempty set and $X : \Omega \to H$ be a map. We define outer distribution of X (cz. vnější rozdělení)*

$$\mu^*_X : \mathcal{P}(H) \to [0,1] : A \in \mathcal{P}(H) \mapsto \mathsf{P}^*(X \in A),$$

inner distribution of X (cz. vnitřní rozdělení)

$$\mu_{*X} : \mathcal{P}(H) \to [0,1] : A \in \mathcal{P}(H) \mapsto \mathsf{P}_*(X \in A)$$

and measurability region of X (cz. oblast měřitelnosti)

$$\Sigma_X = \{A \in \mathcal{P}(H) \,:\, \mathsf{P}^*(X \in A) = \mathsf{P}_*(X \in A)\}.$$

Lemma 1.82 *Always, Σ_X is a σ-algebra and μ_X^* is a probability on Σ_X.*

Definition 1.83 *Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space, H be a nonempty set, $\mathcal{B} \subset \mathcal{P}(H)$ be a σ-algebra and $X : \Omega \to H$ be a map. If $\mathcal{B} \subset \Sigma_X$, we say, X is a $\mathcal{B}$-measurable random variable (cz. $\mathcal{B}$-měřitelná náhodná veličina) and outer distribution of X restricted to $\mathcal{B}$ is called distribution of X (cz. rozdělení); notation $\mu_X = \mu_X^* |_{\mathcal{B}}$.*

Measurability is usually denoted by $X : (\Omega, \mathcal{A}) \to (H, \mathcal{B})$ and, often, term random variable with values in $(H, \mathcal{B})$ is used.

Definition 1.84 *Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space, H be a topological space and $X : \Omega \to H$ be a map. We say X is a random variable (or, Borel random variable) (cz. náhodná veličina, borelovská náhodná veličina) if $\mathcal{B}(H) \subset \Sigma_X$, i.e. map X is $\mathcal{B}(H)$-measurable.*

Definition 1.85 *Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space, H be a nonempty set, $\mathcal{B} \subset \mathcal{P}(H)$ be a σ-algebra $X, Y : \Omega \to H$ be maps with $\mathcal{B} \subset \Sigma_X$, $\mathcal{B} \subset \Sigma_Y$. We say, distributions of X, Y coincide on $\mathcal{B}$ (cz. rozdělení se shodují) whenever $\mu_X(B) = \mu_Y(B)$ for each $B \in \mathcal{B}$.*

This fact will be denoted by $X \stackrel{\mathcal{D}}{\equiv} Y$ on $\mathcal{B}$.

1.9.2 Topology and randomness

Topology is combined with randomness in this section. We consider a net of random variables $X_\lambda : (\Omega, \mathcal{A}) \to (\mathcal{X}, \mathcal{B}(\mathcal{X}))$, $\lambda \in \Lambda$, where $(\Omega, \mathcal{A}, \mathsf{P})$ is a probability space and $\mathcal{X}$ is a topological space.

We focus to three most important convergences used for random variables; i.e. convergence almost surely, in probability, in distribution. Convergence in distribution of random variables is defined as the weak convergence of distributions of these random variables. Thus, the weak convergence of probability measures must be also remembered.

Definition 1.86 *Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space, $\mathcal{X}$ be a topological space, $< \mu_\lambda >_{\lambda \in \Lambda}$ be a net of probability measures and $\mu \in \mathcal{M}_1(\mathcal{X})$. We say $< \mu_\lambda >_{\lambda \in \Lambda}$ converges weakly to μ (cz. konverguje slabě), whenever for each $G \in \mathcal{G}(\mathcal{X})$ it is fulfilled*

$$\liminf_{\lambda \in \Lambda} \mu_\lambda(G) \geq \mu(G).$$

The convergence will be denoted $\mu_\lambda \xrightarrow[\lambda \in \Lambda]{\mathrm{w}} \mu$ in $\mathcal{X}$.

Definition 1.87 *Let $\mathcal{X}$ be a topological space. We consider $\mathcal{M}_1(\mathcal{X})$ as a topological space $\mathcal{M}_1(\mathcal{X}) = (\mathcal{M}_1(\mathcal{X}), \mathrm{w})$, where topology* w *is induced by weak convergence and is called weak topology.*

Definition 1.88 *Let $\mathcal{X}$ be a topological space and $\mathcal{M} \subset \mathcal{M}_1(\mathcal{X})$. We say $\mathcal{M}$ is a weakly relative compact (cz. slabě relativní kompakt) if from each net in $\mathcal{M}$ we are able to select a subnet convergent in $\mathcal{M}_1(\mathcal{X})$.*

In other words, $\mathrm{clo}\,(\mathcal{M}) \in \mathcal{K}\,(\mathcal{M}_1(\mathcal{X}))$.

Definition 1.89 *Let $\mathcal{X}$ be a topological space and $\mathcal{M} \subset \mathcal{M}_1(\mathcal{X})$. We say $\mathcal{M}$ is tight (cz. těsná) if for each $\varepsilon > 0$ there exists $K \in \mathcal{K}\,(\mathcal{X})$ such that $\mu(K) > 1 - \varepsilon$ for each $\mu \in \mathcal{M}$.*

We have an immediate simple observation.

Lemma 1.90 *Let $\mathcal{X}$ be a topological space and $\mathcal{M} \subset \mathcal{M}_1(\mathcal{X})$. If $\mathcal{M}$ is tight, then $\mathcal{M} \subset \mathcal{M}_{1,t}(\mathcal{X})$.*

Definition 1.91 *Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space, $\mathcal{X}$ be a topological space. Consider a net of random variables $< X_\lambda >_{\lambda \in \Lambda}$ with values in $\mathcal{X}$ and a random variable X with values in $\mathcal{X}$.*

1. *We say that $< X_\lambda >_{\lambda \in \Lambda}$ converges almost surely to X in $\mathcal{X}$, whenever there exists $\Omega_0 \in \mathcal{A}$, $\mathsf{P}\,(\Omega_0) = 1$ such that (cz. konverguje skoro jistě, konverguje s.j.) for each $\omega \in \Omega_0$ it is fulfilled $X_\lambda(\omega) \xrightarrow[\lambda \in \Lambda]{} X(\omega)$ in $\mathcal{X}$.*

   The convergence will be denoted $X_\lambda \xrightarrow[\lambda \in \Lambda]{a.s.} X$ in $\mathcal{X}$.

2. *We say that $< X_\lambda >_{\lambda \in \Lambda}$ converges in t-probability to X in $\mathcal{X}$ (cz. konverguje v t-pravděpodobnosti), whenever for each $G \in \mathcal{G}\,(\mathcal{X})$*

$$\lim_{\lambda \in \Lambda} \mathsf{P}\,(X_\lambda \notin G, X \in G) = 0.$$

   The convergence will be denoted $X_\lambda \xrightarrow[\lambda \in \Lambda]{t-P} X$ in $\mathcal{X}$.

3. *We say that $< X_\lambda >_{\lambda \in \Lambda}$ converges in distribution to X in $\mathcal{X}$ (cz. konverguje v distribuci), whenever for each $G \in \mathcal{G}\,(\mathcal{X})$ it is fulfilled*

$$\liminf_{\lambda \in \Lambda} \mathsf{P}\,(X_\lambda \in G) \geq \mathsf{P}\,(X \in G)\,.$$

   The convergence will be denoted $X_\lambda \xrightarrow[\lambda \in \Lambda]{\mathcal{D}} X$ in $\mathcal{X}$.

Definition 1.92 *We say that a random variable X is Radon (or tight), whenever its distribution is a Radon probability measure; see Definition 1.72.*

Theorem 1.93: *Let X be a Radon random variable and $\mathcal{G}$ be a subbasis $\mathcal{X}$ then*

$$X_\lambda \xrightarrow[\lambda\in\Lambda]{t-P} X \text{ in } \mathcal{X} \quad \Longleftrightarrow \quad \forall\, G \in \mathcal{G} : \lim_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin G, X \in G\right) = 0 \; .$$

Proof: Let $G \in \mathcal{G}\left(\mathcal{X}\right)$.
Then, it can be written as $G = \bigcup_{\psi\in\Psi} G_\psi$, where $G_\psi = \bigcap_{j\in J_\psi} Q_{j,\psi}$ for some $Q_{j,\psi} \in \mathcal{G}$ and J_ψ is a finite set for each $\psi \in \Psi$.
Fix $\varepsilon > 0$.
Since X is a Radon random variable, there is $K \in \mathcal{K}\left(\mathcal{X}\right)$ such that $K \subset G$ and $\mathsf{P}\left(X \in K\right) > \mathsf{P}\left(X \in G\right) - \varepsilon$.
Since $K \in \mathcal{K}\left(\mathcal{X}\right)$ and $K \subset G = \bigcup_{\psi\in\Psi} G_\psi$, there exists $I \in \mathsf{Fin}\left(\Psi\right)$ such that $K \subset \bigcup_{\psi\in I} G_\psi \subset G$. Therefore,

$$\mathsf{P}\left(X \in \bigcup_{\psi\in I} G_\psi\right) \geq \mathsf{P}\left(X \in K\right) > \mathsf{P}\left(X \in G\right) - \varepsilon,$$

Then,

$$\begin{aligned}
&\limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin G, X \in G\right) \\
&\leq \limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin \bigcup_{\psi\in I} G_\psi, X \in G\right) \\
&\leq \limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin \bigcup_{\psi\in I} G_\psi, X \in \bigcup_{\psi\in I} G_\psi\right) + \mathsf{P}\left(X \in G\right) - \mathsf{P}\left(X \in \bigcup_{\psi\in I} G_\psi\right) \\
&\leq \sum_{\psi\in I} \limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin \bigcup_{\psi\in I} G_\psi, X \in G_\psi\right) + \varepsilon \\
&\leq \sum_{\psi\in I} \limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin G_\psi, X \in G_\psi\right) + \varepsilon \\
&= \sum_{\psi\in I} \limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin \bigcap_{j\in J_\psi} Q_{j,\psi}, X \in \bigcap_{j\in J_\psi} Q_{j,\psi}\right) + \varepsilon \\
&\leq \sum_{\psi\in I} \sum_{j\in J_\psi} \limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin Q_{j,\psi}, X \in \bigcap_{j\in J_\psi} Q_{j,\psi}\right) + \varepsilon \\
&\leq \sum_{\psi\in I} \sum_{j\in J_\psi} \limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin Q_{j,\psi}, X \in Q_{j,\psi}\right) + \varepsilon = \varepsilon .
\end{aligned}$$

Q.E.D.

Theorem 1.94: *Let X be Radon random variable and $\mathcal{G}$ be a basis $\mathcal{X}$ then*

$$X_\lambda \xrightarrow[\lambda\in\Lambda]{\mathcal{D}} X \text{ in } \mathcal{X} \iff \begin{array}{c} \liminf_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \in \bigcup_{i\in I} G_i\right) \geq \mathsf{P}\left(X \in \bigcup_{i\in I} G_i\right) \\ \forall\ I \text{ finite and } G_i \in \mathcal{G} \text{ for } i \in I. \end{array}$$

Proof: Let $G \in \mathcal{G}(\mathcal{X})$.
Then, it can be written as $G = \bigcup_{\psi\in\Psi} G_\psi$, where $G_\psi \in \mathcal{G}$.
Fix $\varepsilon > 0$.
Since X is a Radon random variable, there is $K \in \mathcal{K}(\mathcal{X})$ such that $K \subset G$ and $\mathsf{P}(X \in K) > \mathsf{P}(X \in G) - \varepsilon$.
Since $K \in \mathcal{K}(\mathcal{X})$ and $K \subset G = \bigcup_{\psi\in\Psi} G_\psi$, there exists $I \in \mathsf{Fin}(\Psi)$ such that $K \subset \bigcup_{\psi\in I} G_\psi \subset G$. Therefore,

$$\mathsf{P}\left(X \in \bigcup_{\psi\in I} G_\psi\right) > \mathsf{P}(X \in G) - \varepsilon.$$

Then,

$$\begin{aligned} \liminf_{\lambda\in\Lambda} \mathsf{P}(X_\lambda \in G) &\geq \liminf_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \in \bigcup_{\psi\in I} G_\psi\right) \\ &\geq \mathsf{P}\left(X \in \bigcup_{\psi\in I} G_\psi\right) > \mathsf{P}(X \in G) - \varepsilon. \end{aligned}$$

Q.E.D.

Theorem 1.95 (Portmanteau lemma): *Let $X_\lambda, \lambda \in \Lambda$ and X be random variables with values in a topological space $\mathcal{X}$. Then, following statements are equivalent:*

i) $X_\lambda \xrightarrow[\lambda\in\Lambda]{\mathcal{D}} X$ *in* $\mathcal{X}$.

ii) $\mu_{X_\lambda} \xrightarrow[\lambda\in\Lambda]{\mathrm{w}} \mu_X$ *in* $\mathcal{X}$.

iii) For all $G \in \mathcal{G}(\mathcal{X})$ it is fulfilled

$$\liminf_{\lambda\in\Lambda} \mathsf{P}(X_\lambda \in G) \geq \mathsf{P}(X \in G).$$

iv) For all $F \in \mathcal{F}(\mathcal{X})$ it is fulfilled

$$\limsup_{\lambda \in \Lambda} \mathsf{P}(X_\lambda \in F) \leq \mathsf{P}(X \in F).$$

v) For all lower bounded and lower semicontinuous functions $f : \mathcal{X} \to \mathbb{R}$ it is fulfilled

$$\liminf_{\lambda \in \Lambda} \mathsf{E}[f(X_\lambda)] \geq \mathsf{E}[f(X)].$$

vi) For all upper bounded and upper semicontinuous functions $f : \mathcal{X} \to \mathbb{R}$ it is fulfilled

$$\limsup_{\lambda \in \Lambda} \mathsf{E}[f(X_\lambda)] \leq \mathsf{E}[f(X)].$$

Bounded continuous functions are not giving an equivalent characterization in general case. A general topological space can possess only a few of bounded continuous functions. It can happen that constant functions are the only bounded continuous functions on the space.

Convergence in distribution is preserved by a mapping with discontinuity points of probability zero.

Definition 1.96 *Let $\mathcal{X}$, $\mathcal{Y}$ be topological spaces and $F : \mathcal{X} \to \mathcal{Y}$. We denote by*

$$T_F = \{x \in \mathcal{X} \,:\, F \text{ is discontinuous in } x\}$$

the set of all discontinuity points of F.

Theorem 1.97 (preserving of convergence in distribution): *Let $\mathcal{X}$, $\mathcal{Y}$ be topological spaces, $< X_\lambda >_{\lambda \in \Lambda}$ be a net of random variables with values in $\mathcal{X}$, X be a random variable with values in $\mathcal{X}$, and, $F : \mathcal{X} \to \mathcal{Y}$ be a function.*

If

$$X_\lambda \xrightarrow[\lambda \in \Lambda]{\mathcal{D}} X \text{ in } \mathcal{X} \quad \text{and} \quad \mathsf{P}(X \in T_F) = 0,$$

then

$$F(X_\lambda) \xrightarrow[\lambda \in \Lambda]{\mathcal{D}} F(X) \text{ in } \mathcal{Y}.$$

Proof: Take $H \in \mathcal{F}(\mathcal{Y})$.

1. Take $x \in \operatorname{clo}(F^{-1}(H))$.

   There are two possibilities:

   (a) $x \in T_F$.

   (b) $x \notin T_F$; i.e. F is continuous at x.
   Since $x \in \operatorname{clo}(F^{-1}(H))$, there is a net $x_\lambda \in \mathcal{X}$, $x_\lambda \xrightarrow[\lambda\in\Lambda]{} x$ with $F(x_\lambda) \in H$.
   Then $F(x_\lambda) \xrightarrow[\lambda\in\Lambda]{} F(x)$, because, F is continuous at x.
   Therefore $F(x) \in H$.

   Consequently, $\operatorname{clo}(F^{-1}(H)) \subset F^{-1}(H) \cup T_F$.

2. Convergence in distribution follows from

$$\limsup_{\lambda\in\Lambda} \mathsf{P}\left(F\left(X_\lambda\right) \in H\right) = \limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \in F^{-1}(H)\right) \leq$$
$$\leq \limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \in \operatorname{clo}\left(F^{-1}(H)\right)\right) \leq \mathsf{P}\left(X \in \operatorname{clo}\left(F^{-1}(H)\right)\right)$$
$$\leq \mathsf{P}\left(X \in F^{-1}(H) \cup T_F\right) = \mathsf{P}\left(X \in F^{-1}(H)\right) = \mathsf{P}\left(F\left(X\right) \in H\right).$$

Q.E.D.

Let us mention, the classical version of Theorem on weak convergence preservation by a continuous mapping is a particular case of Theorem 1.97.

Chapter 2

Metric spaces

2.1 Definition and basic properties

Let us start with definition.

Definition 2.1 *Let* $\mathsf{E} \neq \emptyset$ *and* $\rho : \mathsf{E} \times \mathsf{E} \to \mathbb{R}_{+,0}$. *We say,* ρ *is a* *metric* *on* E *(cz. metrika), if*

a) $\rho(x, y) = 0$ *if and only if* $x = y$.

b) $\forall\, x, y \in \mathsf{E} : \ \rho(y, x) = \rho(x, y)$.

c) $\forall\, x, y, z \in \mathsf{E} : \ \rho(x, y) \leq \rho(x, z) + \rho(z, y)$

and it is a *pseudometric* *on* E *(cz. pseudometrika), if*

a) $\rho(x, x) = 0$.

b) $\forall\, x, y \in \mathsf{E} : \ \rho(y, x) = \rho(x, y)$.

c) $\forall\, x, y, z \in \mathsf{E} : \ \rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Definition 2.2 *Space* $\mathsf{E} = (\mathsf{E}, \rho)$ *is called a* *metric space* *(pseudometric space) (cz. metriký prostor, pseudometriký prostor) whenever* $\mathsf{E} \neq \emptyset$ *and* ρ *is a metric (pseudometric) on* E.

Definition 2.3 *Let* (E, ρ) *be a metric space. A set*

$$\mathcal{U}(x, \varepsilon) = \{y \in \mathsf{E} : \rho(y, x) < \varepsilon\} \tag{2.1}$$

is called *open ball* *with a center* $x \in \mathsf{E}$ *and a radius* $\varepsilon > 0$ *(cz. otevřené okolí bodu* x *s poloměrem* $\varepsilon > 0$*) and a set*

$$\mathcal{V}(x, \varepsilon) = \{y \in \mathsf{E} : \rho(y, x) \leq \varepsilon\} \tag{2.2}$$

is called closed ball with a center $x \in \mathsf{E}$ *and a radius* $\varepsilon > 0$ *(cz. uzavřené okolí bodu* x *s poloměrem* $\varepsilon > 0$*).*

Remark 2.4: Consider, that always $\operatorname{clo}(\mathcal{U}(x,\varepsilon)) \subset \mathcal{V}(x,\varepsilon)$. Unfortunately, equality can be violated.

♠

Each metric space is a topological space.

Lemma 2.5 *A metric space* (E,ρ) *is a topological space with a base*

$$\{\mathcal{U}(x,\varepsilon) \,:\, x \in \mathsf{E},\ \varepsilon > 0\}.$$

Let us recapitulate basic topological notions for metric spaces.

Remark 2.6: For a metric space (E,ρ) and $A \subset \mathsf{E}$ we have:

- $A \in \mathcal{G}(\mathsf{E})$ if and only if for each $x \in A$ there exists $\varepsilon > 0$ such that $\mathcal{U}(x,\varepsilon) \subset A$.
- $A \in \mathcal{F}(\mathsf{E})$ if and only if $\mathsf{E} \setminus A \in \mathcal{G}(\mathsf{E})$.
- Recall Borel σ-algebra $\mathcal{B}(\mathsf{E}) = \sigma(\mathcal{G}(\mathsf{E}))$.

♠

Topology of metric spaces is nice.

Theorem 2.7: *A metric space* (E,ρ) *fulfills the I.axiom of countability.*

Proof: A countable basis of neighborhoods at x can be taken for example as $\mathcal{U}_x = \{\mathcal{U}(x, 2^{-n}) \,:\, n \in \mathbb{N}\}$.

Q.E.D.

Theorem 2.8: *A metric space* (E,ρ) *is a normal* T_1 *topological space.*

Proof:

1. If $x, y \in \mathsf{E}$, $x \neq y$, take an open ball $G = \mathcal{U}(x, \rho(x,y)) \in \mathcal{G}(\mathsf{E})$. Then, $x \in G$ and $y \notin G$. Therefore, E is T_1,

2. Let $F, H \in \mathcal{F}(\mathsf{E})$ with $F \cap H = \emptyset$.

   For each $f \in F$ and $h \in H$ there are $\epsilon_f, \eta_h > 0$ such that $\mathcal{U}(f, \epsilon_f) \cap H = \emptyset$, $\mathcal{U}(h, \eta_h) \cap F = \emptyset$.

   Set $G = \bigcup_{f\in F} \mathcal{U}\left(f, \frac{\epsilon_f}{2}\right)$, $Q = \bigcup_{h\in H} \mathcal{U}\left(h, \frac{\eta_h}{2}\right)$. After that, $G, Q \in \mathcal{G}(\mathsf{E})$, $F \subset G$ and $H \subset Q$.

   Assume $f \in G \cap Q$.

   Then there are $u \in F$ and $v \in H$ such that $\rho(u, f) < \frac{\varepsilon_u}{2}$ and $\rho(v, f) < \frac{\eta_v}{2}$.

   Consequently,

   $$\max\{\varepsilon_u, \eta_v\} \le \rho(u, v) \le \rho(u, f) + \rho(v, f) < \frac{\varepsilon_u + \eta_v}{2} \le \max\{\varepsilon_u, \eta_v\}.$$

   This is a contradiction, therefore, $G \cap Q = \emptyset$.

   Normality of E is verified.

Q.E.D.

Theorem 2.9: *Let $T \neq \emptyset$ and a metric space $\mathsf{E}_t = (\mathsf{E}_t, \rho_t)$ be given for each $t \in T$.*

- *If T is a finite set then product topological space $\left(\bigtimes_{t\in T} \mathsf{E}_t, \bigotimes_{t\in T} \tau(\mathsf{E}_t)\right)$ is metrizable. If E_t is complete for each $t \in T$ then there is a metric making the product topological space to be a complete metric space.*

  A convenient metric is for example

  $$\rho(e, f) = \sum_{t\in T} \rho_t(e_t, f_t) \quad \text{for } e, f \in \bigtimes_{t\in T} \mathsf{E}_t.$$

- *If T is a countable set then product topological space $\left(\bigtimes_{t\in T} \mathsf{E}_t, \bigotimes_{t\in T} \tau(\mathsf{E}_t)\right)$ is metrizable. If E_t is complete for each $t \in T$ then there is a metric making the product topological space to be a complete metric space.*

  A convenient metric is for example

  $$\rho(e, f) = \sum_{i=1}^{+\infty} 2^{-i} \frac{\rho_{t_i}(e_{t_i}, f_{t_i})}{\rho_{t_i}(e_{t_i}, f_{t_i}) + 1} \quad \text{for } e, f \in \bigtimes_{t\in T} \mathsf{E}_t,$$

  where we number indexes in T as $T = \{t_i : i \in \mathbb{N}\}$.

- *If T is an uncountable set and E_t contains at least two different points for each $t \in T$ then product topological space $\left(\bigtimes_{t\in T} \mathsf{E}_t, \bigotimes_{t\in T} \tau(\mathsf{E}_t)\right)$ cannot be metrized.*

Proof:

1. For T at most countable, the statement is evident.
2. If T is uncountable and E_t contains at least two different points for each $t \in T$, then product topological space $\left(\bigtimes_{t\in T} \mathsf{E}_t, \bigotimes_{t\in T} \tau(\mathsf{E}_t)\right)$ does not fulfill the I. axiom of countability; see Lemma 1.60. Therefore, it cannot be metrized.

Q.E.D.

2.2 Convergence in metric spaces

According to Theorem 2.7, metric spaces fulfill the I.axiom of countability. Therefore, we do not need nets to handle with topology of metric spaces. Sequences are sufficient for that; see Lemma 1.59.

Convergence is determined by topology of metric spaces. But, there is another description using metric.

Lemma 2.10 *Let (E, ρ) be a metric space, Λ be a directed set and $x_\lambda, x \in \mathsf{E}$ for $\lambda \in \Lambda$. Then,*

$$x_\lambda \xrightarrow[\lambda\in\Lambda]{} x \text{ in } \mathsf{E} \iff \lim_{\lambda\in\Lambda} \rho\left(x_\lambda, x\right) = 0. \tag{2.3}$$

Proposition 2.11 *If (E, ρ) is a metric space, then metric $\rho : \mathsf{E} \times \mathsf{E} \to \mathbb{R}$ is a continuous function. Where $(\mathsf{E} \times \mathsf{E}, \psi)$ is a metric space with metric $\psi((x_1, x_2), (y_1, y_2)) = \rho\left(x_1, y_1\right) + \rho\left(x_2, y_2\right)$.*

Proof: Metric space fulfills I. axiom of countability, therefore, continuity of metric can be shown using sequences, only; see Lemma 1.59.
Let $x_n, y_n, x, y \in \mathsf{E}$ and $\lim_{n\to+\infty} \psi((x_n, y_n), (x, y)) = 0$.
Then $x_n \to x$, $y_n \to y$ in E and

$$\begin{aligned} \rho\left(x_n, y_n\right) &\leq \rho\left(x, y\right) + \rho\left(x_n, x\right) + \rho\left(y_n, y\right), \\ \rho\left(x, y\right) &\leq \rho\left(x_n, y_n\right) + \rho\left(x_n, x\right) + \rho\left(y_n, y\right). \end{aligned}$$

Consequently,

$$-\rho\left(x_n, x\right) - \rho\left(y_n, y\right) \leq \rho\left(x_n, y_n\right) - \rho\left(x, y\right) \leq \rho\left(x_n, x\right) + \rho\left(y_n, y\right)$$

and, therefore, $\lim_{n\to+\infty} \rho\left(x_n, y_n\right) = \rho\left(x, y\right)$.

Q.E.D.

Consider σ-algebra generated by metric ρ

$$\sigma(\rho) = \{\{(x,y) \in \mathsf{E} \times \mathsf{E} \,:\, \rho(x,y) \in B\} \,:\, B \in \mathbb{B}\}.$$

Always $\sigma(\rho) \subset \mathcal{B}(\mathsf{E} \times \mathsf{E})$. A connection to product σ-algebra $\mathcal{B}(\mathsf{E})^2$ is a question.

Theorem 2.12: *If (E, ρ) is separable metric space, then $\sigma(\rho) \subset \mathcal{B}(\mathsf{E})^2$.*

Proof: If is E separable, then there is an at most countable set $D \subset \mathsf{E}$ dense in E. Set for $\varepsilon > 0$ and $n \in \mathbb{N}$

$$Q_{n,\varepsilon} = \bigcup_{\substack{b,\, d \in D \\ \rho(b,d) \le \varepsilon + 2^{-n}}} \mathcal{U}\left(b, 2^{-n}\right) \times \mathcal{U}\left(d, 2^{-n}\right).$$

We have $Q_{1,\varepsilon} \supset Q_{2,\varepsilon} \supset Q_{3,\varepsilon} \supset \ldots$ and
$\bigcap_{n=1}^{+\infty} Q_{n,\varepsilon} = \{(x,y) \in \mathsf{E} \times \mathsf{E} \,:\, \rho(x,y) \le \varepsilon\}$.
Consequently, $\{(x,y) \in \mathsf{E} \times \mathsf{E} \,:\, \rho(x,y) \le \varepsilon\} \in \mathcal{B}(\mathsf{E})^2$ since $Q_{n,\varepsilon} \in \mathcal{B}(\mathsf{E})^2$ $\forall\, n \in \mathbb{N}$.
Finally, $\sigma(\rho) \subset \mathcal{B}(\mathsf{E})^2$.

Q.E.D.

Definition 2.13 *A sequence x_n, $n \in \mathbb{N}$ in a metric space (E, ρ) is called Cauchy (cz. Cauchyovská), whenever $\lim_{(n,m) \in \mathcal{M}} \rho(x_n, x_m) = 0$, where $\mathcal{M} = \mathbb{N}^2$ is directed by ordering $(n_1, n_2) \le (m_1, m_2) \iff n_1 \le m_1, n_2 \le m_2$.*

Lemma 2.14 *A Cauchy sequence x_n, $n \in \mathbb{N}$ in a metric space (E, ρ) possesses at most one limit in (E, ρ).*

Definition 2.15 *A metric space (E, ρ) is called complete (cz. úplný), whenever all Cauchy sequences possess a limit in (E, ρ).*

Definition 2.16 *Let (E, ρ), $(\widetilde{\mathsf{E}}, \widetilde{\rho})$ be metric spaces. We say $(\widetilde{\mathsf{E}}, \widetilde{\rho})$ is a completion of (E, ρ) (cz. zúplnění) if there is an imbedding $\iota : \mathsf{E} \to \widetilde{\mathsf{E}}$ such that $(\iota(\mathsf{E}), \widetilde{\rho})$ is a complete metric space and $\rho(x,y) = \widetilde{\rho}(\iota(x), \iota(y))$ for all $x, y \in \mathsf{E}$.*

Theorem 2.17: *Each metric space possesses a completion.*

Proof: Consider a metric space (E, ρ) and

$$\mathcal{M} = \left\{(x_n, n \in \mathbb{N}) \in \mathsf{E}^{\mathbb{N}} \, : \, (x_n, n \in \mathbb{N}) \text{ is Cauchy in } \mathsf{E}\right\}.$$

Take $(x_n, n \in \mathbb{N}), (y_n, n \in \mathbb{N}) \in \mathcal{M}$. Then $\rho\,(x_n, y_n)$, $n \in \mathbb{N}$ is Cauchy in $\mathbb{R}$, since

$$|\rho\,(x_k, y_k) - \rho\,(x_n, y_n)| \leq \rho\,(x_k, x_n) + \rho\,(y_k, y_n)\,.$$

Therefore, we can correctly define $\psi : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ as

$$\psi\,((x_n, n \in \mathbb{N}), (y_n, n \in \mathbb{N})) = \lim_{n \to +\infty} \rho\,(x_n, y_n)\,.$$

ψ is a pseudometric on $\mathcal{M}$.
We define an equivalence $\sim$ on $\mathcal{M}$ by $(x_n, n \in \mathbb{N}) \sim (y_n, n \in \mathbb{N})$ whenever $\psi\,((x_n, n \in \mathbb{N}), (y_n, n \in \mathbb{N})) = 0$.
Set $\widetilde{\mathsf{E}} = \mathcal{M}|_{\sim}$ and $\widetilde{\rho}\,(u, v) = \psi\,((x_n, n \in \mathbb{N}), (y_n, n \in \mathbb{N}))$ for each $(x_n, n \in \mathbb{N}) \in u$, $(y_n, n \in \mathbb{N}) \in v$.
Set $u_x = \{(y_n, n \in \mathbb{N}) \in \mathcal{M} \, : \, \lim_{n \to +\infty} \rho\,(y_n, x) = 0\}$ for all $x \in \mathsf{E}$.
We have a natural imbedding $\iota : \mathsf{E} \to \widetilde{\mathsf{E}} : x \in \mathsf{E} \mapsto u_x$ and $(\widetilde{\mathsf{E}}, \widetilde{\rho})$ is a completion of (E, ρ).

Q.E.D.

Definition 2.18 *A topological space $\mathcal{X}$ is called Polish (cz. Polský prostor), whenever there is a metric $\rho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that $(\mathcal{X}, \rho)$ is a complete separable metric space and both topologies coincide.*

"To be Polish" is a topological notion. A topological space can be equipped with two different metrics such that topologies coincide and the space is complete in one of them and non-complete in the other one.

Example 2.19: Consider open interval $(0, 1)$ and two metrics ρ_1, ρ_2 defined for $x, y \in (0, 1)$ by

$$\begin{aligned} \rho_1(x, y) &= |x - y|, \\ \rho_2(x, y) &= \left|\frac{2x - 1}{x(1 - x)} - \frac{2y - 1}{y(1 - y)}\right|. \end{aligned}$$

ρ_2 is also metric, since function

$$\psi(x) = \frac{2x - 1}{x(1 - x)} = \frac{1}{1 - x} - \frac{1}{x}$$

is increasing bijection between $(0,1)$ and $\mathbb{R}$.

Space $((0,1),\rho_1)$ is a separable metric, but, non-complete, since a sequence $\frac{1}{n}$, $n \in \mathbb{N}$ is Cauchy and does not converge. Its limit "lays outside of" the interval $(0,1)$.

Space $((0,1),\rho_2)$ is a complete separable metric space, since it is isomorfic with Euclidean space $\mathbb{R}$.

△

Construction in example leads to a general characterization of Polish spaces.

Theorem 2.20: *A topological space is Polish if and only if it is isomorphic with a $\mathcal{G}_{\delta\sigma}$-subset of some complete separable metric space.*

Definition 2.21 *Let (E,ρ) be a metric space and $A \subset \mathsf{E}$. We say, A is totally bounded (cz. totálně omezená) if for each $\varepsilon > 0$ there is a finite set $H_\varepsilon \subset \mathsf{E}$ such that for each $x \in \mathsf{E}$ one can find $y \in H_\varepsilon$ with property $\rho(x,y) < \varepsilon$.*

Lemma 2.22 *Let (E,ρ) be a metric space and $A \subset \mathsf{E}$.*

$$A \in \mathcal{K}(\mathsf{E}) \Longrightarrow A \in \mathcal{F}(\mathsf{E}) \text{ and } A \text{ is a totally bounded set.} \tag{2.4}$$

Lemma 2.23 *Let (E,ρ) be a complete metric space and $A \subset \mathsf{E}$.*

$$A \in \mathcal{K}(\mathsf{E}) \iff A \in \mathcal{F}(\mathsf{E}) \text{ and } A \text{ is a totally bounded set.} \tag{2.5}$$

2.3 Metric space and randomness

This chapter combines metric spaces and randomness. We will consider a net of random variables $< X_\lambda >_{\lambda\in\Lambda}$ with values in a metric space and its convergence almost surely, in probability and in distribution will be studied.

Lemma 2.24 *If E is a metric space, then each $\mu \in \mathcal{M}_1(\mathsf{E})$ is regular.*

Lemma 2.25 *If E is a Polish space, then each $\mu \in \mathcal{M}_1(\mathsf{E})$ is Radon and τ-additive.*

Convergence almost surely possesses no new relation in a metric space. But, there is another natural notion of convergence in probability.

Definition 2.26 *Let* E *be a metric space,* Λ *be a directed set,* X_λ*,* $\lambda \in \Lambda$ *be random variables in* E *and* X *be a a random variable in* E*. We say,* $< X_\lambda >_{\lambda\in\Lambda}$ *converges in probability to* X *in* E *(cz. konverguje v pravděpodobnosti), whenever for each* $\varepsilon > 0$ *it is fulfilled*

$$\lim_{\lambda\in\Lambda} \mathsf{P}\left(\rho\left(X_\lambda, X\right) > \varepsilon\right) = 0.$$

The convergence will be denoted $X_\lambda \xrightarrow[\lambda\in\Lambda]{P} X$ *in* E*.*

Theorem 2.27: *Let* E *be a metric space,* Λ *be a directed set,* X_λ*,* $\lambda \in \Lambda$ *be random variables in* E *and* X *be a random variable in* E*. Then*

$$X_\lambda \xrightarrow[\lambda\in\Lambda]{P} X \text{ in } \mathsf{E} \Longrightarrow X_\lambda \xrightarrow[\lambda\in\Lambda]{t-P} X \text{ in } \mathsf{E}\,.$$

Proof: Let $X_\lambda \xrightarrow[\lambda\in\Lambda]{P} X$ in E.
Fix $x \in \mathsf{E}$ and $\varepsilon > 0$.
Take $0 < \delta < \varepsilon$.
We can estimate

$$\begin{aligned}
&\limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin \mathcal{U}\left(x,\varepsilon\right),\ X \in \mathcal{U}\left(x,\varepsilon\right)\right) \\
&\leq \limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin \mathcal{U}\left(x,\varepsilon\right),\ X \in \mathcal{U}\left(x,\delta\right)\right) + \mathsf{P}\left(X \in \mathcal{U}\left(x,\varepsilon\right) \setminus \mathcal{U}\left(x,\delta\right)\right) \\
&\leq \limsup_{\lambda\in\Lambda} \mathsf{P}\left(\rho\left(X_\lambda, X\right) > \varepsilon - \delta\right) + \mathsf{P}\left(X \in \mathcal{U}\left(x,\varepsilon\right) \setminus \mathcal{U}\left(x,\delta\right)\right) \\
&= \mathsf{P}\left(X \in \mathcal{U}\left(x,\varepsilon\right) \setminus \mathcal{U}\left(x,\delta\right)\right),
\end{aligned}$$

since $X_\lambda \xrightarrow[\lambda\in\Lambda]{P} X$ in E.
We have shown $X_\lambda \xrightarrow[\lambda\in\Lambda]{t-P} X$ in E, since

$$\lim_{\delta\to\varepsilon-} \mathsf{P}\left(X \in \mathcal{U}\left(x,\varepsilon\right) \setminus \mathcal{U}\left(x,\delta\right)\right) = 0\,.$$

Q.E.D.

Theorem 2.28: *Let* E *be a metric space,* Λ *be a directed set,* X_λ*,* $\lambda \in \Lambda$ *be random variables in* E *and* X *be a a random variable in* E*. If* X *is Radon then*

$$X_\lambda \xrightarrow[\lambda\in\Lambda]{t-P} X \text{ in } \mathsf{E} \Longleftrightarrow X_\lambda \xrightarrow[\lambda\in\Lambda]{P} X \text{ in } \mathsf{E}\,.$$

Proof: Let $X_\lambda \xrightarrow[\lambda\in\Lambda]{t-P} X$ in E.
Fix $\varepsilon > 0$.
Take $\delta > 0$.
There is a $K \in \mathcal{K}(\mathsf{E})$ such that $\mathsf{P}(X \in K) \geq 1 - \delta$; since X is Radon.
According to Lemma 2.22, there is a finite set $H \subset \mathsf{E}$ such that $K \subset \bigcup_{x\in H} \mathcal{U}\left(x, \frac{\varepsilon}{2}\right)$.
Hence, we can estimate

$$\begin{aligned}
&\limsup_{\lambda\in\Lambda} \mathsf{P}\left(\rho\left(X_\lambda, X\right) > \varepsilon\right) \leq \limsup_{\lambda\in\Lambda} \mathsf{P}\left(\rho\left(X_\lambda, X\right) > \varepsilon,\ X \in K\right) + \delta \\
&\leq \limsup_{\lambda\in\Lambda} \mathsf{P}\left(\rho\left(X_\lambda, X\right) > \varepsilon,\ X \in \bigcup_{x\in H} \mathcal{U}\left(x, \frac{\varepsilon}{2}\right)\right) + \delta \\
&\leq \sum_{x\in H} \limsup_{\lambda\in\Lambda} \mathsf{P}\left(\rho\left(X_\lambda, X\right) > \varepsilon,\ X \in \mathcal{U}\left(x, \frac{\varepsilon}{2}\right)\right) + \delta \\
&\leq \sum_{x\in H} \limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \notin \mathcal{U}\left(x, \frac{\varepsilon}{2}\right),\ X \in \mathcal{U}\left(x, \frac{\varepsilon}{2}\right)\right) + \delta \\
&= \delta,
\end{aligned}$$

since $X_\lambda \xrightarrow[\lambda\in\Lambda]{t-P} X$ in E.
We have shown $X_\lambda \xrightarrow[\lambda\in\Lambda]{P} X$ in E.

Q.E.D.

List of equivalent descriptions of convergence in distribution is a bit larger than in general topological space.

Theorem 2.29 (Portmanteau lemma): *Let Λ be a directed set, X_λ, $\lambda \in \Lambda$ and X be random variables with values in a metric space $\mathsf{E} = (\mathsf{E}, \rho)$. Then the following statements are equivalent:*

i) $X_\lambda \xrightarrow[\lambda\in\Lambda]{\mathcal{D}} X$ *in* E.

ii) For each $G \in \mathcal{G}(\mathsf{E})$ we have

$$\liminf_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \in G\right) \geq \mathsf{P}\left(X \in G\right).$$

iii) For each $F \in \mathcal{F}(\mathsf{E})$ we have

$$\limsup_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda \in F\right) \leq \mathsf{P}\left(X \in F\right).$$

iv) For each lower bounded and lower semicontinuous function $f : \mathsf{E} \to \mathbb{R}$ *it is fulfilled*

$$\liminf_{\lambda \in \Lambda} \mathsf{E}\left[f(X_\lambda)\right] \geq \mathsf{E}\left[f(X)\right].$$

v) For each upper bounded and upper semicontinuous function $f : \mathsf{E} \to \mathbb{R}$ *it is fulfilled*

$$\limsup_{\lambda \in \Lambda} \mathsf{E}\left[f(X_\lambda)\right] \leq \mathsf{E}\left[f(X)\right].$$

vi) For each continuous bounded function $f : \mathsf{E} \to \mathbb{R}$ *we have*

$$\lim_{\lambda \in \Lambda} \mathsf{E}\left[f(X_\lambda)\right] = \mathsf{E}\left[f(X)\right].$$

vii) For each $B \in \mathcal{B}(\mathsf{E})$ *with* $\mathsf{P}(X \in \partial(B)) = 0$ *we have*

$$\lim_{\lambda \in \Lambda} \mathsf{P}(X_\lambda \in B) = \mathsf{P}(X \in B).$$

Proof:

1. From Theorem 1.95 we know $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$.
2. Immediately $(iv), (v) \Rightarrow (vi)$,
3. We have $(ii), (iii) \Rightarrow (vii)$, since for $B \in \mathcal{B}(\mathsf{E})$, $\mathsf{P}(X \in \partial(B)) = 0$

$$\begin{aligned} &\mathsf{P}(X \in B) = \mathsf{P}(X \in \operatorname{int}(B)) \\ &\leq \liminf_{\lambda \in \Lambda} \mathsf{P}(X_\lambda \in \operatorname{int}(B)) \leq \liminf_{\lambda \in \Lambda} \mathsf{P}(X_\lambda \in B) \\ &\leq \limsup_{\lambda \in \Lambda} \mathsf{P}(X_\lambda \in \operatorname{clo}(B)) \\ &\leq \mathsf{P}(X \in \operatorname{clo}(B)) = \mathsf{P}(X \in B). \end{aligned}$$

4. It remains to show $(iv), (v) \Leftarrow (vi)$ and $(ii), (iii) \Leftarrow (vii)$.

Q.E.D.

Theorem 2.30 (Prochorovova věta): *Let* E *be a Polish space and* $\mathcal{M} \subset \mathcal{M}_1(\mathsf{E})$. *Then it is equivalent:*

i) $\mathcal{M}$ *is a weak relative compact.*

ii) $\mathcal{M}$ is tight.

(Recall Definitions 1.88 and 1.89)

Proof: See textbook [5].

Q.E.D.

Example 2.31: Consider a metric space (E, ρ) and define a topology τ generated by a subbasis

$$\mathcal{G} = \{\mathsf{E} \setminus \mathcal{V}(x, \varepsilon) : x \in \mathsf{E},\ \varepsilon > 0\}.$$

Then, (E, τ) is T_1. Let be $\sup\{\rho(x, y) : x, y \in \mathsf{E}\} = +\infty$, then (E, τ) cannot be Hausdorff.

Topology τ is called ball topology.

Let us give a short proof.

1. Let $x, y \in \mathsf{E}$, $x \neq y$ then $G = \mathsf{E} \setminus \mathcal{V}\left(y, \frac{1}{2}\rho(x, y)\right)$ fulfills $G \in \mathcal{G}(\mathsf{E}, \tau)$, $x \in G$ and $y \notin G$. Therefore (E, τ) is T_1.

2. Let $x, y \in \mathsf{E}$, $x \neq y$, $G, Q \in \mathcal{G}(\mathsf{E})$, $x \in G$, $x \in Q$ and $G \cap Q = \emptyset$.

   Then, there are $I, J \in \mathbb{N}$, $x_1, x_2, \ldots, x_I \in \mathsf{E}$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_I \in \mathbb{R}_{+,0}$, $y_1, y_2, \ldots, y_J \in \mathsf{E}$, $\varphi_1, \varphi_2, \ldots, \varphi_J \in \mathbb{R}_{+,0}$ such that

$$x \in \mathsf{E} \setminus \bigcup_{i=1}^{I} \mathcal{V}(x_i, \varepsilon_i) \subset G,$$
$$y \in \mathsf{E} \setminus \bigcup_{j=1}^{J} \mathcal{V}(y_j, \varphi_j) \subset Q.$$

   We assume $G \cap Q = \emptyset$, therefore,

$$\mathsf{E} \subset \bigcup_{i=1}^{I} \mathcal{V}(x_i, \varepsilon_i) \cup \bigcup_{j=1}^{J} \mathcal{V}(y_j, \varphi_j).$$

   Hence, $\sup\{\rho(x, y) : x, y \in \mathsf{E}\} \leq 2\left(\sum_{i=1}^{I} \varepsilon_i + \sum_{j=1}^{J} \varphi_j\right) < +\infty$.

   Therefore, if diameter of E is infinite then E cannot be Hausdorff.

△

2.4 Normed spaces

Consider normed spaces a particular case of metric spaces.

Definition 2.32 *A space* $\mathsf{E} = (\mathsf{E}, \|\cdot\|)$ *is called a normed space (or, a space with norm) (cz. normovaný prostor, nebo prostor s normou) if* E *is a linear vector space and* $\|\cdot\|$ *is a norm.*

Lemma 2.33 *Let* $\mathsf{E} = (\mathsf{E}, \|\cdot\|)$ *be a normed space. Then,* $\rho : \mathsf{E} \times \mathsf{E} \to \mathbb{R}$ *defined by* $\rho(x, y) = \|x - y\|$ *is a metric.*

Definition 2.34 *A normed space* $\mathsf{E} = (\mathsf{E}, \|\cdot\|)$ *is considered as a topological space with topology of the metric space* $\mathsf{E} = (\mathsf{E}, \rho)$*, where*
$\rho : \mathsf{E} \times \mathsf{E} \to \mathbb{R} : (x, y) \in \mathsf{E} \times \mathsf{E} \to \|x - y\|$.

Lemma 2.35 *Let* $\mathsf{E} = (\mathsf{E}, \|\cdot\|)$ *be a normed space. Then,* $\|\cdot\|$ *is continuous.*

Proof: Statement follows an estimate $|\|x\| - \|y\|| \leq \|x - y\|$.

Q.E.D.

Definition 2.36 *A space* $\mathsf{E} = (\mathsf{E}, \|\cdot\|)$ *is called a Banach space (cz. Banachův prostor) if* $(\mathsf{E}, \|\cdot\|)$ *is a complete normed space.*

Theorem 2.37: *Let* $T \neq \emptyset$ *and a normed space* $\mathsf{E}_t = (\mathsf{E}_t, \|\cdot\|_t)$ *be given for each* $t \in T$.

- *If* T *is a finite set then product topological space* $\left(\times_{t\in T} \mathsf{E}_t, \bigotimes_{t\in T} \tau(\mathsf{E}_t)\right)$ *can be equipped with a norm to be a normed space. If moreover* E_t *is a Banach space for each* $t \in T$ *then there is a norm making the product topological space to be a Banach space.*

  A convenient norm is for example

  $$\|f\| = \sum_{t\in T} \|f_t\|_t \quad \text{for } f \in \times_{t\in T} \mathsf{E}_t$$

  or

  $$\|f\| = \sqrt{\sum_{t\in T} \|f_t\|_t^2} \quad \text{for } f \in \times_{t\in T} \mathsf{E}_t.$$

- *If T is a countable set and E_t contains at least two different points for each $t \in T$ then there is no norm making product topological space $\left(\bigtimes_{t\in T} \mathsf{E}_t, \bigotimes_{t\in T} \tau(\mathsf{E}_t) \right)$ to be a normed space. Nevertheless, product topological space $\left(\bigtimes_{t\in T} \mathsf{E}_t, \bigotimes_{t\in T} \tau(\mathsf{E}_t) \right)$ is metrizable. If moreover E_t is a Banach space for each $t \in T$ then there is a metric making the product topological space to be a complete metric space.*

  Such a convenient metric is

$$\rho(e,f) = \sum_{i=1}^{+\infty} 2^{-i} \frac{\|e_{t_i} - f_{t_i}\|_{t_i}}{\|e_{t_i} - f_{t_i}\|_{t_i} + 1} \quad \text{for } e,f \in \bigtimes_{t\in T} \mathsf{E}_t,$$

  where we number members of T as $T = \{t_i \, : \, i \in \mathbb{N}\}$.

- *If T is an uncountable set and E_t contains at least two different points for each $t \in T$ then product topological space $\left(\bigtimes_{t\in T} \mathsf{E}_t, \bigotimes_{t\in T} \tau(\mathsf{E}_t) \right)$ cannot be metrized.*

Proof:

1. If T is a finite set, the statement is evident.

2. Let T be a countable set and E_t contains at least two different points for each $t \in T$.

   Assume $\left(\bigtimes_{t\in T} \mathsf{E}_t, \|\cdot\| \right)$ is a normed space.

   Without any loss of generality we can expect $T = \mathbb{N}$.

   Take some $e_i \in \mathsf{E}_i$, $e_i \neq 0$, $i \in \mathbb{N}$.

   Now, one can recursively construct $\alpha_i > 0$, $i \in \mathbb{N}$ such that $\|x_k\| = k$ for each $k \in \mathbb{N}$, where $x_{k,i} = \alpha_i e_i$ for $i = 1, 2, \ldots, k$ and $x_{k,i} = 0$ for $i = k+1, k+2, \ldots$.

   Then, $x_k \to \xi$ in product topology, where $\xi_i = \alpha_i e_i$ for all $i \in \mathbb{N}$.

   Hence, $\xi \in \bigtimes_{t\in T} \mathsf{E}_t$ would have $\|\xi\| = +\infty$, because we should have $\lim_{k\to+\infty} \|x_k\| = \|\xi\|$ in the normed space.

   That is a contradiction, since norm must be real-valued.

3. If T is countable the product space is metrizable with a metric

$$\rho(e,f) = \sum_{i=1}^{+\infty} 2^{-i} \frac{\|e_{t_i} - f_{t_i}\|_{t_i}}{\|e_{t_i} - f_{t_i}\|_{t_i} + 1} \quad \text{for } e,f \in \bigtimes_{t\in T} \mathsf{E}_t \,.$$

If E_t is a Banach space for each $t \in T$ then the metric is making the product space to be a complete metric.

4. If T is uncountable, product space cannot be metrized, see Theorem 2.9.

Q.E.D.

Chapter 3

Space of all functions

3.1 $\mathbb{R}^T$ - topology

We consider the space of all real functions $\mathbb{R}^T$ equipped with product topology; i.e. $\mathbb{R}^T = (\mathbb{R}^T, \tau(\mathbb{R})^{\otimes T})$. Therefore convergence in this space is a convergence via coordinates.

Theorem 3.1: *Let $T \neq \emptyset$.*

- *If T is a finite set then product topological space $(\mathbb{R}^T, \tau(\mathbb{R})^{\otimes T})$ can be equipped with a norm to be Banach. A convenient norm is for example*

$$\|f\| \quad = \quad \sum_{t \in T} |f_t| \ \text{ for } f \in \mathbb{R}^T$$

  or

$$\|f\| \quad = \quad \sqrt{\sum_{t \in T} |f_t|^2} \ \text{ for } f \in \mathbb{R}^T.$$

- *If T is a countable set then there is no norm making product topological space $(\mathbb{R}^T, \tau(\mathbb{R})^{\otimes T})$ to be a normed space. Nevertheless, the product topological space is a Polish space. A suitable metric is*

$$\rho\,(e, f) \quad = \quad \sum_{i=1}^{+\infty} 2^{-i} \frac{|e_{t_i} - f_{t_i}|}{|e_{t_i} - f_{t_i}| + 1} \ \text{ for } e, f \in \mathbb{R}^T,$$

  where we numbered members of T as $T = \{t_i \,:\, i \in \mathbb{N}\}$.

- *If T is an uncountable set then product topological space $(\mathbb{R}^T, \tau(\mathbb{R})^{\otimes T})$ cannot be metrized.*

Proof: Theorem is a straightforward application of Theorem 2.37 for the case $\mathsf{E}_t = \mathbb{R}$ for all $t \in T$.

Q.E.D.

Convergence via coordinates is inducing a product topology; see Section 1.5. Product topology is determined by a subbase

$$\left\{ {}_T\Pi_{\{t\}}^{-1}(a,b) \,:\, t \in T,\ a < b,\ a,b \in \mathbb{R} \right\}.$$

Upper index -1 denotes inverse mapping; i.e. preimage of a given set.

Also, other subbases are available, for example

$$\left\{ {}_T\Pi_{\{t\}}^{-1}(G) \,:\, t \in T,\ G \in \mathcal{G}(\mathbb{R}) \right\},$$
$$\left\{ {}_T\Pi_I^{-1}(G) \,:\, I \in \mathsf{Fin}(T),\ G \in \mathcal{G}\left(\mathbb{R}^I\right) \right\},$$
$$\left\{ {}_T\Pi_I^{-1}\left(\mathop{\times}_{i\in I} (a_i,b_i) \right) \,:\, I \in \mathsf{Fin}(T),\ a_i < b_i \text{ for } i \in I \right\}.$$

Definition 3.2 *For $I \in \mathsf{Fin}(T)$ and $B \in \mathbb{B}^I$, a set ${}_T\Pi_I^{-1}(B)$ is called cylinder with a finite base (cz. válec s konečněrozměrnou podstavou).*

A σ-algebra generated by all cylinders with a finite base is called cylindric σ-algebra (cz. válcová σ-algebra) and will be denoted by $\mathsf{Cylindric}(T)$ in this text.

It is interesting that cylindric σ-algebra coincides with Baire σ-algebra.

Theorem 3.3: *We have a relation*

$$\mathcal{B}\left(\mathbb{R}^T\right) \supset \mathsf{Cylindric}(T) = \mathbb{B}^T = \mathsf{Baire}\left(\mathbb{R}^T\right).$$

Proof: Inclusion is trivial, the first equality follows definition of cylindric σ-algebra and last equality follows [6].

Q.E.D.

In later sections, we will need open sets and the other topological notions with a base restricted to a given index set.

Definition 3.4 *Let $S \subset T$, $S \neq \emptyset$. We will denote*

$${}_T\mathcal{G}_S = {}_T\Pi_S^{-1}\left(\mathcal{G}\left(\mathbb{R}^S\right)\right),\ {}_T\mathcal{F}_S = {}_T\Pi_S^{-1}\left(\mathcal{F}\left(\mathbb{R}^S\right)\right),\ {}_T\mathcal{B}_S = {}_T\Pi_S^{-1}\left(\mathcal{B}\left(\mathbb{R}^S\right)\right),$$
$${}_T\mathcal{K}_S = {}_T\Pi_S^{-1}\left(\mathcal{K}\left(\mathbb{R}^S\right)\right),\ {}_T\mathcal{G}_{S;\delta\sigma} = {}_T\Pi_S^{-1}\left(\mathcal{G}_{\delta\sigma}\left(\mathbb{R}^S\right)\right).$$

Closed, open and compact sets in $\mathbb{R}^T$ possess nice helpful characterizations by means of finite number of coordinates.

Lemma 3.5 *Let $A \subset \mathbb{R}^T$ and $x \in A$. Then,*

$$x \in \bigcap_{I \in \mathsf{Fin}(T)} {}_T\Pi_I^{-1}\left({}_T\Pi_I\left(A\right)\right). \tag{3.1}$$

Proof: The statement is evident.

Q.E.D.

Lemma 3.6 *Let $F \in \mathcal{F}\left(\mathbb{R}^T\right)$ and $x \in \mathbb{R}^T$. Then,*

$$x \in F \iff x \in \bigcap_{I \in \mathsf{Fin}(T)} {}_T\Pi_I^{-1}\left({}_T\Pi_I\left(F\right)\right). \tag{3.2}$$

Proof: We will show separately each of both implication.

1. For $x \in F$ we immediately have $x \in \bigcap_{I \in \mathsf{Fin}(T)} {}_T\Pi_I^{-1}\left({}_T\Pi_I\left(F\right)\right)$; see Lemma 3.5.
2. Assume $x \in \bigcap_{I \in \mathsf{Fin}(T)} {}_T\Pi_I^{-1}\left({}_T\Pi_I\left(F\right)\right)$.

   Then, for each $I \in \mathsf{Fin}\left(T\right)$ there is $\xi^I \in F$ such that $\xi^I_I = x_I$.

   Index set $\mathsf{Fin}\left(T\right)$ is directed by preorder

   $$I \leq J \iff I \subset J.$$

   Then, $< \xi^I >_{I \in \mathsf{Fin}(T)}$ is a net in F.

   For $t \in T$ and for each $I \in \mathsf{Fin}\left(T\right)$ such that $t \in I$, we have $\xi^I_t = x_t$.

   We have verified convergence $\xi^I \xrightarrow[I \in \mathsf{Fin}(T)]{} x$ in $\mathbb{R}^T$.

   We know $\xi^I \in F$ and $F \in \mathcal{F}\left(\mathbb{R}^T\right)$ therefore $x \in F$.

Q.E.D.

Lemma 3.7 *Let $F \in \mathcal{F}\left(\mathbb{R}^T\right)$ and $x \in \mathbb{R}^T$. Then,*

$$x \in F \iff x \in \bigcap_{I \in \mathsf{Fin}(T)} {}_T\Pi_I^{-1}\left(\mathrm{clo}\left({}_T\Pi_I\left(F\right)\right)\right). \tag{3.3}$$

Proof: We will show separately each of both implication.

1. If $x \in F$, then immediately $x \in \bigcap_{I \in \mathsf{Fin}(T)} {}_T\Pi_I^{-1}\left(\mathrm{clo}\left({}_T\Pi_I\left(F\right)\right)\right)$.
2. Take $x \in \bigcap_{I \in \mathsf{Fin}(T)} {}_T\Pi_I^{-1}\left(\mathrm{clo}\left({}_T\Pi_I\left(F\right)\right)\right)$.

   Then, for each $I \in \mathsf{Fin}\left(T\right)$ and $\varepsilon > 0$ there is $\xi^{I,\varepsilon} \in F$ such that

   $$\forall\, t \in I \text{ we have } |\xi_t^{I,\varepsilon} - x_t| < \varepsilon.$$

   Denote $\Lambda = \{(I, \varepsilon) \,:\, I \in \mathsf{Fin}\left(T\right), \varepsilon > 0\}$ and consider a preorder

   $$(I, \varepsilon) \le (J, \eta) \iff I \subset J,\ \varepsilon \ge \eta.$$

   Then, $\Lambda = (\Lambda, \le)$ is directed and $< \xi^{I,\varepsilon} >_{(I,\varepsilon)\in\Lambda}$ is a net in F.

   For $t \in T$, $\varepsilon > 0$ and for each $I \in \mathsf{Fin}\left(T\right)$ such that $t \in I$, we have $|\xi_t^{I,\varepsilon} - x_t| < \varepsilon$.

   That is $\xi^{I,\varepsilon} \xrightarrow[(I,\varepsilon)\in\Lambda]{} x$.

   We know $\xi^{I,\varepsilon} \in F$ and $F \in \mathcal{F}\left(\mathbb{R}^T\right)$.

   Consequently, $x \in F$.

Q.E.D.

Theorem 3.8: *Let $A \subset \mathbb{R}^T$. Then, $A \in \mathcal{F}\left(\mathbb{R}^T\right)$ if and only if*

$$A = \bigcap_{I \in \mathsf{Fin}(T)} {}_T\Pi_I^{-1}(\mathrm{clo}\left({}_T\Pi_I\left(A\right)\right)). \tag{3.4}$$

Proof:

1. The right-hand side of (3.4) is a closed set being an intersection of closed sets. Therefore (3.4) implies $A \in \mathcal{F}\left(\mathbb{R}^T\right)$.
2. Assume $A \in \mathcal{F}\left(\mathbb{R}^T\right)$. According to Lemma 3.7, (3.4) is fulfilled.

Q.E.D.

Theorem 3.9: *Let $A \subset \mathbb{R}^T$. Then, $A \in \mathcal{G}\left(\mathbb{R}^T\right)$ if and only if*

$$A = \bigcup_{I \in \mathsf{Fin}(T)} {}_T\Pi_I^{-1}(\mathrm{int}\left({}_T\Pi_I\left(A\right)\right)). \tag{3.5}$$

Proof: Characterization is a consequence of Theorem 3.8 and of the fact that open set is complement of a closed set.

Q.E.D.

Theorem 3.10: *Let $A \subset \mathbb{R}^T$. Then, $A \in \mathcal{K}\left(\mathbb{R}^T\right)$ if and only if*

$$\forall\ I \in \mathsf{Fin}\,(T) \ \text{ we have } {}_T\Pi_I\,(A) \in \mathcal{K}\left(\mathbb{R}^I\right), \tag{3.6}$$

$$A = \bigcap_{I \in \mathsf{Fin}(T)} {}_T\Pi_I^{-1}({}_T\Pi_I\,(A)). \tag{3.7}$$

Proof:

1. Let $A \in \mathcal{K}\left(\mathbb{R}^T\right)$.
   (a) Take $I \in \mathsf{Fin}\,(T)$ and consider a sequence $x^n \in {}_T\Pi_I\,(A)$, $n \in \mathbb{N}$.
   Then, for each $n \in \mathbb{N}$ there is $a^n \in A$ such that $a_I^n = x^n$.
   Since A is a compact, there is a subnet such that
   $$a^{\phi(\psi)} \xrightarrow[\psi \in \Psi]{} b \in A.$$
   Then,
   $$x^{\phi(\psi)} = a_I^{\phi(\psi)} \xrightarrow[\psi \in \Psi]{} b_I \in {}_T\Pi_I\,(A)\,.$$
   Now, we can select a subsequence $x^{\phi(\psi_k)}$, $k \in \mathbb{N}$ such that
   $$\|x^{\phi(\psi_k)} - b_I\| < \frac{1}{k},\ \phi(\psi_k) < \phi(\psi_{k+1}).$$
   We have found a convergent subsequence of the sequence x^n, $n \in \mathbb{N}$ with limit in ${}_T\Pi_I\,(A)$.
   Thus we have verified ${}_T\Pi_I\,(A) \in \mathcal{K}\left(\mathbb{R}^I\right)$.
   (b) Now, we have verified ${}_T\Pi_I\,(A) \in \mathcal{K}\left(\mathbb{R}^I\right)$ for each $I \in \mathsf{Fin}\,(T)$.
   Therefore according to Theorem 3.8, we have description (3.7).

2. Assume (3.6) and (3.7).
   Then according to Theorem 3.8, $A \in \mathcal{F}\left(\mathbb{R}^T\right)$.
   Evidently,
   $$A \subset \bigtimes_{t \in T} {}_T\Pi_{\{t\}}\,(A)\,.$$
   Product of compacts is a compact in $\mathbb{R}^T$, according to Theorem 1.41 (Tikhonov Theorem).
   Therefore, $A \in \mathcal{K}\left(\mathbb{R}^T\right)$.

Q.E.D.

3.2 $\mathbb{R}^T$ and randomness

Before proceeding to the subject, we have to fix terminology on random processes. We assume a probability space $(\Omega, \mathcal{A}, \mathsf{P})$ and we will deal with collections of mappings $X = (X(t), t \in T)$, where T is a nonempty index set and $X(t) : \Omega \to \mathbb{R}$ is a map for all $t \in T$.

In accordance to Definitions 1.81, 1.83, 1.84, we will say.

Definition 3.11 *We call $X = (X(t), t \in T)$:*

- *a random process if $X(t)$ is a real random variable for all $t \in T$; i.e. $X(t) : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, or equivalently,* $\mathsf{Cylindric}(T) \subset \Sigma_X$.
- *a $\mathcal{B}$-measurable random process if $H \subset \mathbb{R}^T$ is a nonempty set, $\mathcal{B} \subset \mathcal{P}(H)$ is a σ-algebra and $\mathcal{B} \subset \Sigma_X$.*
- *a Borel random process in C if $C \subset \mathbb{R}^T$ is a topological space and $\mathcal{B}(C) \subset \Sigma_X$.*

Theorem 3.12 (Daniell-Kolmogoroff): *Let for each $I \in \mathsf{Fin}(T)$ a probability $\mu_I \in \mathcal{M}_1(\mathbb{R}^I)$ is given and $\{\mu_I : I \in \mathsf{Fin}(T)\}$ forms a consistent system, i.e. for each $I, J \in \mathsf{Fin}(T)$, $I \subset J$ we have $\mu_I = \mu_J \circ {}_J\Pi_I^{-1}$.*

Then, there is a probability ν defined on $\mathsf{Cylindric}(T)$ such that for each $I \in \mathsf{Fin}(T)$ we have $\nu = \mu_I \circ {}_T\Pi_I^{-1}$.

Proof: A proof is an application of Hopf's Theorem on measure extension from an algebra to σ-algebra.

Q.E.D.

Definition 3.13 *Let X and Y be a couple of random processes. We say finite dimensional distributions of X and Y coincide (cz. konečněrozměrné distribuce se shodují) if for each $I \in \mathsf{Fin}(T)$, $B \in \mathcal{B}(\mathbb{R}^I)$ we have*

$$\mathsf{P}(Y(I) \in B) = \mathsf{P}(X(I) \in B).$$

The fact will be denoted by $X \overset{fidi}{\equiv} Y$ in $\mathbb{R}^T$.

Lemma 3.14 *If* $X \overset{fidi}{\equiv} Y$ *in* $\mathbb{R}^T$ *then* $X \overset{\mathcal{D}}{\equiv} Y$ *on* $\mathsf{Cylindric}(T)$.

Definition 3.15 *We say finite dimensional distributions of a net of random processes* $< X_\lambda >_{\lambda\in\Lambda}$ *converge to a random process* X *(cz. konvergence konečněrozměrných distribucí) if for each* $I \in \mathsf{Fin}(T)$ *we have* $X_\lambda(I) \xrightarrow[\lambda\in\Lambda]{\mathcal{D}} X(I)$ *in* $\mathbb{R}^I$.

The convergence will be denoted by $X_\lambda \xrightarrow[\lambda\in\Lambda]{fidi} X$ *in* $\mathbb{R}^T$.

Lemma 3.16 *Let* $< X_\lambda >_{\lambda\in\Lambda}$ *be a net of random processes in* $\mathbb{R}^T$ *and* Y, Z *be random processes in* $\mathbb{R}^T$.

If $X_\lambda \xrightarrow[\lambda\in\Lambda]{fidi} Y$ *in* $\mathbb{R}^T$ *and* $X_\lambda \xrightarrow[\lambda\in\Lambda]{fidi} Z$ *in* $\mathbb{R}^T$ *then* $Y \overset{fidi}{\equiv} Z$.

Immediately, convergence in distribution and convergence of finite dimensional distributions are connected.

Theorem 3.17: *Whenever,* $X_\lambda \xrightarrow[\lambda\in\Lambda]{\mathcal{D}} X$ *in* $\mathbb{R}^T$, *then* $X_\lambda \xrightarrow[\lambda\in\Lambda]{fidi} X$ *in* $\mathbb{R}^T$.

Proof: The argument is ${}_T\Pi_I^{-1}(G) \in \mathcal{G}\left(\mathbb{R}^T\right)$ for all $G \in \mathcal{G}\left(\mathbb{R}^I\right)$.

Q.E.D.

Theorem 3.18: *Let* $< X_\lambda >_{\lambda\in\Lambda}$ *be a net of random processes. If for each* $I \in \mathsf{Fin}(T)$ *there is a random vector* $Y_I \in \mathbb{R}^I$ *such that* $X_\lambda(I) \xrightarrow[\lambda\in\Lambda]{\mathcal{D}} Y_I$ *in* $\mathbb{R}^I$, *then there is a random process* X *such that* $X_\lambda \xrightarrow[\lambda\in\Lambda]{fidi} X$ *in* $\mathbb{R}^T$.

Moreover, for each $I \in \mathsf{Fin}(T)$ *distributions of random vectors* Y_I *and* $X(I)$ *coincide.*

Proof: For each $I \in \mathsf{Fin}(T)$ we denote μ_I the distribution of random vector Y_I. Consider $I, J \in \mathsf{Fin}(T)$, $I \subset J$ and $G \in \mathcal{G}\left(\mathbb{R}^I\right)$. Then

$$\begin{aligned}\liminf_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda(I) \in G\right) &= \liminf_{\lambda\in\Lambda} \mathsf{P}\left(X_\lambda(J) \in {}_J\Pi_I^{-1}(G)\right) \\ &\geq \mathsf{P}\left(Y_J \in {}_J\Pi_I^{-1}(G)\right) = \mu_J\left({}_J\Pi_I^{-1}(G)\right) = \mu_J \circ {}_J\Pi_I^{-1}(G).\end{aligned}$$

Distribution of the limit is uniquely determined, therefore, $\mu_I = \mu_J \circ {}_J\Pi_I^{-1}$.
Hence, $\{\mu_I : I \in \mathsf{Fin}(T)\}$ forms a consistent system.
According Theorem 3.12, there is a probability ν defined on $\mathsf{Cylindric}(T)$ such that for each $I \in \mathsf{Fin}(T)$ we have $\nu = \mu_I \circ {}_T\Pi_I^{-1}$.
Then, there is a random process X such that $\mu_X = \nu$ on $\mathsf{Cylindric}(T)$ and $X_\lambda \xrightarrow[\lambda\in\Lambda]{fidi} X$ in $\mathbb{R}^T$.
Moreover, for each $I \in \mathsf{Fin}(T)$ distributions of random vectors Y_I and $X(I)$ coincide.

Q.E.D.

Chapter 4

Space of bounded functions

Largeness of a real function $f \in \mathbb{R}^T$ can be measured by its supremum

$$\|f\|_T = \sup \{|f(t)| : t \in T\}. \tag{4.1}$$

Supremum exhibits properties of a norm except real-values. More precisely, $\|\cdot\|_T$ is a norm for T finite, only. If T is infinite $\|\cdot\|_T$ attains value $+\infty$.

Consider the set of all bounded functions

$$l^{+\infty}(T) \quad = \quad \left\{ f \in \mathbb{R}^T : \|f\|_T < +\infty \right\}. \tag{4.2}$$

Lemma 4.1

i) If T is a finite set, then $l^{+\infty}(T) = \mathbb{R}^T$.

ii) If T is an infinite set, then $l^{+\infty}(T) \neq \mathbb{R}^T$.

Proof: The case T is a finite set is clear.
If T is an infinite set, then without any loss of generality we assume $\mathbb{N} \subset T$. Hence $\|f\|_T = +\infty$ for $f \in \mathbb{R}^T$, where $f(s) = s$ for $s \in \mathbb{N}$ and $f(s) = 0$ for $s \in T \setminus \mathbb{N}$.

Q.E.D.

Theorem 4.2: *Space $l^{+\infty}(T) = (l^{+\infty}(T), \|\cdot\|_T)$ is a Banach space which is separable only if T is a finite set.*

Proof: Space $(l^{+\infty}(T), \|\cdot\|_T)$ is normed. We have to show completeness and discuss separability.

1. Let $f_n \in l^{+\infty}(T)$, $n \in \mathbb{N}$ is a Cauchy sequence.

   Fix $t \in T$.

   Hence, $f_n(t)$, $n \in \mathbb{N}$ is a Cauchy sequence of reals, therefore, possessing a limit, say $g(t) \in \mathbb{R}$.

   Thus, we are receiving $g \in \mathbb{R}^T$. It remains to show convergence in $l^{+\infty}(T)$.

   Take $\varepsilon > 0$.

   Then there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $m \geq n_0$, $n, m \in \mathbb{N}$ we have $\|f_n - f_m\|_T < \varepsilon$.

   For $t \in T$ and $m > n \geq n_0$ we have

$$\begin{aligned} |f_n(t) - g(t)| &= |(f_n(t) - f_m(t)) + (f_m(t) - g(t))| \\ &\leq |f_n(t) - f_m(t)| + |f_m(t) - g(t)| \\ &\leq \|f_n - f_m\|_T + |f_m(t) - g(t)| \\ &< \varepsilon + |f_m(t) - g(t)|. \end{aligned}$$

   Letting $m \to +\infty$ we are receiving

$$\forall t \in T \quad |f_n(t) - g(t)| \leq \varepsilon.$$

   We have shown $\lim_{n \to +\infty} \|f_n - g\|_T = 0$ and $g \in l^{+\infty}(T)$, since for n large enough

$$\|g\|_T \leq \|f_n - g\|_T + \|f_n\|_T < +\infty.$$

2. (a) If T is a finite set, then $l^{+\infty}(T) = \mathbb{R}^T$. Thus, it is separable.

   (b) If T is an infinite set, then for $S \subset T$ consider functions

$$\begin{aligned} \psi_S(t) &= 1 \quad \text{if} \quad t \in S, \\ &= 0 \quad \text{if} \quad t \notin S. \end{aligned}$$

   For $S, U \subset T$, $S \neq U$, we have $\|\psi_S - \psi_U\|_T = 1$.
   Family of functions ψ_S, $S \subset T$ is uncountable.
   Consequently, space $l^{+\infty}(T)$ cannot be separable.

Q.E.D.

Lemma 4.3 *If* $f_\lambda \xrightarrow[\lambda \in \Lambda]{} f$ *in* $l^{+\infty}(T)$, *then* $f_\lambda \xrightarrow[\lambda \in \Lambda]{} f$ *in* $\mathbb{R}^T$.

Proof: Statement is straightforward.

Q.E.D.

The opposite implication is true, only, for T being a finite set.

Example 4.4: Let T be an infinite set, i.e. $\mathbb{N} \subset T$. Consider a sequence $f_n \in l^{+\infty}(T)$, $n \in \mathbb{N}$, where $f_n(n) = 1$ and $f_n(t) = 0$ otherwise.

The sequence $f_n \in l^{+\infty}(T)$, $n \in \mathbb{N}$ is not Cauchy in $l^{+\infty}(T)$, but, $f_n \xrightarrow[n\in\mathbb{N}]{} \mathbf{0}$ in $\mathbb{R}^T$, where $\mathbf{0}$ denotes zero function.

△

Example 4.5: Let T be an infinite set, i.e. $\mathbb{N} \subset T$. Consider a sequence $f_n \in l^{+\infty}(T)$, $n \in \mathbb{N}$, where $f_n(n) = n$ and $f_n(t) = 0$ otherwise.

The sequence $f_n \in l^{+\infty}(T)$, $n \in \mathbb{N}$ is not Cauchy in $l^{+\infty}(T)$, even, $\|f_n\|_T = n \nearrow +\infty$. But, $f_n \xrightarrow[n\in\mathbb{N}]{} \mathbf{0}$ in $\mathbb{R}^T$, where $\mathbf{0}$ denotes zero function.

△

Chapter 5

Spaces of continuous functions

5.1 $\mathsf{C}([0,1])$

We consider $\mathsf{C}([0,1])$ the set of all continuous real functions defined on the interval $[0,1]$. The space is naturally equipped with the supremal norm.

Theorem 5.1: *Topology of* $\left(\mathsf{C}([0,1]), \|\cdot\|_{[0,1]}\right)$ *coincides with relative topology on* $\mathsf{C}([0,1])$ *induced by topology of* $l^{+\infty}([0,1])$.

Proof:

1. For any $f \in \mathsf{C}([0,1])$ and $\varepsilon > 0$, we observe

$$\mathcal{U}(f;\varepsilon|\,\mathsf{C}([0,1])) = \mathcal{U}\left(f;\varepsilon|\,l^{+\infty}([0,1])\right) \cap \mathsf{C}([0,1])$$

   Thus, $\mathcal{G}(\mathsf{C}([0,1])) \subset \mathcal{G}(l^{+\infty}([0,1])) \cap \mathsf{C}([0,1])$.

2. Take $G \in \mathcal{G}(l^{+\infty}([0,1]))$ and $g \in G \cap \mathsf{C}([0,1])$.

   Then, there is $\delta > 0$ such that

$$\mathcal{U}\left(g;\delta|\,l^{+\infty}([0,1])\right) \subset G.$$

   Hence,

$$\mathcal{U}(g;\delta|\,\mathsf{C}([0,1])) = \mathcal{U}\left(g;\delta|\,l^{+\infty}([0,1])\right) \cap \mathsf{C}([0,1]) \subset G \cap \mathsf{C}([0,1]).$$

   Thus, $\mathcal{G}(\mathsf{C}([0,1])) \supset \mathcal{G}(l^{+\infty}([0,1])) \cap \mathsf{C}([0,1])$.

We have shown topology of $\mathsf{C}([0,1])$ coincides with relative topology on $\mathsf{C}([0,1])$ induced by topology of $l^{+\infty}([0,1])$.

Q.E.D.

Theorem 5.2: *Space* $\left(\mathsf{C}\left([0,1]\right), \|\cdot\|_{[0,1]}\right)$ *is a separable Banach space.*

Proof: Space $\left(\mathsf{C}\left([0,1]\right), \|\cdot\|_{[0,1]}\right)$ is a normed subspace of Banach space $\left(l^{+\infty}\left([0,1]\right), \|\cdot\|_{[0,1]}\right)$. We have to show completeness and recall separability.

1. Let $f_n \in \mathsf{C}\left([0,1]\right)$, $n \in \mathbb{N}$ is a Cauchy sequence.

   Sequence is also Cauchy in $l^{+\infty}\left([0,1]\right)$. Therefore according to Theorem 4.2, there is $g \in l^{+\infty}\left([0,1]\right)$ such that $\lim_{n\to+\infty} \|f_n - g\|_{[0,1]} = 0$.

   Fix $\varepsilon > 0$.

   Then, there is $n \in \mathbb{N}$ such that $\|f_n - g\|_{[0,1]} < \varepsilon$.

   Since $f_n \in \mathsf{C}\left([0,1]\right)$, one can find $\delta > 0$ such that for all $t, s \in [0,1]$, $|t-s| < \delta$ is $|f_n(t) - f_n(s)| < \varepsilon$.

   Then, for all $t, s \in [0,1]$, $|t-s| < \delta$ we can estimate

   $|g(t) - g(s)| \le |f_n(t) - g(t)| + |f_n(s) - g(s)| + |f_n(t) - f_n(s)| < 3\varepsilon.$

   Hence, $g \in \mathsf{C}\left([0,1]\right)$ and $\left(\mathsf{C}\left([0,1]\right), \|\cdot\|_{[0,1]}\right)$ is complete, thus a Banach space.

2. Space $\left(\mathsf{C}\left([0,1]\right), \|\cdot\|_{[0,1]}\right)$ is separable, since polynomials with rational coefficients are dense in it.

Q.E.D.

To be able properly describe continuous functions, we introduce a continuity modulus.

Definition 5.3 *We define continuity modulus* $\mathsf{w} : \mathbb{R}^{[0,1]} \times \mathbb{R}_+ \to \mathbb{R}_{+,0}$ *such that for* $x \in \mathbb{R}^{[0,1]}$, $\delta > 0$ *we set*

$$\mathsf{w}\left(x, \delta\right) = \sup\left\{|x(t) - x(s)| \;:\; |t-s| < \delta,\; t, s \in [0,1]\right\}. \tag{5.1}$$

The continuity modulus characterizes continuous functions.

Theorem 5.4: *Let* $f \in \mathbb{R}^{[0,1]}$. *Then,*
$f \in \mathsf{C}\left([0,1]\right)$ *if and only if* $\lim_{\delta\to 0+} \mathsf{w}\left(f, \delta\right) = 0$.

Since $[0,1]$ is a compact, each continuous function defined on it is uniformly continuous. That enables to characterize compacts of $\mathsf{C}\left([0,1]\right)$.

Theorem 5.5 (Ascela-Ascolli): *Let* $A \subset \mathsf{C}\left([0,1]\right)$. *Then,*
$\operatorname{clo}(A) \in \mathcal{K}\left(\mathsf{C}\left([0,1]\right)\right)$ *if and only if*

$$\sup_{f\in A} |f(0)| < +\infty, \quad \lim_{\delta\to 0+} \sup_{f\in A} \mathsf{w}\left(f, \delta\right) = 0\,.$$

Or in short, $A \subset \mathsf{C}\left([0,1]\right)$ is relatively compact in $\mathsf{C}\left([0,1]\right)$ if and only if functions from A are equicontinuous and uniformly bounded at 0.

5.1.1 Relation between topologies of $\mathsf{C}([0,1])$ and $\mathbb{R}^{[0,1]}$

Theorem 5.6: *We have* $\mathcal{G}\left(\mathbb{R}^{[0,1]}\right) \cap \mathsf{C}([0,1]) \subset \mathcal{G}(\mathsf{C}([0,1]))$.

Proof: It is sufficient to show the property for sets from a topological subbasis of $\mathbb{R}^{[0,1]}$, only.
Take $-\infty < a < b < +\infty$, $0 \le t \le 1$ and consider set

$$G = \left\{x \in \mathbb{R}^{[0,1]} \,:\, a < x(t) < b\right\}.$$

For $y \in G \cap \mathsf{C}([0,1])$ and $\varepsilon = \min\{b - y(t), y(t) - a\}$ is $\mathcal{U}(y; \varepsilon | \mathsf{C}([0,1])) \subset G$.
We have verified that $G \cap \mathsf{C}([0,1])$ is an open set in $\mathsf{C}([0,1])$.

Q.E.D.

The inclusion is sharp.

Lemma 5.7 *Let* $x \in \mathsf{C}([0,1])$ *and* $\varepsilon > 0$ *then*
$\mathcal{U}(x; \varepsilon | \mathsf{C}([0,1])) \notin \mathcal{G}\left(\mathbb{R}^{[0,1]}\right) \cap \mathsf{C}([0,1])$.

Proof: It is sufficient to consider a set from a topological basis of $\mathbb{R}^{[0,1]}$.
Take $I \in \mathsf{Fin}([0,1])$, $Q \in \mathcal{G}\left(\mathbb{R}^{I}\right)$ and $Q \neq \emptyset$.
We consider $H = {}_{[0,1]}\Pi_I^{-1}(Q) \cap \mathsf{C}([0,1])$.
Since H controls values of continuous functions in a finite number of arguments, we have $\sup_{x \in H} \|x\|_{[0,1]} = +\infty$.
Therefore, H cannot be contained in any ball in $\mathsf{C}([0,1])$.

Q.E.D.

Recall a notation introduced in Definition 3.4.

Theorem 5.8: *Let* $T \subset [0,1]$ *be countable and dense in* $[0,1]$. *Then, we have* ${}_{[0,1]}\mathcal{G}_{T;\delta\sigma} \cap \mathsf{C}([0,1]) \supset \mathcal{G}(\mathsf{C}([0,1]))$.

Proof: The space $\mathsf{C}([0,1])$ is separable. Therefore, it is sufficient to show the property only for open balls, since general open set is an union of countable many open balls.
Then for $x \in \mathsf{C}([0,1])$ and $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathcal{U}(x; \varepsilon | \mathsf{C}([0,1])) \\
&= \bigcup_{k=1}^{+\infty} \bigcap_{t \in T} \left\{ y \in \mathbb{R}^{[0,1]} \,:\, x(t) - \varepsilon + \frac{1}{k} < y(t) < x(t) + \varepsilon - \frac{1}{k} \right\} \cap \mathsf{C}([0,1]) \\
&\in {}_{[0,1]}\mathcal{G}_{T;\delta\sigma} \cap \mathsf{C}([0,1]).
\end{aligned}$$

Q.E.D.

Lemma 5.9 *For all $x \in \mathsf{C}([0,1])$ and $\varepsilon > 0$ we have*

$$\mathcal{V}(x;\varepsilon|\,\mathsf{C}([0,1])) \in \mathcal{F}\left(\mathbb{R}^{[0,1]}\right) \cap \mathsf{C}([0,1])\,. \tag{5.2}$$

Proof:

$$\begin{aligned}
&\mathcal{V}(x;\varepsilon|\,\mathsf{C}([0,1])) \\
&= \bigcap_{t\in[0,1]} \left\{y \in \mathbb{R}^{[0,1]} \,:\, x(t)-\varepsilon \le y(t) \le x(t)+\varepsilon\right\} \cap \mathsf{C}([0,1]) \\
&\in \mathcal{F}\left(\mathbb{R}^{[0,1]}\right) \cap \mathsf{C}([0,1])\,.
\end{aligned}$$

Q.E.D.

Theorem 5.10: *We have $\mathcal{K}(\mathsf{C}([0,1])) \subset \left\{K \in \mathcal{K}\left(\mathbb{R}^{[0,1]}\right) \,:\, K \subset \mathsf{C}([0,1])\right\}$.*

Proof: Let $K \in \mathcal{K}(\mathsf{C}([0,1]))$.
Assume an covering $K \subset \bigcup_{\lambda\in\Lambda} G_\lambda$, where $G_\lambda \in \mathcal{G}\left(\mathbb{R}^{[0,1]}\right)$.
Then, $K \subset \bigcup_{\lambda\in\Lambda} (G_\lambda \cap \mathsf{C}([0,1]))$.
We know $G_\lambda \cap \mathsf{C}([0,1]) \in \mathcal{G}(\mathsf{C}([0,1]))$ according to Theorem 5.6.
Hence, we can select $I \in \mathsf{Fin}(\Lambda)$ such that

$$K \subset \bigcup_{\lambda\in i} (G_\lambda \cap \mathsf{C}([0,1])) \subset \bigcup_{\lambda\in i} G_\lambda.$$

We have selected a finite covering of K, thus, we have verified $K \in \mathcal{K}\left(\mathbb{R}^{[0,1]}\right)$; we employ Lemma 1.32 and Theorem 2.8.

Q.E.D.

Lemma 5.11 *Consider real functions on $[0,1]$, $f_0 \equiv 0$ and for $n \in \mathbb{N}$ piecewise linear continuous function f_n determined by values $f_n(0) = 0$, $f_n\left(\frac{1}{n+2}\right) = 1$, $f_n\left(\frac{2}{n+2}\right) = 0$, $f_n(1) = 0$.*

Then, $\{f_n, n \in \mathbb{N}\}$ is a compact in $\mathbb{R}^{[0,1]}$, but, it is no compact neither in $\mathsf{C}([0,1])$ nor in $l^{+\infty}([0,1])$.

Theorem 5.12: *We have*

$$\mathsf{Cylindric}([0,1]) \cap \mathsf{C}([0,1]) = \mathcal{B}\left(\mathbb{R}^{[0,1]}\right) \cap \mathsf{C}([0,1]) = \mathcal{B}(\mathsf{C}([0,1]))\,. \tag{5.3}$$

Proof: Theorem is a consequence of theorems 3.3, 5.6 and 5.8.

Q.E.D.

5.1.2 $\mathsf{C}([0,1])$ and randomness

This part is taken from [2], Chapter 2.8. Let us start with a theorem from [2], T8.2, p.83.

Lemma 5.13 *Let* $T \subset [0,1]$ *then* ${}_{[0,1]}\Pi_T : \mathsf{C}([0,1]) \to \mathbb{R}^T$ *is continuous.*

Proof: Consider a sequence $f_n \in \mathsf{C}([0,1])$, $n \in \mathbb{N}$ such that $f_n \xrightarrow[n\to+\infty]{} f$ in $\mathsf{C}([0,1])$. Then for any $t \in T$,

$$\left| {}_{[0,1]}\Pi_T(f_n)(t) - {}_{[0,1]}\Pi_T(f)(t) \right| = |f_n(t) - f(t)| \le \|f_n - f\|_{[0,1]}.$$

It means ${}_{[0,1]}\Pi_T : \mathsf{C}([0,1]) \to \mathbb{R}^T$ is continuous.
Even, we proved ${}_{[0,1]}\Pi_T : \mathsf{C}([0,1]) \to l^{+\infty}(T)$ is continuous.

Q.E.D.

Theorem 5.14: *Let* X_n*,* $n \in \mathbb{N}$ *be a sequence of random processes in* $\mathsf{C}([0,1])$ *and* X *be a random process in* $\mathsf{C}([0,1])$*. Then,*

$$X_n \xrightarrow[n\to+\infty]{\mathcal{D}} X \text{ in } \mathsf{C}([0,1]) \qquad \text{implies} \qquad X_n \xrightarrow[n\to+\infty]{fidi} X \text{ in } \mathbb{R}^{[0,1]}.$$

Proof: Take $I \in \mathsf{Fin}([0,1])$.
Hence, ${}_{[0,1]}\Pi_I : \mathsf{C}([0,1]) \to \mathbb{R}^I$ is continuous, according to Lemma 5.13.
Applying Theorem 1.97, we have

$${}_{[0,1]}\Pi_I(X_n) \xrightarrow[n\to+\infty]{\mathcal{D}} {}_{[0,1]}\Pi_I(X) \text{ in } \mathbb{R}^I.$$

Thus, convergence of finite distributions is proved.

Q.E.D.

Lemma 5.15 *Let* X*,* Y *be random processes in* $\mathsf{C}([0,1])$*. If* $X \overset{fidi}{\equiv} Y$ *in* $\mathbb{R}^{[0,1]}$ *then* $X \overset{\mathcal{D}}{\equiv} Y$ *on* $\mathcal{B}(\mathsf{C}([0,1]))$*.*

Proof: Space $\mathsf{C}([0,1])$ is a Polish space. Hence, distributions μ_X, μ_Y on $\mathcal{B}(\mathsf{C}([0,1]))$ are Radon τ-additive probabilities; see Theorem 2.25.
Take, $K \in \mathcal{K}(\mathsf{C}([0,1]))$.
According to Theorem 5.10,

$$\forall\ I \in \mathsf{Fin}([0,1]) \quad \text{we have} \quad {}_{[0,1]}\Pi_I(K) \in \mathcal{K}\left(\mathbb{R}^I\right),$$
$$K = \bigcap_{I \in \mathsf{Fin}([0,1])} {}_{[0,1]}\Pi_I^{-1}({}_{[0,1]}\Pi_I(K)).$$

Hence,

$$\begin{aligned}
\mu_X(K) &= \mu_X(K \cap \mathsf{C}([0,1])) \\
&= \mu_X\left(\bigcap_{I \in \mathsf{Fin}([0,1])} {}_{[0,1]}\Pi_I^{-1}({}_{[0,1]}\Pi_I(K)) \cap \mathsf{C}([0,1])\right) \\
&= \inf\left\{\mu_X\left({}_{[0,1]}\Pi_I^{-1}({}_{[0,1]}\Pi_I(K)) \cap \mathsf{C}([0,1])\right) \,:\, I \in \mathsf{Fin}([0,1])\right\} \\
&= \inf\left\{\mathsf{P}\left(X \in {}_{[0,1]}\Pi_I^{-1}({}_{[0,1]}\Pi_I(K)) \cap \mathsf{C}([0,1])\right) \,:\, I \in \mathsf{Fin}([0,1])\right\} \\
&= \inf\left\{\mathsf{P}\left(X \in {}_{[0,1]}\Pi_I^{-1}({}_{[0,1]}\Pi_I(K))\right) \,:\, I \in \mathsf{Fin}([0,1])\right\} \\
&= \inf\left\{\mathsf{P}\left(X(I) \in {}_{[0,1]}\Pi_I(K)\right) \,:\, I \in \mathsf{Fin}([0,1])\right\}.
\end{aligned}$$

Similarly, we receive

$$\mu_Y(K) = \inf\left\{\mathsf{P}\left(Y(I) \in {}_{[0,1]}\Pi_I(K)\right) \,:\, I \in \mathsf{Fin}([0,1])\right\}.$$

We have derived μ_X, μ_Y coincide on $\mathcal{K}(\mathsf{C}([0,1]))$, since $X \overset{fidi}{\equiv} Y$ in $\mathbb{R}^{[0,1]}$.
We know μ_X, μ_Y are Radon in $\mathsf{C}([0,1])$. Therefore, $X \overset{\mathcal{D}}{\equiv} Y$ on $\mathcal{B}(\mathsf{C}([0,1]))$.

Q.E.D.

Theorem 5.16: *Let X_n, $n \in \mathbb{N}$ be a sequence of random processes in $\mathsf{C}([0,1])$ and X be a random process in $\mathbb{R}^{[0,1]}$. Suppose*

i) $X_n \xrightarrow[n \to +\infty]{fidi} X$ in $\mathbb{R}^{[0,1]}$.

ii) The sequence X_n, $n \in \mathbb{N}$ is tight in $\mathsf{C}([0,1])$.

Then there is $\widetilde{X}$ *a random process in* $\mathsf{C}([0,1])$ *such that*

$$X_n \xrightarrow[n\to+\infty]{\mathcal{D}} \widetilde{X} \text{ in } \mathsf{C}([0,1])\,.$$

The distribution $\mu_{\widetilde{X}}$ *on* $\mathcal{B}(\mathsf{C}([0,1]))$ *is a Radon* τ*-additive probability uniquely determined by coincidence* $\widetilde{X} \overset{fidi}{\equiv} X$ *in* $\mathbb{R}^{[0,1]}$.

Proof: The sequence is tight in $\mathsf{C}([0,1])$, therefore according to Prochoroff theorem 2.30, it is relatively weakly compact. This means that each subsequence possesses at least one weak cluster point. We have to show that all weak cluster points of the given sequence possess the same distribution on $\mathcal{B}(\mathsf{C}([0,1]))$.
Let Z be a weak cluster point of the sequence X_n, $n \in \mathbb{N}$ in $\mathsf{C}([0,1])$. Thus, we have a subsequence such that

$$X_{n_k} \xrightarrow[k\to+\infty]{\mathcal{D}} Z \text{ in } \mathsf{C}([0,1])\,.$$

Take $I \in \mathsf{Fin}([0,1])$ and $G \in \mathcal{G}\left(\mathbb{R}^I\right)$.
According to Theorem 5.6 we have ${}_{[0,1]}\Pi_I^{-1}(G) \cap \mathsf{C}([0,1]) \in \mathcal{G}(\mathsf{C}([0,1]))$. Therefore,

$$\begin{aligned}
\liminf_{k\to+\infty} \mathsf{P}\left(X_{n_k}(I) \in G\right) &= \liminf_{k\to+\infty} \mathsf{P}\left(X_{n_k} \in {}_{[0,1]}\Pi_I^{-1}(G)\right) = \\
&= \liminf_{k\to+\infty} \mathsf{P}\left(X_{n_k} \in {}_{[0,1]}\Pi_I^{-1}(G) \cap \mathsf{C}([0,1])\right) \\
&\geq \mathsf{P}\left(Z \in {}_{[0,1]}\Pi_I^{-1}(G) \cap \mathsf{C}([0,1])\right) = \\
&= \mathsf{P}\left(Z \in {}_{[0,1]}\Pi_I^{-1}(G)\right) = \mathsf{P}\left(Z(I) \in G\right).
\end{aligned}$$

We have shown $X_{n_k} \xrightarrow[k\to+\infty]{fidi} Z$ in $\mathbb{R}^{[0,1]}$, we also know $X_{n_k} \xrightarrow[n\to+\infty]{fidi} X$ in $\mathbb{R}^{[0,1]}$.
According to Lemma 3.16, $Z \overset{fidi}{\equiv} X$ in $\mathbb{R}^T$.
According to Lemma 5.15, all cluster points of our sequence possesses the same distribution uniquely determined by finite dimensional distributions of X.

Q.E.D.

Theorem 5.17: *Sequence $X_n = (X_n(t), t \in [0,1])$ of random processes in $\mathsf{C}([0,1])$ is tight in $\mathsf{C}([0,1])$ if and only if for all $\varepsilon > 0$, $\eta > 0$ there are $\alpha \geq 0$, $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$ we have*

$$\mathsf{P}(|X_n(0)| > \alpha) < \varepsilon, \tag{5.4}$$
$$\mathsf{P}(\mathsf{w}(X_n, \delta) > \eta) < \varepsilon. \tag{5.5}$$

Proof:

1. Let the sequence is tight in $\mathsf{C}([0,1])$ and $\varepsilon > 0$, $\eta > 0$.

   Then, there exists a compact $K \in \mathcal{K}(\mathsf{C}([0,1]))$ such that for all $n \in \mathbb{N}$ we have $\mathsf{P}(X_n \notin K) < \varepsilon$.

   According to Theorem 5.5,
   $$\sup_{f \in K} |f(0)| < +\infty, \quad \lim_{\delta \to 0+} \sup_{f \in K} \mathsf{w}(f, \delta) = 0\,.$$

   We denote $\alpha = \sup_{f \in K} |f(0)|$ and find $\delta > 0$ such that $\mathsf{w}(f, \delta) < \eta$.

   Then, we receive for all $n \in \mathbb{N}$
   $$\mathsf{P}(|X_n(0)| > \alpha) \leq \mathsf{P}(X_n \notin K) < \varepsilon,$$
   $$\mathsf{P}(\mathsf{w}(X_n, \delta) > \eta) \leq \mathsf{P}(X_n \notin K) < \varepsilon\,.$$

   Thus, the property is shown, even with $n_0 = 1$.

2. Let the property holds and $\varepsilon > 0$ is given.

   (a) Space $\mathsf{C}([0,1])$ is Polish, therefore, each Borel probability on $\mathsf{C}([0,1])$ is Radon; see Theorem 2.25.
   Hence for $\varepsilon > 0$, $\eta > 0$, $n \in \{1, 2, \ldots, n_0 - 1\}$ we are able to find $\alpha_n \geq 0$, $\delta_n > 0$ such that

   i) $\mathsf{P}(|X_n(0)| > \alpha_n) < \varepsilon$;
   ii) $\mathsf{P}(\mathsf{w}(X_n, \delta_n) > \eta) < \varepsilon$.

   Set $\tilde{\alpha} = \max\{\alpha_1, \alpha_2, \ldots, \alpha_{n_0-1}, \alpha\}$, $\tilde{\delta} = \max\{\delta_1, \delta_2, \ldots, \delta_{n_0-1}, \delta\}$. Then $\tilde{\alpha} > 0$, $\tilde{\delta} > 0$ and for all $n \in \mathbb{N}$ we have

   i) $\mathsf{P}(|X_n(0)| > \tilde{\alpha}) < \varepsilon$;
   ii) $\mathsf{P}\left(\mathsf{w}\left(X_n, \tilde{\delta}\right) > \eta\right) < \varepsilon$.

   Thus, we can assume $n_0 = 1$.

(b) As shown above we can assume $n_0 = 1$.
Then, we can find $\alpha \geq 0$ and $\delta_k > 0$, $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$\mathsf{P}\left(|X_n(0)| > a\right) < \varepsilon,$$
$$\mathsf{P}\left(\mathsf{w}\left(X_n, \delta_k\right) > 2^{-k}\right) < \varepsilon 2^{-k}.$$

We set

$$K = \left\{x \in \mathsf{C}([0,1]) \, : \, |x(0)| \leq a, \ \forall k \in \mathbb{N} \quad \mathsf{w}(x, \delta_k) \leq 2^{-k}\right\}.$$

Evidently $K \in \mathcal{F}(\mathsf{C}([0,1]))$.
Therefore according to Theorem 5.5, $K \in \mathcal{K}(\mathsf{C}([0,1]))$.
We have to estimate the probability

$$\mathsf{P}(X_n \notin K) \leq \mathsf{P}(|X_n(0)| > a) + \sum_{k=1}^{+\infty} \mathsf{P}\left(\mathsf{w}(X_n, \delta_k) > 2^{-k}\right)$$
$$< \varepsilon + \varepsilon \sum_{k=1}^{+\infty} 2^{-k} = 2\varepsilon.$$

Q.E.D.

Theorem 5.18: *Let sequence $X_n = (X_n(t), t \in [0,1])$, $n \in \mathbb{N}$ of random processes in $\mathsf{C}([0,1])$ fulfill:*

i) Sequence $X_n(0)$, $n \in \mathbb{N}$ is tight in $\mathbb{R}$.

ii) For each $\varepsilon > 0$, $\eta > 0$ there are $\delta \in (0, \frac{1}{2})$ and $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$ and for all $t \in [0, 1-\delta]$ it is fulfilled

$$\mathsf{P}\left(\sup\left\{|X_n(t) - X_n(s)| \, : \, s \in [t, t+\delta]\right\} > \eta\right) \ < \ \varepsilon\delta. \qquad (5.6)$$

Then the sequence is tight in $\mathsf{C}([0,1])$.

Proof: We have to verify assumptions of Theorem 5.17.
Tightness of the sequence $X_n(0)$ is equivalent with i) of Theorem 5.17.
We have to verify the assumption ii) of Theorem 5.17, only.
Take $\eta > 0$ and $\varepsilon > 0$.
According to (5.6), there are $\delta \in (0, \frac{1}{2})$ and $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$ and $t \in [0, 1-\delta]$ we have

$$\mathsf{P}\left(\sup\left\{|X_n(t) - X_n(s)| \, : \, s \in [t, t+\delta]\right\} > \eta\right) < \varepsilon\delta.$$

Take $M \in \mathbb{N}$, $M \geq 3$ such that $\frac{1}{M} \leq \delta < \frac{1}{M-1}$. Then for all $n \in \mathbb{N}$, $n \geq n_0$ and $t \in [0, 1 - \frac{1}{M}]$ we have

$$\begin{aligned}
&\mathsf{P}\left(\sup\left\{|X_n(t) - X_n(s)| \,:\, s \in \left[t, t + \frac{1}{M}\right]\right\} > \eta\right) \\
&\leq \mathsf{P}\left(\sup\left\{|X_n(t) - X_n(s)| \,:\, s \in [t, t+\delta]\right\} > \eta\right) \\
&< \varepsilon\delta < \varepsilon\frac{1}{M-1} = \frac{M}{M-1}\frac{\varepsilon}{M} < \frac{2\varepsilon}{M}.
\end{aligned}$$

Let us estimate the probability

$$\begin{aligned}
&\mathsf{P}\left(\mathsf{w}\left(X_n, \frac{1}{M}\right) > 3\eta\right) \\
&= \mathsf{P}\left(\sup\left\{|X_n(t) - X_n(s)| \,:\, |t-s| < \frac{1}{M},\ t, s \in [0,1]\right\} > 3\eta\right) \\
&\leq \mathsf{P}\left(\sup\left\{\left|X_n(t) - X_n\left(\frac{\lfloor tM \rfloor}{M}\right)\right| + \left|X_n(s) - X_n\left(\frac{\lfloor sM \rfloor}{M}\right)\right| + \right.\right. \\
&\left.\left. + \left|X_n\left(\frac{\lfloor tM \rfloor}{M}\right) - X_n\left(\frac{\lfloor sM \rfloor}{M}\right)\right|\right\} \,:\, |t-s| < \frac{1}{M},\ t, s \in [0,1]\right\} > 3\eta\right) \\
&\leq \sum_{k=0}^{M-1} \mathsf{P}\left(\sup\left\{\left|X_n(s) - X_n\left(\frac{k}{M}\right)\right| \,:\, \frac{k}{M} \leq s \leq \frac{k+1}{M}\right\} > \eta\right) \\
&< M\frac{2\varepsilon}{M} = 2\varepsilon.
\end{aligned}$$

Assumptions of Theorem 5.17 are fulfilled, then, the sequence is tight in $\mathsf{C}([0,1])$.

Q.E.D.

Theorem 5.19: *Let sequence $X_n = (X_n(t), t \in [0,1])$ of random processes in $\mathsf{C}([0,1])$ fulfill:*

i) Sequence $X_n(0)$, $n \in \mathbb{N}$ is tight in $\mathbb{R}$.

ii) There are $\alpha > 0$, $\beta \geq 0$ and nondecreasing continuous function $F : [0,1] \to \mathbb{R}$ such that for all $n \in \mathbb{N}$, $0 \leq t < s \leq 1$, $\lambda > 0$ we have

$$\mathsf{P}\left(\sup\left\{|X_n(u) - X_n(t)| \,:\, u \in [t,s]\right\} > \lambda\right) \leq \lambda^{-\beta}(F(s) - F(t))^{1+\alpha}. \tag{5.7}$$

Then, the sequence is tight in $\mathsf{C}([0,1])$.

Proof: We have to verify assumptions of Theorem 5.17.
We know tightness of the sequence $X_n(0)$ is equivalent with assumption i) of Theorem 5.17.
We have to verify the second property, only.
Take $\eta > 0$ and $M \in \mathbb{N}$.
Then, for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\mathsf{P}\left(\mathsf{w}\left(X_n, \frac{1}{M}\right) > 3\eta\right) \\
&= \mathsf{P}\left(\sup\left\{|X_n(t) - X_n(s)| \,:\, |t-s| < \frac{1}{M},\ t,s \in [0,1]\right\} < 3\eta\right) \\
&\leq \mathsf{P}\left(\sup\left\{\left|X_n(t) - X_n\left(\frac{\lfloor tM \rfloor}{M}\right)\right| + \left|X_n(s) - X_n\left(\frac{\lfloor sM \rfloor}{M}\right)\right| + \right.\right. \\
&+ \left.\left.\left|X_n\left(\frac{\lfloor tM \rfloor}{M}\right) - X_n\left(\frac{\lfloor sM \rfloor}{M}\right)\right| \,:\, |t-s| < \frac{1}{M},\ t,s \in [0,1]\right\} > 3\eta\right) \\
&\leq \sum_{k=0}^{M-1} \mathsf{P}\left(\sup\left\{\left|X_n(s) - X_n\left(\frac{k}{M}\right)\right| \,:\, \frac{k}{M} \leq s \leq \frac{k+1}{M}\right\} > \eta\right) \\
&\leq \eta^{-\beta} \sum_{k=0}^{M-1} \left(F\left(\frac{k+1}{M}\right) - F\left(\frac{k}{M}\right)\right)^{1+\alpha} \\
&\leq \eta^{-\beta} \mathsf{w}\left(F, \frac{1}{M}\right)^{\alpha} (F(1) - F(0)).
\end{aligned}$$

Increasing M, the estimate can be made arbitrary small, since function F is continuous.
Assumptions of Theorem 5.17 are verified. Therefore, the sequence is tight in $\mathsf{C}([0,1])$.

Q.E.D.

Now, we are approaching to a criterion for tightness of a sequence of random processes in $\mathsf{C}([0,1])$ which can be easily checked. Theorem is introduced in [2], Th 12.3, p. 136.

Theorem 5.20: *Let sequence $X_n = (X_n(t), t \in [0,1])$ of random processes in $\mathsf{C}([0,1])$ fulfill:*

i) Sequence $X_n(0)$, $n \in \mathbb{N}$ is tight in $\mathbb{R}$.

ii) There are $\alpha > 0$, $\beta \geq 0$ and nondecreasing continuous function $F : [0,1] \to \mathbb{R}$ such that for all $n \in \mathbb{N}$, $t, s \in [0,1]$, $\lambda > 0$ we have

$$\mathsf{P}\left(|X_n(t) - X_n(s)| \geq \lambda\right) \leq \lambda^{-\beta} |F(t) - F(s)|^{1+\alpha}. \tag{5.8}$$

Then, the sequence is tight in $\mathsf{C}([0,1])$.

Proof: We have to verify assumptions of Theorem 5.19.
The first condition is identical with the assumption i) of Theorem 5.19.
We have to verify the second condition, only.
Take $\delta > 0$, $\eta > 0$, $t \in [0, 1-\delta]$ and $n \in \mathbb{N}$.
Processes are continuous and, thus, we have

$$\lim_{m \to +\infty} \max \left\{ \left| X_n(t) - X_n\left(t + \frac{k}{m}\delta\right) \right| \, : \, k = 1, 2, \ldots, m \right\}$$
$$= \sup \left\{ |X_n(t) - X_n(s)| \, : \, s \in [t, t+\delta] \right\}.$$

Now, we apply Theorem 7.7 for choice for $k = 1, 2, \ldots, m$

$$\begin{aligned} \xi_k &= X_n\left(t + \frac{k}{m}\delta; -\right) X_n\left(t + \frac{k-1}{m}\delta; ,\right) \\ u_k &= F\left(t + \frac{k}{m}\delta\right) - F\left(t + \frac{k-1}{m}\delta\right) \end{aligned}$$

According to (5.8), we have for all $i, j = 1, 2, \ldots, m$, $i \leq j$ and $\lambda > 0$ an estimate

$$\mathsf{P}\left(\left| \sum_{k=i}^{j} \xi_k \right| \geq \lambda \right) = \mathsf{P}\left(\left| X_n\left(t + \frac{j}{m}\delta\right) - X_n\left(t + \frac{i-1}{m}\delta\right) \right| \geq \lambda \right)$$
$$\leq \lambda^{-\beta} \left(F\left(t + \frac{j}{m}\delta\right) - F\left(t + \frac{i-1}{m}\delta\right) \right)^{1+\alpha} = \lambda^{-\beta} \left(\sum_{k=i}^{j} u_k \right)^{1+\alpha}.$$

Then according to Theorem 7.7, we have

$$\mathsf{P}\left(\max \left\{ \left| X_n(t) - X_n\left(t + \frac{k}{m}\delta\right) \right| \, : \, k = 1, 2, \ldots, m \right\} \geq \eta \right)$$
$$\leq \mathsf{K}(\alpha, \beta)\, \eta^{-\beta} \left(\sum_{k=1}^{m} \left(F\left(t + \frac{k}{m}\delta\right) - F\left(t + \frac{k-1}{m}\delta\right) \right) \right)^{1+\alpha}$$
$$= \mathsf{K}(\alpha, \beta)\, \eta^{-\beta} \left(F(t+\delta) - F(t) \right)^{1+\alpha}.$$

Letting $m \to +\infty$, we are receiving

$$\begin{aligned}&\mathsf{P}\left(\sup\left\{\left|X_n(t) - X_n(s)\right| \,:\, s \in [t, t+\delta]\right\} \geq \eta\right)\\ &\leq \mathsf{K}(\alpha,\beta)\,\eta^{-\beta}\left(F(t+\delta) - F(t)\right)^{1+\alpha}.\end{aligned}$$

All assumptions of Theorem 5.19 are verified. Therefore, the sequence is tight in $\mathsf{C}([0,1])$.

Q.E.D.

5.2 Interpolated random walk

I this section, we present an application of the weak convergence in $\mathsf{C}([0,1])$. Random processes arising by interpolation of partial sums of random walks will be treated here.

A sequence of real random variables ξ_i, $i \in \mathbb{N}$ and a positive real constant σ are assumed. Partial sums are denoted $\mathsf{S}_k = \sum_{i=1}^{k} \xi_i$, $k \in \mathbb{N}$, where $\mathsf{S}_0 = 0$.

Partial sums determine a sequence of random processes with jumps $(V_n(t), 0 \leq t \leq 1)$, $n \in \mathbb{N}$ given by

$$V_n(t) = \frac{1}{\sigma\sqrt{n}} \mathsf{S}_{\lfloor nt \rfloor} \quad \text{for} \quad n \in \mathbb{N},\ 0 \leq t \leq 1, \tag{5.9}$$

and, a sequence of continuous processes $(Z_n(t), 0 \leq t \leq 1)$, $n \in \mathbb{N}$ given by

$$\begin{aligned} Z_n(t) &= \frac{1}{\sigma\sqrt{n}}\left(\mathsf{S}_k + n\left(t - \frac{k}{n}\right)\left(\mathsf{S}_{k+1} - \mathsf{S}_k\right)\right) \\ &\qquad \text{for} \quad n \in \mathbb{N},\ k = 0, 1, \ldots, n-1,\ \frac{k}{n} \leq t \leq \frac{k+1}{n}. \end{aligned} \tag{5.10}$$

Finite-dimensional distributions of these random processes are close each to the other.

Lemma 5.21 *There is a simple estimate*

$$\forall t \in [0,1] \quad \left|V_n(t) - Z_n(t)\right| \leq \frac{1}{\sigma\sqrt{n}} \left|\xi_{\lfloor nt \rfloor + 1}\right|. \tag{5.11}$$

Let us begin with measurability.

Lemma 5.22 *Always, Z_n, $n \in \mathbb{N}$ are random processes in $\mathsf{C}([0,1])$ and V_n, $n \in \mathbb{N}$ are random processes in $l^{+\infty}([0,1])$.*

Proof: Fix $n \in \mathbb{N}$ and consider mappings $\kappa_1 : \mathbb{R}^n \to l^{+\infty}([0,1])$ and $\kappa_2 : \mathbb{R}^n \to \mathsf{C}([0,1])$ given by

$$\kappa_1(x_1, x_2, \ldots, x_n)(t) = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{k} x_i,$$

$$\kappa_2(x_1, x_2, \ldots, x_n)(t) = \frac{1}{\sigma\sqrt{n}} \left(\sum_{i=1}^{k} x_i + n\left(t - \frac{k}{n}\right) x_{k+1} \right)$$

$$\text{for} \quad k = 0, 1, \ldots, n-1, \ \frac{k}{n} \le t \le \frac{k+1}{n}.$$

Both mappings $\kappa_1 : \mathbb{R}^n \to l^{+\infty}([0,1])$ and $\kappa_2 : \mathbb{R}^n \to \mathsf{C}([0,1])$ are continuous, but, in different spaces.
Therefore, $V_n = \kappa_1(\xi_1, \xi_2, \ldots, \xi_n)$ is a random process in $l^{+\infty}([0,1])$ and $Z_n = \kappa_2(\xi_1, \xi_2, \ldots, \xi_n)$ is a random process in $\mathsf{C}([0,1])$.

Q.E.D.

Theorem 5.23: *If for all $\varepsilon > 0$ there is $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \ge n_0$ and for all $k \in \mathbb{N}$ it is fulfilled*

$$\mathsf{P}\left(\max\{|\mathsf{S}_{k+i} - \mathsf{S}_k| : i = 1, 2, \ldots, n\} \ge \lambda\sigma\sqrt{n}\right) \le \lambda^{-2}\varepsilon, \tag{5.12}$$

then the sequence Z_n, $n \in \mathbb{N}$ is tight in $\mathsf{C}([0,1])$.

Proof: We will verify assumptions of Theorem 5.18. Tightness of the sequence $Z_n(0)$, $n \in \mathbb{N}$ is evident, since it is a sequence of zeros. We have to verify the second assumption, only.

1. For $\delta \in (0, \frac{1}{2})$, $t \in [0, 1-\delta]$, $s \in [t, t+\delta]$ and $n \in \mathbb{N}$ we prepare an estimate
$$\frac{\lfloor tn \rfloor}{n} \le t \le s \le t + \delta \le \frac{\lfloor tn \rfloor + 1}{n} + \frac{\lfloor \delta n \rfloor + 1}{n} = \frac{\lfloor tn \rfloor + \lfloor \delta n \rfloor + 2}{n}.$$

2. Fix $\delta \in (0, \frac{1}{2})$, $t \in [0, 1-\delta]$ and $n \in \mathbb{N}$. Using previous estimate we are receiving
$$\sup\{|Z_n(s) - Z_n(t)| : s \in [t, t+\delta]\}$$
$$\le \sup\left\{ \left| Z_n(s) - Z_n\left(\frac{\lfloor tn \rfloor}{n}\right) \right| + \left| Z_n(t) - Z_n\left(\frac{\lfloor tn \rfloor}{n}\right) \right| : s \in [t, t+\delta] \right\}$$
$$\le \frac{2}{\sigma\sqrt{n}} \max\left\{ \left| \mathsf{S}_{\lfloor tn \rfloor + j} - \mathsf{S}_{\lfloor tn \rfloor} \right| : j = 1, 2, \ldots, \lfloor \delta n \rfloor + 2 \right\}.$$

3. For $\varepsilon > 0$, $\eta > 0$ there is $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$ and for all $k \in \mathbb{N}$ it is fulfilled

$$\mathsf{P}\left(\max\left\{\left|\mathsf{S}_{k+i} - \mathsf{S}_k\right| : i = 1, 2, \ldots, n\right\} \geq \lambda\sigma\sqrt{n}\right) \leq \frac{1}{8}\lambda^{-2}\eta^2\varepsilon,$$

Let us denote $\delta = \frac{\eta^2}{8\lambda^2}$. Then for all $t \in [0, 1-\delta]$ and $n \geq \max\left\{n_0, \frac{2}{\delta}\right\}$ we have

$$\begin{aligned}
&\mathsf{P}\left(\sup\left\{\left|Z_n(s) - Z_n(t)\right| : s \in [t, t+\delta]\right\} \geq \eta\right) \\
&\leq \mathsf{P}\left(\frac{2}{\sigma\sqrt{n}}\max\left\{\left|\mathsf{S}_{\lfloor tn\rfloor+j} - \mathsf{S}_{\lfloor tn\rfloor}\right| : j = 1, 2, \ldots, \lfloor\delta n\rfloor + 2\right\} \geq \eta\right) \\
&\leq \mathsf{P}\left(\max\left\{\left|\mathsf{S}_{\lfloor tn\rfloor+j} - \mathsf{S}_{\lfloor tn\rfloor}\right| : j = 1, 2, \ldots, \lfloor n\delta\rfloor + 2\right\} \geq \frac{\eta}{2}\sigma\sqrt{n}\right) \\
&\leq \mathsf{P}\left(\max\left\{\left|\mathsf{S}_{\lfloor tn\rfloor+j} - \mathsf{S}_{\lfloor tn\rfloor}\right| : j = 1, 2, \ldots, \lfloor n\delta\rfloor + 2\right\}\right. \\
&\qquad\left.\geq \frac{\eta}{2}\sqrt{\frac{n}{\lfloor n\delta\rfloor + 2}}\sigma\sqrt{\lfloor n\delta\rfloor + 2}\right) \\
&\leq \mathsf{P}\left(\max\left\{\left|\mathsf{S}_{\lfloor tn\rfloor+j} - \mathsf{S}_{\lfloor tn\rfloor}\right| : j = 1, 2, \ldots, \lfloor n\delta\rfloor + 2\right\}\right. \\
&\qquad\left.\geq \frac{\eta}{2\sqrt{2\delta}}\sigma\sqrt{\lfloor n\delta\rfloor + 2}\right) \\
&\leq \mathsf{P}\left(\max\left\{\left|\mathsf{S}_{\lfloor tn\rfloor+j} - \mathsf{S}_{\lfloor tn\rfloor}\right| : j = 1, 2, \ldots, \lfloor n\delta\rfloor + 2\right\}\right. \\
&\qquad\left.\geq \lambda\sigma\sqrt{\lfloor n\delta\rfloor + 2}\right) \\
&\leq \frac{1}{8}\lambda^{-2}\eta^2\varepsilon = \delta\varepsilon.
\end{aligned}$$

Assumptions of Theorem 5.18 are fulfilled, thus, the sequence of processes is tight in $\mathsf{C}([0,1])$.

Q.E.D.

5.2.1 Donsker invariance principle

We start with convergence of finite dimensional distributions.

Definition 5.24 *A random process $W = (W(t), 0 \leq t \leq 1)$ with continuous sample paths and finite dimensional distributions $W(I) \sim \mathsf{N}(\mathbf{0}, \Sigma_I)$ for all $I \in \mathsf{Fin}([0,1])$, where $\Sigma_I = (i \wedge j)_{i,j\in I}$ is called Wiener process.*

A random process $\widetilde{W} = \left(\widetilde{W}(t), 0 \leq t \leq 1\right)$ with finite dimensional distributions $\widetilde{W}(I) \sim \mathsf{N}(\mathbf{0}, \Sigma_I)$ for all $I \in \mathsf{Fin}([0,1])$, where $\Sigma_I = (i \wedge j)_{i,j\in I}$ is called pre-Wiener process.

Lemma 5.25 *A pre-Wiener process* $\widetilde{W} = \left(\widetilde{W}(t), 0 \leq t \leq 1\right)$ *exists and its distribution* $\mu_{\widetilde{W}}$ *on* $\mathsf{Cylindric}([0,1])$ *is a probability and is uniquely defined.*

Proof: Existence of the process follows Theorem 3.12.

Q.E.D.

Theorem 5.26: *Let* ξ_i, $i \in \mathbb{N}$ *be i.i.d. real random variables with* $\mathsf{E}[\xi_1] = 0$ *and* $\mathsf{var}(\xi_1) = \sigma^2 \in \mathbb{R}_+$. *Then* $Z_n \xrightarrow[n\to+\infty]{fidi} \widetilde{W}$ *in* $\mathbb{R}^{[0,1]}$, $V_n \xrightarrow[n\to+\infty]{fidi} \widetilde{W}$ *in* $\mathbb{R}^{[0,1]}$.

Proof: Consider step process at first.

1. $V_n(0) \equiv 0$ and $\widetilde{W}(0) = 0$ almost surely. Therefore, convergence $V_n(0) \xrightarrow[n\to+\infty]{\mathcal{D}} \widetilde{W}(0)$ is trivial.

2. Take $0 \leq t < s \leq 1$. Then,

$$\begin{aligned} V_n(s) - V_n(t) &= \frac{1}{\sigma\sqrt{n}}\left(\mathsf{S}_{\lfloor ns\rfloor} - \mathsf{S}_{\lfloor nt\rfloor}\right) = \frac{1}{\sigma\sqrt{n}}\sum_{i=\lfloor nt\rfloor+1}^{\lfloor ns\rfloor}\xi_i \\ &= \sqrt{\frac{\lfloor ns\rfloor - \lfloor nt\rfloor}{n}}\frac{1}{\sigma\sqrt{\lfloor ns\rfloor - \lfloor nt\rfloor}}\sum_{i=\lfloor nt\rfloor+1}^{\lfloor ns\rfloor}\xi_i. \end{aligned}$$

   According to CLT for i.i.d. real random variables, we have

$$V_n(s) - V_n(t) \xrightarrow[n\to+\infty]{\mathcal{D}} \sqrt{s-t}\,\widetilde{W}(1) \sim \widetilde{W}(s) - \widetilde{W}(t).$$

3. Fix $K \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_K \leq 1$.
   Then, $V_n(t_k) - V_n(t_{k-1})$, $k \in \{1, 2, \ldots, K\}$ are independent. Hence,

$$\begin{aligned} &\left(V_n(t_k) - V_n(t_{k-1}), k \in \{1, 2, \ldots, K\}\right)^\top \\ &\xrightarrow[n\to+\infty]{\mathcal{D}} \left(\widetilde{W}(t_k) - \widetilde{W}(t_{k-1}), k \in \{1, 2, \ldots, K\}\right)^\top. \end{aligned}$$

   Multiplying by a matrix $M \in \mathbb{R}^{K\times K}$, where $M_{i,j} = 1$ if $i \leq j$ and $M_{i,j} = 0$ if $i > j$, we are receiving

$$\left(V_n(t_k), k \in \{1, 2, \ldots, K\}\right)^\top \xrightarrow[n\to+\infty]{\mathcal{D}} \left(\widetilde{W}(t_k), k \in \{1, 2, \ldots, K\}\right)^\top.$$

   We have shown $V_n \xrightarrow[n\to+\infty]{fidi} \widetilde{W}$ in $\mathbb{R}^{[0,1]}$.

4. Because of the estimate (5.11), we observe $V_n \xrightarrow[n\to+\infty]{fidi} \widetilde{W}$ in $\mathbb{R}^{[0,1]}$ implies $Z_n \xrightarrow[n\to+\infty]{fidi} \widetilde{W}$ in $\mathbb{R}^{[0,1]}$.

Q.E.D.

Theorem 5.27: *Let $\xi_1, \xi_2, \ldots, \xi_N$ are independent real random variables and for all $i = 1, 2, \ldots, N$ we have $\mathsf{E}[\xi_i] = 0$, $\mathsf{var}(\xi_i) = \sigma_i^2 < +\infty$.*

Denoting $s_N^2 = \sum_{i=1}^N \sigma_i^2$ and $s_N = \sqrt{s_N^2}$, following estimate take place for all $\lambda > 0$:

$$\mathsf{P}(\max\{|S_1|, |S_2|, \ldots, |S_N|\} \geq \lambda s_N) \leq 2\mathsf{P}\left(|S_N| \geq (\lambda - \sqrt{2}) s_N\right). \quad (5.13)$$

Proof: Let us denote

$$M_k = \max\{|S_1|, |S_2|, \ldots, |S_k|\} \quad \text{for all } k \in \mathbb{N}.$$

If $\lambda \leq \sqrt{2}$ or $s_N = 0$ the estimate is trivial, since its right-hand side of (5.13) is equal to 2. Thus, it is sufficient to consider the case $\lambda > \sqrt{2}$, $s_N > 0$, only. For all $i = 1, 2, \ldots, N$, we denote

$$E_i = [M_{i-1} < \lambda s_N \leq |S_i|].$$

Evidently,

$$\bigcup_{i=1}^{N} E_i = [M_N \geq \lambda s_N].$$

Therefore, we can write

$$\begin{aligned}
&\mathsf{P}(M_N \geq \lambda s_N) \\
&\leq \mathsf{P}\left(|S_N| \geq (\lambda - \sqrt{2}) s_N\right) + \mathsf{P}\left([M_N \geq \lambda s_N] \cap \left[|S_N| < (\lambda - \sqrt{2}) s_N\right]\right) \\
&\leq \mathsf{P}\left(|S_N| \geq (\lambda - \sqrt{2}) s_N\right) + \sum_{i=1}^{N-1} \mathsf{P}\left(E_i \cap \left[|S_N| < (\lambda - \sqrt{2}) s_N\right]\right).
\end{aligned}$$

Let us estimate the second term

$$\begin{aligned}
&\sum_{i=1}^{\mathsf{N}-1} \mathsf{P}\left(E_i \cap \left[|\mathsf{S}_\mathsf{N}| < (\lambda - \sqrt{2}) s_\mathsf{N}\right]\right) \leq \\
&\leq \sum_{i=1}^{\mathsf{N}-1} \mathsf{P}\left(E_i \cap \left[|\mathsf{S}_i| - |\mathsf{S}_\mathsf{N} - \mathsf{S}_i| < (\lambda - \sqrt{2}) s_\mathsf{N}\right]\right) \leq \\
&\leq \sum_{i=1}^{\mathsf{N}-1} \mathsf{P}\left(E_i \cap \left[|\mathsf{S}_\mathsf{N} - \mathsf{S}_i| > \sqrt{2} s_\mathsf{N}\right]\right) = \\
&= \sum_{i=1}^{\mathsf{N}-1} \mathsf{P}\left(E_i\right) \mathsf{P}\left(\left[|\mathsf{S}_\mathsf{N} - \mathsf{S}_i| > \sqrt{2} s_\mathsf{N}\right]\right) \leq \\
&\leq \sum_{i=1}^{\mathsf{N}-1} \mathsf{P}\left(E_i\right) \frac{1}{2 s_\mathsf{N}^2} \sum_{k=i+1}^{\mathsf{N}} \sigma_i^2 \leq \\
&\leq \frac{1}{2} \sum_{i=1}^{\mathsf{N}-1} \mathsf{P}\left(E_i\right) \leq \frac{1}{2} \mathsf{P}\left(\mathsf{M}_\mathsf{N} \geq \lambda s_\mathsf{N}\right).
\end{aligned}$$

We have derived

$$\mathsf{P}\left(\mathsf{M}_\mathsf{N} \geq \lambda s_\mathsf{N}\right) \leq \mathsf{P}\left(|\mathsf{S}_\mathsf{N}| \geq (\lambda - \sqrt{2}) s_\mathsf{N}\right) + \frac{1}{2} \mathsf{P}\left(\mathsf{M}_\mathsf{N} \geq \lambda s_\mathsf{N}\right).$$

This is precisely (5.13).

Q.E.D.

Theorem 5.28 (Donsker): *Let ξ_i, $i \in \mathbb{N}$ be i.i.d. real random variables with $\mathsf{E}\left[\xi_1\right] = 0$ and $\mathsf{var}\left(\xi_1\right) = \sigma^2 \in \mathbb{R}_+$. Then a Wiener process $W = \left(W(t), t \in [0,1]\right)$ exists and $Z_n \xrightarrow[n \to +\infty]{\mathcal{D}} W$ in $\mathsf{C}\left([0,1]\right)$.*

Proof: In Theorem 5.26 we have shown $Z_n \xrightarrow[n \to +\infty]{fidi} \widetilde{W}$ in $\mathbb{R}^{[0,1]}$.
Take $k \in \mathbb{N}$ and $\lambda > 2\sqrt{2}$. Summands are i.i.d. and, thus, the statement of Theorem 5.27 is in power. Therefore, we have

$$\begin{aligned}
&\mathsf{P}\left(\max\left\{|\mathsf{S}_{k+i} - \mathsf{S}_k| \,:\, i = 1, 2, \ldots, n\right\} \geq \lambda \sigma \sqrt{n}\right) \\
&= \mathsf{P}\left(\max\left\{|\mathsf{S}_i| \,:\, i = 1, 2, \ldots, n\right\} \geq \lambda \sigma \sqrt{n}\right) \leq \\
&\leq 2\mathsf{P}\left(|\mathsf{S}_n| \geq (\lambda - \sqrt{2}) \sigma \sqrt{n}\right) \leq \\
&\leq 2\mathsf{P}\left(|Z_n(1)| \geq (\lambda - \sqrt{2})\right) \leq \\
&\leq 2\mathsf{P}\left(|Z_n(1)| \geq \frac{\lambda}{2}\right).
\end{aligned}$$

According to Theorem 5.26, we have

$$\lim_{n\to+\infty} \mathsf{P}\left(|Z_n(1)| \geq \frac{\lambda}{2}\right) = \mathsf{P}\left(|\widetilde{W}(1)| \geq \frac{\lambda}{2}\right).$$

Using Tchebisheff inequality, we are receiving a rough estimate

$$\mathsf{P}\left(|\widetilde{W}(1)| \geq \frac{\lambda}{2}\right) \leq 8\mathsf{E}\left[|W(1)|^3\right]\lambda^{-3} = \lambda^{-2}\frac{8\mathsf{E}\left[|W(1)|^3\right]}{\lambda}.$$

Let $\varepsilon > 0$ be given.
Take $\lambda > 2\sqrt{2}$ such that $\lambda > \frac{16\mathsf{E}\left[|W(1)|^3\right]}{\varepsilon}$. Then,

$$\begin{aligned}\limsup_{n\to+\infty}\sup_{k\in\mathbb{N}}&\left\{\mathsf{P}\left(\max\left\{|\mathsf{S}_{k+i}-\mathsf{S}_k| \,:\, i=1,2,\ldots,n\right\} \geq \lambda\sigma\sqrt{n}\right)\right\}\\ &\leq \lambda^{-2}\frac{16\mathsf{E}\left[|W(1)|^3\right]}{\lambda} < \lambda^{-2}\varepsilon.\end{aligned}$$

Then, there is a $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$ and all $k \in \mathbb{N}$ we have

$$\mathsf{P}\left(\max\left\{|\mathsf{S}_{k+i}-\mathsf{S}_k| \,:\, i=1,2,\ldots,n\right\} \geq \lambda\sigma\sqrt{n}\right) < \lambda^{-2}\varepsilon.$$

Assumptions of Theorem 5.23 are fulfilled. Therefore, the sequence Z_n, $n \in \mathbb{N}$ is tight in $\mathsf{C}([0,1])$. From Theorem 5.26 we know $Z_n \xrightarrow[n\to+\infty]{fidi} \widetilde{W}$ in $\mathbb{R}^{[0,1]}$. Assumptions of Theorem 5.16 are fulfilled, hence, there is a continuous version of $\widetilde{W}$, i.e. a Wiener process $W = (W(t), t \in [0,1])$ exists, and $Z_n \xrightarrow[n\to+\infty]{\mathcal{D}} W$ in $\mathsf{C}([0,1])$.

Q.E.D.

5.3 Space $\mathsf{C}([a,b])$

Consider compact intervals given by a couple of points $-\infty \leq a < b \leq +\infty$ and $\mathsf{C}([a,b])$ a set of all continuous functions defined on a segment $[a,b]$. Since $[a,b]$ is a compact, each continuous function defined on it is uniformly continuous. Then, we consider the space naturally equipped with the supremal norm $\mathsf{C}([a,b]) = \left(\mathsf{C}([a,b]), \|\cdot\|_{[a,b]}\right)$.

Theorem 5.29: $\mathsf{C}([a,b]) = \left(\mathsf{C}([a,b]), \|\cdot\|_{[a,b]}\right)$ *is a separable Banach space.*

Proof: Take an increasing bijection $\iota : [0,1] \to [a,b]$ and consider a mapping $\zeta : \mathsf{C}([a,b]) \to \mathsf{C}([0,1]) : f \in \mathsf{C}([a,b]) \to f \circ \iota$. Maps ζ, ζ^{-1} are continuous, hence, topological spaces $\mathsf{C}([a,b])$, $\mathsf{C}([0,1])$ are isomorphic. Therefore, $\mathsf{C}([a,b])$ is a separable Banach space, as $\mathsf{C}([0,1])$ is.

Q.E.D.

5.4 Space $\mathsf{C}([0,+\infty))$

Continuous functions defined on $[0,+\infty)$ are typically not uniformly continuous, e.g. $t \mapsto t^2$. Let us introduce a natural topology on $\mathsf{C}([0,+\infty))$.

Definition 5.30 *A natural topology on* $\mathsf{C}([0,+\infty))$ *is introduced by a subbasis*

$$\mathcal{G} = \{\mathcal{U}(x,k,\varepsilon) : x \in \mathsf{C}([0,+\infty)), k \in \mathbb{N}, \varepsilon > 0\}, \tag{5.14}$$
$$\text{where } \mathcal{U}(x,k,\varepsilon) = \left\{y \in \mathsf{C}([0,+\infty)) : \|y-x\|_{[k,k+1]} < \varepsilon\right\}.$$

Consider metrics defined for $x, y \in \mathsf{C}([0,+\infty))$ by

$$\rho_1(x,y) = \sum_{k=0}^{+\infty} 2^{-k} \frac{\|y-x\|_{[k,k+1]}}{1+\|y-x\|_{[k,k+1]}}, \tag{5.15}$$

$$\rho_2(x,y) = \sum_{k=1}^{+\infty} 2^{-k} \frac{\|y-x\|_{[0,k]}}{1+\|y-x\|_{[0,k]}}, \tag{5.16}$$

$$\rho_3(x,y) = \int_1^{+\infty} 2^{-t} \frac{\|y-x\|_{[0,t]}}{1+\|y-x\|_{[0,t]}}\,\mathrm{d}t. \tag{5.17}$$

Theorem 5.31: *Topological space* $\mathsf{C}([0,+\infty))$ *is Polish.*

Proof: For example each of the above defined metrics ρ_1, ρ_2, ρ_3 is making $\mathsf{C}([0,+\infty))$ to be a complete separable metric space.

Q.E.D.

Chapter 6

Skorokhod space of discontinuous functions

6.1 Càdlàg functions

This chapter is devoted to a particular space of discontunious functions.

Definition 6.1 *A set of all real functions defined on the interval* $[0, 1]$*, which are continuous from right at each point of* $[0, 1)$ *and with a limit from left at each point of* $(0, 1]$*, are called càdlàg functions on* $[0, 1]$ *and will be denoted by* $\mathsf{cadlag}([0, 1])$*.*

These functions are traditionally called càdlàg, due to Bourbaki as an abbreviation from French. Having introduced a particular topology on them, incurred topological space is called Skorokhod space, see section 6.3.

6.1.1 Properties of càdlàg functions

At first, consider basic properties of càdlàg functions.

Lemma 6.2 *We have inclusion* $\mathsf{cadlag}([0, 1]) \subset l^{+\infty}([0, 1])$*.*

Proof: Take $f \in \mathsf{cadlag}([0, 1])$ and assume it is unbounded.
Take a sequence $t_n \in [0, 1]$, $n \in \mathbb{N}$ such that

$$\lim_{n \to +\infty} |f(t_n)| = +\infty.$$

Interval $[0, 1]$ is a compact, then, there is a convergent subsequence

$$\lim_{k \to +\infty} t_{n_k} = \xi \in [0, 1].$$

Denote

$$\begin{aligned} E &= \{k \in \mathbb{N} : t_{n_k} = \xi\}, \\ L &= \{k \in \mathbb{N} : t_{n_k} > \xi\}, \\ S &= \{k \in \mathbb{N} : t_{n_k} < \xi\}. \end{aligned}$$

Set E is either empty or finite because $\lim_{k\to+\infty} |f(t_{n_k})| = +\infty$. Hence, at least one of sets L, S must be countable.

1. Let L be countable. Then,

$$\lim_{k\in L} t_{n_k} = \xi, \ \lim_{k\in L} |f(t_{n_k})| = +\infty.$$

Function f is right continuous at ξ, hence

$$\lim_{k\in L} f(t_{n_k}) = f(\xi) \in \mathbb{R}.$$

It is a contradiction.

2. Let S be countable. Then,

$$\lim_{k\in S} t_{n_k} = \xi, \ \lim_{k\in S} |f(t_{n_k})| = +\infty.$$

Function f possesses a left limit at ξ, hence

$$\lim_{k\in S} f(t_{n_k}) = f(\xi-) \in \mathbb{R}.$$

It is a contradiction.

Our assumption on unboundness led to a contradiction. Therefore, $\|f\|_{[0,1]} \in \mathbb{R}$ for each càdlàg function.

Q.E.D.

Let us investigate jumps of càdlàg functions. For $f \in \mathsf{cadlag}([0,1])$, let us consider sets of its jumps

$$\begin{aligned} \mathcal{D}(f,\varepsilon) &= \{t \in (0,1] : |f(t) - f(t-)| > \varepsilon\} \quad \text{for} \quad \varepsilon > 0, \\ \mathcal{D}(f) &= \{t \in (0,1] : f(t) \neq f(t-)\}. \end{aligned}$$

Lemma 6.3 *If $f \in \mathsf{cadlag}([0,1])$ and $\varepsilon > 0$ then $\mathcal{D}(f,\varepsilon)$ is finite and $\mathcal{D}(f)$ is at most countable.*

Proof: Assume $\mathcal{D}(f,\varepsilon)$ is infinite.
Take a sequence $t_n \in \mathcal{D}(f,\varepsilon)$, $n \in \mathbb{N}$ such that no point is repeated.
Interval $[0,1]$ is a compact, then, there is a convergent subsequence

$$\lim_{k\to+\infty} t_{n_k} = \xi \in [0,1].$$

Denote

$$\begin{aligned} E &= \{k \in \mathbb{N} : t_{n_k} = \xi\}, \\ L &= \{k \in \mathbb{N} : t_{n_k} > \xi\}, \\ S &= \{k \in \mathbb{N} : t_{n_k} < \xi\}. \end{aligned}$$

At least one of the sets L, S must be countable, since E can contain at most one point.

1. Let L be countable.

   Then, for each $k \in L$ there is a point $u_k \in [0,1]$ such that

   $$\xi < u_k < t_{n_k}, \ |f(t_{n_k}) - f(u_k)| > \varepsilon.$$

   Function f is right continuous at ξ, hence

   $$\lim_{k\in L} f(t_{n_k}) = \lim_{k\in L} f(u_k) = f(\xi) \in \mathbb{R}.$$

   It is a contradiction.

2. Let S be countable.

   Then, for each $k \in L$ there is a point $v_k \in [0,1]$ such that

   $$t_{n_k} - 2^{-k} < v_k < t_{n_k} < \xi, \ |f(t_{n_k}) - f(v_k)| > \varepsilon.$$

   Function f possesses a finite limit from left, hence

   $$\lim_{k\in S} f(t_{n_k}) = \lim_{k\in S} f(v_k) = f(\xi-) \in \mathbb{R}.$$

   It is a contradiction.

We reached a contradiction and, therefore, $\mathcal{D}(f,\varepsilon)$ is finite. Consequently, $\mathcal{D}(f)$ is at most countable, since

$$\mathcal{D}(f) = \bigcup_{n=1}^{+\infty} \mathcal{D}\left(f, 2^{-n}\right).$$

Q.E.D.

Example 6.4: Consider function $f : [0,1] \to \mathbb{R}$ defined by

$$\begin{aligned} f(t) &= (D+1)\left(t-1+\frac{1}{D}\right) \quad \text{for} \quad 1-\frac{1}{D} \le t < 1-\frac{1}{D+1}, \; D \in \mathbb{N}, \\ &= 0 \quad \text{for} \quad t=1. \end{aligned}$$

Function f belongs to $\mathsf{cadlag}([0,1])$.
Their jumps are $f\left(1-\frac{1}{D+1}\right) - f\left(1-\frac{1}{D+1}-\right) = -\frac{1}{D}$ for all $D \in \mathbb{N}$.

All jumps are down and their sum is infinite, therefore, the function f cannot be expressed as a sum of a continuous function and a step function.

△

6.1.2 Characterization of càdlàg functions

Denote $\Delta(\delta)$ a set of all partitions $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $t_i - t_{i-1} > \delta$ for all $i = 1, 2, \ldots, k$. For a partition $\mathcal{D} \in \Delta(\delta)$ we denote by $t_i(\mathcal{D})$ its points, $K(\mathcal{D})$ number of its points and $\mathsf{I}_i(\mathcal{D}) = [t_{i-1}(\mathcal{D}), t_i(\mathcal{D}))$ intervals determined by the partition.

For $x \in \mathbb{R}^{[0,1]}$, we define moduli

$$\begin{aligned} \widetilde{\mathsf{w}}(x,A) &= \sup\left\{|x(t)-x(s)| \,:\, t,s \in A\right\} \text{ for } A \subset [0,1], && (6.1)\\ \mathsf{w}'(x,\delta) &= \inf\left\{\max\left\{\widetilde{\mathsf{w}}(x,\mathsf{I}_i(\mathcal{D})) \,:\, i=1,2,\ldots,K(\mathcal{D})\right\} \,:\right. && (6.2)\\ &\quad \left. :\, \mathcal{D} \in \Delta(\delta)\right\}, \\ \mathsf{w}''(x,\delta) &= \sup\left\{\min\left\{|x(t)-x(u)|, |x(s)-x(u)|\right\} \,:\right. && (6.3)\\ &\quad \left. :\, 0 \le t < u < s \le 1, \; s-t \le \delta\right\}, \\ \widetilde{\mathsf{w}''}(x,A) &= \sup\left\{\min\left\{|x(t)-x(u)|, |x(s)-x(u)|\right\} \,:\right. && (6.4)\\ &\quad \left. :\, t<u<s, \; t,u,s \in A\right\} \text{ for } A \subset [0,1]. \end{aligned}$$

Introduced moduli are related.

Lemma 6.5 *For $x \in \mathbb{R}^{[0,1]}$ and $\delta > 0$ we have $\mathsf{w}''(x,\delta) \le \mathsf{w}'(x,\delta)$.*

Proof: For $\varepsilon > 0$ we find a partition $\mathcal{D} \in \Delta(\delta)$ such that for all $i = 1, 2, \ldots K(\mathcal{D})$ we have $\widetilde{\mathsf{w}}(x, \mathsf{I}_i(\mathcal{D})) < \mathsf{w}'(x,\delta) + \varepsilon$.
Take $0 \le t < s \le 1, \; s-t \le \delta$. Then, there are only two possibilities:

1. There is i such that $t_{i-1}(\mathcal{D}) \le t < s \le t_i(\mathcal{D})$.

   Then, for all $t < u < s$ we have

$$|x(t) - x(u)| < \mathsf{w}'(x, \delta) + \varepsilon.$$

2. There is i such that $t_{i-1}(\mathcal{D}) < t < t_i(\mathcal{D}) < s < t_{i+1}(\mathcal{D})$.

   Then, for all $t < u < t_i(\mathcal{D})$, we have

$$|x(t) - x(u)| < \mathsf{w}'(x, \delta) + \varepsilon,$$

   and, for all $t_i(\mathcal{D}) \le u < s$ we have

$$|x(s) - x(u)| < \mathsf{w}'(x, \delta) + \varepsilon.$$

We have checked $\mathsf{w}''(x, \delta) \le \mathsf{w}'(x, \delta)$.

Q.E.D.

Introduced moduli are able to characterize functions from $\mathsf{cadlag}([0,1])$.

Theorem 6.6: *Let $f \in \mathbb{R}^{[0,1]}$. Then,*

$$f \in \mathsf{cadlag}([0,1]) \quad \Longleftrightarrow \quad \lim_{\delta \to 0+} \mathsf{w}'(f, \delta) = 0. \tag{6.5}$$

Proof:

1. Let $f \in \mathsf{cadlag}([0,1])$ and $\varepsilon > 0$.

   For each $t \in [0,1]$ there exists $\delta_t > 0$ such that

$$\widetilde{\mathsf{w}}(f, (t - \delta_t, t) \cap [0,1]) \le \varepsilon, \ \ \widetilde{\mathsf{w}}(f, [t, t + \delta_t) \cap [0,1]) \le \varepsilon.$$

   Then,

$$[0,1] \subset \bigcup_{t \in [0,1]} (t - \delta_t, t + \delta_t).$$

   Interval $[0,1]$ is a compact and, therefore, there exists $I \in \mathsf{Fin}([0,1])$ such that

$$[0,1] \subset \bigcup_{t \in I} (t - \delta_t, t + \delta_t).$$

Take a partition $\mathcal{D} \subset I \cup \{0,1\}$ with the smallest number of points such that

$$[0,1] \subset \bigcup_{t\in\mathcal{D}} (t-\delta_t, t+\delta_t).$$

Now for an interval of the partition we have

$$\begin{aligned}[t_{i-1}(\mathcal{D}), t_i(\mathcal{D})] \quad \subset \quad & (t_{i-1}(\mathcal{D}) - \delta_{t_{i-1}(\mathcal{D})}, t_{i-1}(\mathcal{D}) + \delta_{t_{i-1}(\mathcal{D})}) \cup \\ & \cup (t_i(\mathcal{D}) - \delta_{t_i(\mathcal{D})}, t_i(\mathcal{D}) + \delta_{t_i(\mathcal{D})}).\end{aligned}$$

Take $\delta > 0$ such that $\mathcal{D} \in \Delta(\delta)$. Hence,

$$\begin{aligned}&\mathsf{w}'(f,\delta) \le \max\{\widetilde{\mathsf{w}}(f,[t_{i-1}(\mathcal{D}), t_i(\mathcal{D}))) : i = 1,2,\dots,K(\mathcal{D})\} \\ &\le \max\left\{\widetilde{\mathsf{w}}\left(f,[t_{i-1}(\mathcal{D}), t_{i-1}(\mathcal{D}) + \delta_{t_{i-1}(\mathcal{D})})\right) : i = 1,2,\dots,K(\mathcal{D})\right\} \\ &+ \max\left\{\widetilde{\mathsf{w}}\left(f,(t_i(\mathcal{D}) - \delta_{t_i(\mathcal{D})}, t_i(\mathcal{D}))\right) : i = 1,2,\dots,K(\mathcal{D})\right\} \\ &\le 2\varepsilon.\end{aligned}$$

It is because for $u,v \in [t_{i-1}(\mathcal{D}), t_i(\mathcal{D}))$, $u < v$ just one case from the following three is possible

- $u,v \in [t_{i-1}(\mathcal{D}), t_{i-1}(\mathcal{D}) + \delta_{t_{i-1}(\mathcal{D})})$;
- $u,v \in (t_i(\mathcal{D}) - \delta_{t_i(\mathcal{D})}, t_i(\mathcal{D}))$;
- $u \in [t_{i-1}(\mathcal{D}), t_{i-1}(\mathcal{D}) + \delta_{t_{i-1}(\mathcal{D})})$, $v \in (t_i(\mathcal{D}) - \delta_{t_i(\mathcal{D})}, t_i(\mathcal{D}))$.
  Then, for $s \in (t_i(\mathcal{D}) - \delta_{t_i(\mathcal{D})}, t_{i-1}(\mathcal{D}) + \delta_{t_{i-1}(\mathcal{D})})$, we have an estimate

  $$|f(u) - f(v)| \le |f(u) - f(s)| + |f(s) - f(v)|,$$

  since $u,s \in [t_{i-1}(\mathcal{D}), t_{i-1}(\mathcal{D}) + \delta_{t_{i-1}(\mathcal{D})})$, $v,s \in (t_i(\mathcal{D}) - \delta_{t_i(\mathcal{D})}, t_i(\mathcal{D}))$.

Thus, we have shown

$$\lim_{\delta\to 0+} \mathsf{w}'(f,\delta) = 0.$$

2. Let $\lim_{\delta\to 0+} \mathsf{w}'(f,\delta) = 0$ and $\varepsilon > 0$.

   Then, there exists a partition $\mathcal{D}$ such that

   $$\max\{\widetilde{\mathsf{w}}(f,[t_{i-1}(\mathcal{D}), t_i(\mathcal{D}))) : i = 1,2,\dots,K(\mathcal{D})\} \le \varepsilon.$$

   (a) Take $t \in [0,1)$.
   Then, there is a point of the partition such that $t_{i-1}(\mathcal{D}) \le t < t_i(\mathcal{D})$.
   Hence, for all $t < s < t_i(\mathcal{D})$ we have $|f(t) - f(s)| \le \varepsilon$.
   It means f is continuous from right at t.

(b) Take $t \in (0,1]$.
Then, there is a point of the partition such that $t_{i-1}(\mathcal{D}) < t \leq t_i(\mathcal{D})$.
Hence, for all $t_{i-1}(\mathcal{D}) < s < u < t$ we have $|f(s) - f(u)| \leq \varepsilon$.
It means f possesses limit from left at t.

Finally, $f \in \mathsf{cadlag}([0,1])$.

Q.E.D.

Theorem 6.7: *Let $f \in \mathbb{R}^{[0,1]}$. Then,*

$$\lim_{\delta \to 0+} \mathsf{w}''(f, \delta) = 0$$

$$\Updownarrow$$

f possesses a limit from right at each point of $[0,1)$,
possesses a limit from left at each point of $(0,1]$,
is continuous from one side at each point of $(0,1)$.

Proof:

1. Let f possesses a limit from right at each point of $[0,1)$, a limit from left at each point of $(0,1]$, is continuous from one side at each point of $(0,1)$ and let $\varepsilon > 0$.

   For each $t \in [0,1]$ there exists $\delta_t > 0$ such that either

$$\widetilde{\mathsf{w}}\left(f, (t-\delta_t, t) \cap [0,1]\right) \leq \varepsilon, \ \widetilde{\mathsf{w}}\left(f, [t, t+\delta_t) \cap [0,1]\right) \leq \varepsilon$$

   or

$$\widetilde{\mathsf{w}}\left(f, (t-\delta_t, t] \cap [0,1]\right) \leq \varepsilon, \ \widetilde{\mathsf{w}}\left(f, (t, t+\delta_t) \cap [0,1]\right) \leq \varepsilon.$$

   Then,

$$[0,1] \subset \bigcup_{t \in [0,1]} (t-\delta_t, t+\delta_t).$$

   Interval $[0,1]$ is a compact and, therefore, there exists $I \in \mathsf{Fin}\left([0,1]\right)$ such that

$$[0,1] \subset \bigcup_{t \in I} (t-\delta_t, t+\delta_t).$$

Set $\delta = \min\{\delta_t : t \in I\}$.

Then, for $0 \le s < u < t \le 1$, $t - s < \delta$, we have either $|f(t) - f(u)| \le \varepsilon$ or $|f(u) - f(s)| \le \varepsilon$.

Finally, $\mathsf{w}''(f, \delta) \le \varepsilon$.

Thus, we have shown

$$\lim_{\delta \to 0+} \mathsf{w}''(f, \delta) = 0.$$

2. Let $\lim_{\delta \to 0+} \mathsf{w}''(f, \delta) = 0$.

   (a) Let f possess no limit from right at $t \in [0, 1)$.
   Then, there exists $\varepsilon > 0$ such that
   $\limsup_{s \to t} f(s) - \liminf_{s \to t} f(s) > \varepsilon$, and there are points
   $1 \ge h_1 > d_1 > h_2 > d_2 > \cdots > t$ such that

   $$\lim_{n \to +\infty} h_n = \lim_{n \to +\infty} d_n = t,$$
   $$\lim_{n \to +\infty} f(h_n) = \limsup_{s \to t} f(s), \quad \lim_{n \to +\infty} f(d_n) = \liminf_{s \to t} f(s).$$

   Then, for n sufficiently large, we have

   $$f(h_n) - f(d_n) > \varepsilon, \ f(h_{n+1}) - f(d_n) > \varepsilon.$$

   That is a contradiction, since modulus should vanish.
   Finally, f possesses a limit from right at t.

   (b) Similar arguments are giving f possesses a limit from left at each $t \in (0, 1]$.

   (c) Let f be discontinuous from both sides at $t \in (0, 1)$.
   Then, there exists $\varepsilon > 0$ such that
   $|f(t-) - f(t)| > \varepsilon$, $|f(t+) - f(t)| > \varepsilon$,
   and there are points $h_n > t > d_n$, $n \in \mathbb{N}$ such that

   $$\lim_{n \to +\infty} h_n = \lim_{n \to +\infty} d_n = t,$$
   $$\lim_{n \to +\infty} f(h_n) = f(t+), \quad \lim_{n \to +\infty} f(d_n) = f(t-).$$

   Then, for n sufficiently large, we have

   $$f(h_n) - f(t) > \varepsilon, \ f(t) - f(d_n) > \varepsilon,$$

   That is a contradiction, since modulus should vanish.
   Finally, f must be continuous either from right or from left at t.

All required properties of f are checked.

Q.E.D.

Consequence: Let $f \in \mathbb{R}^{[0,1]}$ be continuous from right at each point of $[0,1)$. Then,

$$f \in \mathsf{cadlag}([0,1]) \iff \lim_{\delta \to 0+} \mathsf{w}''(f,\delta) = 0. \tag{6.6}$$

♣

Proof: The statement is a consequence of Theorem 6.7.

Q.E.D.

6.2 Càdlàg functions with supremal norm

Theorem 6.8: $(\mathsf{cadlag}([0,1]), \|\cdot\|_{[0,1]})$ *is a non-separable Banach space.*

Proof: Normed space $(\mathsf{cadlag}([0,1]), \|\cdot\|_{[0,1]})$ is a subspace of Banach space $(l^{+\infty}([0,1]), \|\cdot\|_{[0,1]})$. We have to show its completeness and discuss separability.

1. Let $f_n \in \mathsf{cadlag}([0,1])$, $n \in \mathbb{N}$ be a Cauchy sequence at $\|\cdot\|_{[0,1]}$.

   The sequence is also Cauchy in $l^{+\infty}([0,1])$ and, therefore accordingly to Theorem 4.2, there exists $g \in l^{+\infty}([0,1])$ such that $\lim_{n \to +\infty} \|f_n - g\|_{[0,1]} = 0$.

   For $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\|f_n - g\|_{[0,1]} < \varepsilon$.

   Now there exists $\delta > 0$ and a partition $\mathcal{D} \in \Delta(\delta)$ such that for all $i = 1, 2, \ldots, K(\mathcal{D})$ and each $t, s \in \mathsf{I}_i(\mathcal{D})$ we have $|f_n(t) - f_n(s)| < \varepsilon$.

   Then, for all $i = 1, 2, \ldots, K(\mathcal{D})$ and each $t, s \in \mathsf{I}_i(\mathcal{D})$ we have

   $$|g(t) - g(s)| \leq |f_n(t) - g(t)| + |f_n(s) - g(s)| + |f_n(t) - f_n(s)| < 3\varepsilon.$$

   Thus, $g \in \mathsf{cadlag}([0,1])$ and $(\mathsf{cadlag}([0,1]), \|\cdot\|_{[0,1]})$ is complete, so it is a Banach space.

2. For all $s \in [0,1]$, we introduce a function

$$\begin{aligned} \delta_s(t) &= 0 \quad \text{if} \quad 0 \leq t < s, \\ &= 1 \quad \text{if} \quad s \leq t \leq 1. \end{aligned}$$

For $s, u \in [0,1]$, $s \neq u$ is $\|\delta_s - \delta_u\|_{[0,1]} = 1$.

A collection of functions δ_s, $s \in [0,1]$ is uncountable. Therefore, the space $(\mathsf{cadlag}([0,1]), \|\cdot\|_{[0,1]})$ is non-separable.

Q.E.D.

A random jump can be non-measurable in $(\mathsf{cadlag}([0,1]), \|\cdot\|_{[0,1]})$.

Example 6.9: Consider ξ a random variable uniformly distributed on $(0,1)$ and define

$$\begin{aligned} S(t) &= 1 \text{ if } \xi \leq t \leq 1, \\ &= 0 \text{ if } 0 \leq t < \xi. \end{aligned}$$

Let us denote δ_s a jump at $s \in [0,1]$; i.e.

$$\begin{aligned} \delta_s(t) &= 1 \text{ if } s \leq t \leq 1, \\ &= 0 \text{ if } 0 \leq t < s. \end{aligned}$$

Evidently, jumps are càdlàg functions.

Take $U \subset [0,1]$. Then,

$$\left[S \in \bigcup_{u \in U} \mathcal{U}\left(\delta_u; \frac{1}{3} \middle| (\mathsf{cadlag}([0,1]), \|\cdot\|_{[0,1]}) \right) \right] = \bigcup_{u \in U} [\xi = u] = [\xi \in U].$$

Moreover, always

$$\bigcup_{u \in U} \mathcal{U}\left(\delta_u; \frac{1}{3} \middle| (\mathsf{cadlag}([0,1]), \|\cdot\|_{[0,1]}) \right) \in \mathcal{G}\left((\mathsf{cadlag}([0,1]), \|\cdot\|_{[0,1]}) \right).$$

But,

$$\mathsf{P}_*\left(\xi \in U\right) = \lambda_*\left(U\right) < \lambda^*\left(U\right) = \mathsf{P}^*\left(\xi \in U\right)$$

for any U Lebesgue non-measurable set. Therefore, $[\xi \in U] \notin \mathcal{A}$.

△

Weak convergence of random processes with values in bounded real functions is well defined and can be helpful in some cases. Unfortunately, there is a limit. Empirical processes are typically non-measurable in topology of bounded real functions, see Example 6.9. Therefore, we would have to relax the assumption on measurablity of treated processes. Appropriate theory is introduced and explained in [9]. We will not follow this stream in the lecture. We will overcome the problem with measurablity establishing a finer topology on càdlàg functions.

6.3 Skorokhod space

In this section, we introduce step by step a topology on càdlàg functions defined by Skorokhod in [7].

6.3.1 Time transformations

Denote by Λ a set of all nondecreasing bijections from $[0,1]$ to $[0,1]$. Consequently, $\lambda \in \Lambda$ must fulfill $\lambda(0) = 0$, $\lambda(1) = 1$ and λ is an increasing continuous function. The set Λ contains two important subsets:

$$\Lambda_L = \left\{ \lambda \in \Lambda \,:\, \lambda, \lambda^{-1} \in \mathsf{C}^{0,1} \right\}, \tag{6.7}$$

$$\Lambda_D = \left\{ \lambda \in \Lambda \,:\, \lambda, \lambda^{-1} \in \mathsf{C}^{1,0} \right\}. \tag{6.8}$$

We need to measure deformation of $[0,1]$ made by a particular bijection. Two measures will be employed for that:

$$\|\lambda\|_\Lambda = \|\lambda - \mathrm{Id}\|_{[0,1]} = \sup\left\{ |\lambda(t) - t| \,:\, t \in [0,1] \right\}, \tag{6.9}$$

$$\langle\!\langle \lambda \rangle\!\rangle_\Lambda = \sup\left\{ \left| \log\left(\frac{\lambda(s) - \lambda(t)}{s - t} \right) \right| \,:\, 0 \le t < s \le 1 \right\}. \tag{6.10}$$

These measures possess some properties required for a norm. Unfortunately, they cannot be norms, because, Λ is not a vector space. We will call them half-norms.

At first, we investigate basic properties of above defined half-norms.

Lemma 6.10 *For $\lambda \in \Lambda$ we always have $0 \le \|\lambda\|_\Lambda \le 1$.*

Proof: The statement is evident, since for all $0 \le t \le 1$ we have

$$-t \le \lambda(t) - t \le \lambda(t).$$

Q.E.D.

The second half-norm can attain infinite values.

For example, the half-norm for a transformation $\lambda(t) = t^2$ is $\langle\!\langle \lambda \rangle\!\rangle_\Lambda = |\log(\lambda'(0))| = +\infty$.

Lemma 6.11 *Let $\lambda \in \Lambda$. Then*

$$\langle\!\langle \lambda \rangle\!\rangle_\Lambda = \max\left\{\log(\mathsf{Lip}\,(\lambda)), \log(\mathsf{Lip}\,(\lambda^{-1}))\right\}$$

and hence

$$\langle\!\langle \lambda \rangle\!\rangle_\Lambda < +\infty \iff \lambda \in \Lambda_L.$$

Proof: We know, if $\lambda \in \Lambda$ then also $\lambda^{-1} \in \Lambda$. Remember a definition of Lipschitz constant for λ and λ^{-1}. We have an estimate:

$$\begin{aligned}
\mathsf{Lip}\,(\lambda) &= \sup\left\{\frac{\lambda(s)-\lambda(t)}{s-t} : 0 \le t < s \le 1\right\} \\
&\ge \inf\left\{\frac{\lambda(s)-\lambda(t)}{s-t} : 0 \le t < s \le 1\right\} \\
&= \inf\left\{\frac{\lambda(s)-\lambda(t)}{\lambda^{-1}\circ\lambda(s)-\lambda^{-1}\circ\lambda(t)} : 0 \le t < s \le 1\right\} \\
&= \left(\mathsf{Lip}\,(\lambda^{-1})\right)^{-1}.
\end{aligned}$$

Always $\lambda(0) = 0$, $\lambda(1) = 1$, therefore we have $\mathsf{Lip}\,(\lambda) \ge 1$, $\mathsf{Lip}\,(\lambda^{-1}) \ge 1$. These observations are giving required relation

$$\langle\!\langle \lambda \rangle\!\rangle_\Lambda = \max\left\{\log(\mathsf{Lip}\,(\lambda)),\ \log(\mathsf{Lip}\,(\lambda^{-1}))\right\}.$$

Q.E.D.

Introduced half-norms possess nice and useful properties.

Lemma 6.12 *We have:*

i) For $\lambda \in \Lambda$, it is $\|\lambda\|_\Lambda = 0$ if and only if $\lambda = \mathrm{Id}$.

ii) For $\lambda \in \Lambda$, we have $\|\lambda^{-1}\|_\Lambda = \|\lambda\|_\Lambda$.

iii) For $\lambda, \varphi \in \Lambda$, we have $\|\lambda \circ \varphi\|_\Lambda \le \|\lambda\|_\Lambda + \|\varphi\|_\Lambda$.

Proof:

1. Evidently, $\|\lambda\|_\Lambda = 0$ if and only if $\lambda = \mathrm{Id}$.

2. For $\lambda \in \Lambda$,

$$\left\|\lambda^{-1}\right\|_{\Lambda} = \left\|\lambda^{-1} - \mathrm{Id}\right\|_{[0,1]} = \left\|\lambda^{-1} \circ \lambda - \mathrm{Id} \circ \lambda\right\|_{[0,1]} = \left\|\mathrm{Id} - \lambda\right\|_{[0,1]} = \left\|\lambda\right\|_{\Lambda}.$$

3. For $\lambda, \varphi \in \Lambda$,

$$\begin{aligned} \left\|\lambda \circ \varphi\right\|_{\Lambda} &= \left\|\lambda \circ \varphi - \mathrm{Id}\right\|_{[0,1]} = \left\|(\lambda \circ \varphi - \varphi) + (\varphi - \mathrm{Id})\right\|_{[0,1]} \\ &\leq \left\|\lambda \circ \varphi - \varphi\right\|_{[0,1]} + \left\|\varphi - \mathrm{Id}\right\|_{[0,1]} = \left\|\lambda - \mathrm{Id}\right\|_{[0,1]} + \left\|\varphi - \mathrm{Id}\right\|_{[0,1]} \\ &= \left\|\lambda\right\|_{\Lambda} + \left\|\varphi\right\|_{\Lambda}. \end{aligned}$$

Q.E.D.

Lemma 6.13 *For $\lambda \in \Lambda$ we have an estimate*

$$\left\|\lambda\right\|_{\Lambda} \leq \exp\left\{\langle\!\langle \lambda \rangle\!\rangle_{\Lambda}\right\} - 1. \tag{6.11}$$

Proof:

$$\begin{aligned} \left\|\lambda\right\|_{\Lambda} &= \sup\left\{\left|\lambda(t) - t\right| \,:\, t \in [0,1]\right\}. = \sup\left\{\left|\lambda(t) - t\right| \,:\, t \in (0,1]\right\}. \\ &= \sup\left\{t\left|\frac{\lambda(t)}{t} - 1\right| \,:\, t \in (0,1]\right\} \\ &\leq \sup\left\{\left|\exp\left\{\log\left(\frac{\lambda(t)}{t}\right)\right\} - 1\right| \,:\, t \in (0,1]\right\} \\ &\leq \sup\left\{\exp\left\{\left|\log\left(\frac{\lambda(t)}{t}\right)\right|\right\} - 1 \,:\, t \in (0,1]\right\} \\ &\leq \exp\left\{\langle\!\langle \lambda \rangle\!\rangle_{\Lambda}\right\} - 1. \end{aligned}$$

Q.E.D.

Lemma 6.14 *We have:*

i) For $\lambda \in \Lambda$ is $\langle\!\langle \lambda \rangle\!\rangle_{\Lambda} = 0$ if and only if $\lambda = \mathrm{Id}$.

ii) For $\lambda \in \Lambda_L$ is $\langle\!\langle \lambda^{-1} \rangle\!\rangle_{\Lambda} = \langle\!\langle \lambda \rangle\!\rangle_{\Lambda}$.

iii) For $\lambda, \varphi \in \Lambda_L$ is $\langle\!\langle \lambda \circ \varphi \rangle\!\rangle_{\Lambda} \leq \langle\!\langle \lambda \rangle\!\rangle_{\Lambda} + \langle\!\langle \varphi \rangle\!\rangle_{\Lambda}$.

Proof:

1. Evidently, $\langle\!\langle \mathrm{Id} \rangle\!\rangle_\Lambda = 0$

   According to Lemma 6.13, $\langle\!\langle \lambda \rangle\!\rangle_\Lambda = 0$ implies $\|\lambda\|_\Lambda = 0$. Thus, $\lambda = \mathrm{Id}$.

2. According to Lemma 6.11, $\langle\!\langle \lambda^{-1} \rangle\!\rangle_\Lambda = \langle\!\langle \lambda \rangle\!\rangle_\Lambda$.

3. For $\lambda, \varphi \in \Lambda$,

$$
\begin{aligned}
\|\lambda \circ \varphi\|_\Lambda &= \sup\left\{\left|\log\left(\frac{\lambda\circ\varphi(s) - \lambda\circ\varphi(t)}{s-t}\right)\right| \,:\, 0 \le t < s \le 1\right\} \\
&= \sup\left\{\left|\log\left(\frac{\lambda\circ\varphi(s) - \lambda\circ\varphi(t)}{\varphi(s)-\varphi(t)} \cdot \frac{\varphi(s)-\varphi(t)}{s-t}\right)\right| \,:\, 0 \le t < s \le 1\right\} \\
&\le \sup\left\{\left|\log\left(\frac{\lambda\circ\varphi(s) - \lambda\circ\varphi(t)}{\varphi(s)-\varphi(t)}\right)\right| \,:\, 0 \le t < s \le 1\right\} \\
&\quad + \sup\left\{\left|\log\left(\frac{\varphi(s)-\varphi(t)}{s-t}\right)\right| \,:\, 0 \le t < s \le 1\right\} \\
&= \langle\!\langle \lambda \rangle\!\rangle_\Lambda + \langle\!\langle \varphi \rangle\!\rangle_\Lambda .
\end{aligned}
$$

Q.E.D.

6.3.2 Metrics

For a couple of real functions $x, y \in \mathbb{R}^{[0,1]}$, we will consider two distances:

$$
\begin{aligned}
\mathsf{d}(x,y) &= \inf\left\{\max\left\{\|x\circ\lambda - y\|_{[0,1]}, \|\lambda\|_\Lambda\right\} \,:\, \lambda\in\Lambda\right\}, \qquad (6.12)\\
\mathsf{d}_0(x,y) &= \inf\left\{\max\left\{\|x\circ\lambda - y\|_{[0,1]}, \langle\!\langle \lambda \rangle\!\rangle_\Lambda\right\} \,:\, \lambda\in\Lambda\right\}. \qquad (6.13)
\end{aligned}
$$

For càdlàg functions these distances can be expressed equivalently.

Lemma 6.15 *For $x, y \in \mathsf{cadlag}([0,1])$, we have:*

$$
\begin{aligned}
\mathsf{d}(x,y) &= \inf\left\{\max\left\{\|x\circ\lambda - y\|_{[0,1]}, \|\lambda\|_\Lambda\right\} \,:\, \lambda\in\Lambda\right\} \\
&= \inf\left\{\max\left\{\|x\circ\lambda - y\|_{[0,1]}, \|\lambda\|_\Lambda\right\} \,:\, \lambda\in\Lambda_L\right\} \\
&= \inf\left\{\max\left\{\|x\circ\lambda - y\|_{[0,1]}, \|\lambda\|_\Lambda\right\} \,:\, \lambda\in\Lambda_D\right\}, \\
\mathsf{d}_0(x,y) &= \inf\left\{\max\left\{\|x\circ\lambda - y\|_{[0,1]}, \langle\!\langle \lambda \rangle\!\rangle_\Lambda\right\} \,:\, \lambda\in\Lambda\right\} \\
&= \inf\left\{\max\left\{\|x\circ\lambda - y\|_{[0,1]}, \langle\!\langle \lambda \rangle\!\rangle_\Lambda\right\} \,:\, \lambda\in\Lambda_L\right\} \\
&= \inf\left\{\max\left\{\|x\circ\lambda - y\|_{[0,1]}, \langle\!\langle \lambda \rangle\!\rangle_\Lambda\right\} \,:\, \lambda\in\Lambda_D\right\}.
\end{aligned}
$$

Proof: These expressions are true, since $\Lambda_D \subset \Lambda_L \subset \Lambda$ and Λ_D is dense in Λ in supremal norm.

Q.E.D.

Distances d, d_0 are metrics on $\mathsf{cadlag}([0,1])$.

Theorem 6.16: *Distance* d *is a metric on* $\mathsf{cadlag}([0,1])$.

Proof: We have to verify properties of a metric.

1. Finiteness

   Let $x, y \in \mathsf{cadlag}([0,1])$. Then,

$$\begin{aligned}\mathsf{d}(x,y) &= \inf\left\{\max\left\{\|x\circ\lambda - y\|_{[0,1]}, \|\lambda\|_\Lambda\right\} : \lambda\in\Lambda\right\}\\ &\le \max\left\{\|x\circ \mathrm{Id} - y\|_{[0,1]}, \|\mathrm{Id}\|_\Lambda\right\}\\ &= \max\left\{\|x-y\|_{[0,1]}, 0\right\}\\ &\le \|x\|_{[0,1]} + \|y\|_{[0,1]} < +\infty.\end{aligned}$$

   Finally, $\mathsf{d} : \mathsf{cadlag}([0,1]) \times \mathsf{cadlag}([0,1]) \to \mathbb{R}_{+,0}$.

2. Symmetry

   Let $x, y \in \mathsf{cadlag}([0,1])$. Then,

$$\begin{aligned}\mathsf{d}(y,x) &= \inf\left\{\max\left\{\|y\circ\lambda - x\|_{[0,1]}, \|\lambda\|_\Lambda\right\} : \lambda\in\Lambda\right\}\\ &= \inf\left\{\max\left\{\left\|y - x\circ\lambda^{-1}\right\|_{[0,1]}, \left\|\lambda^{-1}\right\|_\Lambda\right\} : \lambda\in\Lambda\right\}\\ &= \inf\left\{\max\left\{\|x\circ\mu - y\|_{[0,1]}, \|\mu\|_\Lambda\right\} : \mu\in\Lambda\right\}\\ &= \mathsf{d}(x,y).\end{aligned}$$

3. Reflexivity

   (a) For $x \in \mathsf{cadlag}([0,1])$, $\mathsf{d}(x,x) = 0$, evidently.

   (b) Let $x, y \in \mathsf{cadlag}([0,1])$ such that $\mathsf{d}(x,y) = 0$.
   Then, there exists a sequence of bijections $\lambda_n \in \Lambda$, $n \in \mathbb{N}$ such that

$$\lim_{n\to+\infty} \|x\circ\lambda_n - y\|_{[0,1]} = 0, \quad \lim_{n\to+\infty} \|\lambda_n\|_\Lambda = 0.$$

   Take $t \in [0,1]$.

i. If $t = 1$, then

$$0 = \lim_{n\to+\infty} |x \circ \lambda_n(1) - y(1)| = |x(1) - y(1)| .$$

Thus, $x(1) = y(1)$.

ii. If $0 \le t < 1$, then there is a sequence of numbers s_k, $k \in \mathbb{N}$ such that $t < s_k < 1$ for all $k \in \mathbb{N}$ and $\lim_{k\to+\infty} s_k = t$. Then, there is a subsequence n_k, $k \in \mathbb{N}$ such that

$$t < \lambda_{n_k}(s_k) < s_k + \frac{1}{k} \text{ for all } k \in \mathbb{N}.$$

Hence, $\lim_{k\to+\infty} \lambda_{n_k}(s_k) = t$. Since both x, y are continuous from right at t, we have

$$0 = \lim_{k\to+\infty} |x \circ \lambda_{n_k}(s_k) - y(s_k)| = |x(t) - y(t)| .$$

Thus, $x(t) = y(t)$.

We have shown, that $x = y$.

4. Triangular inequality

Let $x, y, z \in \mathsf{cadlag}([0,1])$.

For $\varepsilon > 0$ we find transformation $\lambda, \varphi \in \Lambda$ such that

$$\begin{aligned} \max\left\{\|y \circ \lambda - x\|_{[0,1]}, \|\lambda\|_\Lambda\right\} &< \mathsf{d}(x,y) + \varepsilon, \\ \max\left\{\|z \circ \varphi - y\|_{[0,1]}, \|\varphi\|_\Lambda\right\} &< \mathsf{d}(y,z) + \varepsilon. \end{aligned}$$

There are two cases:

(a) If $\|z \circ \varphi \circ \lambda - x\|_{[0,1]} \ge \|\varphi \circ \lambda\|_\Lambda$, then

$$\begin{aligned} \mathsf{d}(x,z) &\le \|z \circ \varphi \circ \lambda - x\|_{[0,1]} \\ &= \|(z \circ \varphi \circ \lambda - y \circ \lambda) + (y \circ \lambda - x)\|_{[0,1]} \\ &\le \|z \circ \varphi - y\|_{[0,1]} + \|y \circ \lambda - x\|_{[0,1]} \\ &< \mathsf{d}(x,y) + \mathsf{d}(y,z) + 2\varepsilon. \end{aligned}$$

(b) If $\|z \circ \varphi \circ \lambda - x\|_{[0,1]} < \|\varphi \circ \lambda\|_\Lambda$, then

$$\begin{aligned} \mathsf{d}(x,z) &\le \|\varphi \circ \lambda\|_\Lambda \\ &\le \|\lambda\|_\Lambda + \|\varphi\|_\Lambda \\ &< \mathsf{d}(x,y) + \mathsf{d}(y,z) + 2\varepsilon. \end{aligned}$$

Since ε is an arbitrary small positive number, the triangular inequality $\mathsf{d}(x,z) \leq \mathsf{d}(x,y) + \mathsf{d}(y,z)$ is checked.

We have verified d is a metric on $\mathsf{cadlag}([0,1])$.

Q.E.D.

To deal with d_0, we need an estimate.

Lemma 6.17 *For $x, y \in \mathsf{cadlag}([0,1])$, we have an estimate*

$$\mathsf{d}(x,y) \leq \exp\{\mathsf{d}_0(x,y)\} - 1.$$

Proof: Since exponential function is convex, we have an estimate $\mathsf{e}^t \geq t+1$ for all $t \in \mathbb{R}$. Therefore,

$$\|x \circ \lambda - y\|_{[0,1]} \leq \exp\left\{\|x \circ \lambda - y\|_{[0,1]}\right\} - 1.$$

Accordingly to Lemma 6.13,

$$\|\lambda\|_{\Lambda} \quad \leq \quad \exp\{\langle\langle\lambda\rangle\rangle_{\Lambda}\} - 1.$$

These two estimates together imply

$$\max\left\{\|x \circ \lambda - y\|_{[0,1]}, \|\lambda\|_{\Lambda}\right\} \leq \exp\left\{\max\left\{\|x \circ \lambda - y\|_{[0,1]}, \langle\langle\lambda\rangle\rangle_{\Lambda}\right\}\right\} - 1.$$

That is the required estimate.

Q.E.D.

Theorem 6.18: *Distance d_0 is a metric on $\mathsf{cadlag}([0,1])$.*

Proof: We have to verify properties of a metric.

1. Finiteness

   Let $x, y \in \mathsf{cadlag}([0,1])$. Then,

$$\begin{aligned} \mathsf{d}_0(x,y) &= \inf\left\{\max\left\{\|x \circ \lambda - y\|_{[0,1]}, \langle\langle\lambda\rangle\rangle_{\Lambda}\right\} : \lambda \in \Lambda\right\} \\ &\leq \max\left\{\|x \circ \mathrm{Id} - y\|_{[0,1]}, \langle\langle\mathrm{Id}\rangle\rangle_{\Lambda}\right\} \\ &= \max\left\{\|x - y\|_{[0,1]}, 0\right\} \\ &\leq \|x\|_{[0,1]} + \|y\|_{[0,1]} < +\infty. \end{aligned}$$

   Finally, $\mathsf{d}_0 : \mathsf{cadlag}([0,1]) \times \mathsf{cadlag}([0,1]) \to \mathbb{R}_{+,0}$.

2. <u>Symmetry</u>

   Let $x, y \in \mathsf{cadlag}([0,1])$. Then,

$$\begin{aligned} \mathsf{d}_0(y,x) &= \inf\left\{\max\left\{\|y\circ\lambda - x\|_{[0,1]}, \langle\!\langle\lambda\rangle\!\rangle_\Lambda\right\} : \lambda\in\Lambda_L\right\} \\ &= \inf\left\{\max\left\{\left\|y - x\circ\lambda^{-1}\right\|_{[0,1]}, \langle\!\langle\lambda^{-1}\rangle\!\rangle_\Lambda\right\} : \lambda\in\Lambda_L\right\} \\ &= \inf\left\{\max\left\{\|x\circ\mu - y\|_{[0,1]}, \langle\!\langle\mu\rangle\!\rangle_\Lambda\right\} : \mu\in\Lambda_L\right\} \\ &= \mathsf{d}_0(x,y). \end{aligned}$$

3. <u>Reflexivity</u>

   (a) For $x \in \mathsf{cadlag}([0,1])$, $\mathsf{d}_0(x,x) = 0$ is evident.

   (b) Let $x, y \in \mathsf{cadlag}([0,1])$ such that $\mathsf{d}_0(x,y) = 0$.
   According to Lemma 6.17, $\mathsf{d}(x,y) = 0$, too.
   From Theorem 6.16 we know that d is a metric, therefore $x = y$.

4. <u>Triangular inequality</u>

   Let $x, y, z \in \mathsf{cadlag}([0,1])$.

   For $\varepsilon > 0$ we find transformation $\lambda, \varphi \in \Lambda$ such that

$$\begin{aligned} \max\left\{\|y\circ\lambda - x\|_{[0,1]}, \langle\!\langle\lambda\rangle\!\rangle_\Lambda\right\} &< \mathsf{d}_0(x,y) + \varepsilon, \\ \max\left\{\|z\circ\varphi - y\|_{[0,1]}, \langle\!\langle\varphi\rangle\!\rangle_\Lambda\right\} &< \mathsf{d}_0(y,z) + \varepsilon. \end{aligned}$$

   There are two cases:

   (a) If $\|z\circ\varphi\circ\lambda - x\|_{[0,1]} \ge \langle\!\langle\varphi\circ\lambda\rangle\!\rangle_\Lambda$, then

$$\begin{aligned} \mathsf{d}_0(x,z) &\le \|z\circ\varphi\circ\lambda - x\|_{[0,1]} \\ &= \|(z\circ\varphi\circ\lambda - y\circ\lambda) + (y\circ\lambda - x)\|_{[0,1]} \\ &\le \|z\circ\varphi - y\|_{[0,1]} + \|y\circ\lambda - x\|_{[0,1]} \\ &< \mathsf{d}_0(x,y) + \mathsf{d}_0(y,z) + 2\varepsilon. \end{aligned}$$

   (b) If $\|z\circ\varphi\circ\lambda - x\|_{[0,1]} < \langle\!\langle\varphi\circ\lambda\rangle\!\rangle_\Lambda$, then

$$\begin{aligned} \mathsf{d}_0(x,z) &\le \langle\!\langle\varphi\circ\lambda\rangle\!\rangle_\Lambda \\ &\le \langle\!\langle\lambda\rangle\!\rangle_\Lambda + \langle\!\langle\varphi\rangle\!\rangle_\Lambda \\ &< \mathsf{d}_0(x,y) + \mathsf{d}_0(y,z) + 2\varepsilon. \end{aligned}$$

Since, ε is an arbitrary small positive number and, therefore, the triangular inequality $\mathsf{d}_0\left(x,z\right) \leq \mathsf{d}_0\left(x,y\right) + \mathsf{d}_0\left(y,z\right)$ is checked.

We have verified d_0 is a metric on $\mathsf{cadlag}([0,1])$.

Q.E.D.

Lemma 6.19 *Let $f,g \in \mathsf{cadlag}([0,1])$ and $0 < \delta < \frac{1}{2}$. If $\mathsf{d}\left(f,g\right) < \delta^2$, then*

$$\mathsf{d}_0\left(f,g\right) < \mathsf{w}'\left(g,\delta\right) + \left|\log(1-2\delta)\right|.$$

Proof: Because $g \in \mathsf{cadlag}([0,1])$, there exists a partition $\mathcal{D} \in \Delta(\delta)$ such that $\widetilde{\mathsf{w}}\left(g, \mathsf{I}_i(\mathcal{D})\right) < \mathsf{w}'\left(g,\delta\right) + \delta$ for all $i = 1,2,\ldots,K(\mathcal{D})$.
Then, there is a transformation $\lambda \in \Lambda$ such that

$$\max\left\{\left\|f\circ\lambda - g\right\|_{[0,1]}, \left\|\lambda\right\|_\Lambda\right\} < \delta^2.$$

We define a new transformation $\rho \in \Lambda$ such that $\rho(t_i(\mathcal{D})) = \lambda(t_i(\mathcal{D}))$ for all $i = 0,1,2,\ldots,K(\mathcal{D})$. Between neighbors of the partition, we define ρ continuously linear.
Then, for $i = 1,2,\ldots,K(\mathcal{D})$, we have

$$\begin{aligned}
&\frac{\rho(t_i(\mathcal{D})) - \rho(t_{i-1}(\mathcal{D}))}{t_i(\mathcal{D}) - t_{i-1}(\mathcal{D})} = \\
&\qquad = 1 + \frac{(\lambda(t_i(\mathcal{D})) - t_i(\mathcal{D})) - (\lambda(t_{i-1}(\mathcal{D})) - t_{i-1}(\mathcal{D}))}{t_i(\mathcal{D}) - t_{i-1}(\mathcal{D})} \\
&\qquad \leq 1 + \frac{2\left\|\lambda\right\|_\Lambda}{t_i(\mathcal{D}) - t_{i-1}(\mathcal{D})} < 1 + \frac{2\delta^2}{\delta} = 1 + 2\delta, \\
&\frac{\rho(t_i(\mathcal{D})) - \rho(t_{i-1}(\mathcal{D}))}{t_i(\mathcal{D}) - t_{i-1}(\mathcal{D})} \geq 1 - \frac{2\left\|\lambda\right\|_\Lambda}{t_i(\mathcal{D}) - t_{i-1}(\mathcal{D})} > 1 - \frac{2\delta^2}{\delta} = 1 - 2\delta.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle\!\langle \rho \rangle\!\rangle_\Lambda &= \sup\left\{\left|\log\left(\frac{\lambda(s)-\lambda(t)}{s-t}\right)\right| : 0 \leq t < s \leq 1\right\} \\
&= \max\left\{\left|\log\left(\frac{\rho(t_i(\mathcal{D})) - \rho(t_{i-1}(\mathcal{D}))}{t_i(\mathcal{D}) - t_{i-1}(\mathcal{D})}\right)\right| : i = 1,2,\ldots,K(\mathcal{D})\right\} \\
&< \max\left\{\log(1+2\delta), \left|\log(1-2\delta)\right|\right\} = \left|\log(1-2\delta)\right|.
\end{aligned}$$

According to Lemma 6.11, $\rho \in \Lambda_L$. Hence,

$$\begin{aligned}\|f \circ \rho - g\|_{[0,1]} &= \left\|f \circ \lambda - g \circ \rho^{-1} \circ \lambda\right\|_{[0,1]} \\ &= \left\|(f \circ \lambda - g) - (g \circ \rho^{-1} \circ \lambda - g)\right\|_{[0,1]} \\ &\leq \|f \circ \lambda - g\|_{[0,1]} + \left\|(g \circ \rho^{-1} \circ \lambda - g)\right\|_{[0,1]} \\ &\leq \|f \circ \lambda - g\|_{[0,1]} + \max\left\{\widetilde{\mathsf{w}}\left(g, \mathsf{I}_i(\mathcal{D})\right) \,:\, i = 1, 2, \ldots, K(\mathcal{D})\right\} \\ &< \delta^2 + \mathsf{w}'(g, \delta) + \delta.\end{aligned}$$

We have shown that for each $n \geq m$, $n \in \mathbb{N}$, it is fulfilled

$$\begin{aligned}\mathsf{d}_0(f, g) &< \mathsf{w}'(g, \delta) + \max\left\{\delta^2 + \delta,\ |\log(1 - 2\delta)|\right\} \\ &= \mathsf{w}'(g, \delta) + |\log(1 - 2\delta)|\,.\end{aligned}$$

Q.E.D.

Lemma 6.20 *Let* $x, x_n \in \mathsf{cadlag}([0, 1])$ *for* $n \in \mathbb{N}$*. Hence,*

$$\lim_{n\to+\infty} \mathsf{d}(x_n, x) = 0 \iff \lim_{n\to+\infty} \mathsf{d}_0(x_n, x) = 0.$$

Proof:

1. If $\lim_{n\to+\infty} \mathsf{d}_0(x_n, x) = 0$, then $\lim_{n\to+\infty} \mathsf{d}(x_n, x) = 0$, accordingly to Lemma 6.17.

2. Let $\lim_{n\to+\infty} \mathsf{d}(x_n, x) = 0$.

   Take $\varepsilon > 0$.

   Then, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, $n \in \mathbb{N}$ is $\mathsf{d}(x_n, x) < \varepsilon^2$.

   According to Lemma 6.19, for all $n \geq m$, we have an estimate

   $$\mathsf{d}_0(x_n, x) < \mathsf{w}'(x, \varepsilon) + |\log(1 - 2\varepsilon)|\,.$$

   It means $\lim_{n\to+\infty} \mathsf{d}_0(x_n, x) = 0$.

Q.E.D.

Consider also continuity of the modulus w' with respect to introduced metrics.

Lemma 6.21 *Let $x, y \in \mathsf{cadlag}([0,1])$, $\delta > 0$ and $\lambda \in \Lambda$. Then,*

$$\left|\mathsf{w}'(y,\delta) - \mathsf{w}'(x,\delta)\right| \leq 2\left\|y - x\right\|_{[0,1]}, \tag{6.14}$$
$$\mathsf{w}'\left(x, \delta - 2\left\|\lambda\right\|_{\Lambda}\right) \leq \mathsf{w}'(x \circ \lambda, \delta) \leq \mathsf{w}'\left(x, \delta + 2\left\|\lambda\right\|_{\Lambda}\right). \tag{6.15}$$

Proof:

1. For $t, s \in [0,1]$ we have estimates

$$\begin{aligned} |x(t) - x(s)| &\leq |y(t) - y(s)| + 2\left\|y - x\right\|_{[0,1]}, \\ |y(t) - y(s)| &\leq |x(t) - x(s)| + 2\left\|y - x\right\|_{[0,1]}. \end{aligned}$$

   Then, for each set $A \subset [0,1]$ we have estimates

$$\begin{aligned} \widetilde{\mathsf{w}}(x, A) &\leq \widetilde{\mathsf{w}}(y, A) + 2\left\|y - x\right\|_{[0,1]}, \\ \widetilde{\mathsf{w}}(y, A) &\leq \widetilde{\mathsf{w}}(x, A) + 2\left\|y - x\right\|_{[0,1]}. \end{aligned}$$

   Consequently, we are receiving (6.14).

2. It is sufficient to observe that points of partition $\mathcal{D} \in \Delta(\delta)$ are not effected by transformation λ; i.e. $\lambda \circ \mathcal{D} \in \Delta(\delta - 2\left\|\lambda\right\|_{\Lambda})$. Hence, suprema over corresponding intervals coincide and, therefore, contributions to moduli is the same.

Q.E.D.

Lemma 6.22 *Let $x, x_n \in \mathsf{cadlag}([0,1])$ for $n \in \mathbb{N}$. If $\lim_{n \to +\infty} \left\|x_n - x\right\|_{[0,1]} = 0$, hence for all $\delta > 0$,*

$$\lim_{n \to +\infty} \mathsf{w}'(x_n, \delta) = \mathsf{w}'(x, \delta).$$

Proof: The statement is a direct consequence of the estimate (6.14).

Q.E.D.

Lemma 6.23 *Let $x, x_n \in \mathsf{cadlag}([0,1])$ for $n \in \mathbb{N}$. If $\lim_{n \to +\infty} \mathsf{d}(x_n, x) = 0$, hence for all $0 < \eta < \delta < \zeta$, we have*

$$\limsup_{n \to +\infty} \mathsf{w}'(x_n, \eta) \leq \mathsf{w}'(x, \delta) \leq \liminf_{n \to +\infty} \mathsf{w}'(x_n, \zeta).$$

Proof: We know, there is a sequence of transformations $\lambda_n \in \Lambda$, $n \in \mathbb{N}$ such that

$$\lim_{n\to+\infty} \|x_n \circ \lambda_n - x\|_{[0,1]} = 0, \lim_{n\to+\infty} \|\lambda_n\|_\Lambda = 0.$$

According to Lemma 6.22, for all $\delta > 0$,

$$\lim_{n\to+\infty} \mathsf{w}'(x_n \circ \lambda_n, \delta) = \mathsf{w}'(x, \delta).$$

According to the estimate (6.15) we have

$$\begin{aligned} \mathsf{w}'(x_n, \delta) &\leq \mathsf{w}'(x_n \circ \lambda_n, \delta + 2\|\lambda_n\|_\Lambda), \\ \mathsf{w}'(x_n \circ \lambda_n, \delta) &\leq \mathsf{w}'(x_n, \delta + 2\|\lambda_n\|_\Lambda). \end{aligned}$$

Consequently, the statement is shown.

Q.E.D.

6.3.3 Topology

Definition 6.24 *Metric space* $(\mathsf{cadlag}([0,1]), \mathsf{d})$ *induces a topology on* $\mathsf{cadlag}([0,1])$. *The topology is called Skorokhod topology and will be denoted by* τ_S. *A topological space* $\mathsf{D}([0,1]) = (\mathsf{cadlag}([0,1]), \tau_S)$ *is called Skorokhod space; or simply "space D". ("D" comes from "discontinuous")*

Theorem 6.25: *Metric space* $(\mathsf{cadlag}([0,1]), \mathsf{d}_0)$ *also induces the Skorokhod topology on* $\mathsf{cadlag}([0,1])$.

Proof: Both metrics d, d_0 induce the same convergence on $\mathsf{cadlag}([0,1])$; see Lemma 6.20. Consequently, they induce the same topology on $\mathsf{cadlag}([0,1])$; i.e. Skorokhod topology.

Q.E.D.

Theorem 6.26: *Space* $(\mathsf{cadlag}([0,1]), \mathsf{d})$ *is a separable metric space, but, incomplete.*

Proof: According to Theorem 6.16, $(\mathsf{cadlag}([0,1]), \mathsf{d})$ is a metric space. We have to show separability and discuss completeness.

1. Set of functions

$$\left\{ \sum_{i=1}^{K(\mathcal{D})} \alpha_i \mathbb{I}\left[\mathsf{I}_i(\mathcal{D})\right] + \beta \mathbb{I}\left[\{1\}\right] \ : \right.$$
$$\left. : \ \alpha_i, \ \beta \text{ are rational}, \ \mathcal{D} \text{ is a partition with rational points} \right\}$$

   is countable and dense in $(\mathsf{cadlag}([0,1]), \mathsf{d})$. It can be shown in two steps. The set is dense in

$$\left\{ \sum_{i=1}^{K(\mathcal{D})} \alpha_i \mathbb{I}\left[\mathsf{I}_i(\mathcal{D})\right] + \beta \mathbb{I}\left[\{1\}\right] \ : \ \alpha_i, \beta \in \mathbb{R}, \ \mathcal{D} \text{ is a partition} \right\},$$

   which is dense in $(\mathsf{cadlag}([0,1]), \mathsf{d})$, accordingly to Theorem 6.6.

2. Consider a sequence of functions $f_n = \mathbb{I}\left[\left[0, \frac{1}{n}\right)\right]$, $n \in \mathbb{N}$.

   For $n, m \in \mathbb{N}$, we construct a transformation λ_n^m such that $\lambda_n^m(0) = 0$, $\lambda_n^m(1) = 1$, $\lambda_n^m(\frac{1}{m}) = \frac{1}{n}$, and, linear continuous on intervals $\left[0, \frac{1}{m}\right]$, $\left[\frac{1}{m}, 1\right]$.

   Hence, $f_m = f_n \circ \lambda_n^m$ and $\|\lambda_n^m\|_\Lambda = \left|\frac{1}{m} - \frac{1}{n}\right|$.

   Finally, $\mathsf{d}\left(f_n, f_m\right) \leq \left|\frac{1}{m} - \frac{1}{n}\right|$ and the sequence is Cauchy.

   Assume, the sequence possesses a limit in $(\mathsf{cadlag}([0,1]), \mathsf{d})$, say $g \in \mathsf{cadlag}([0,1])$.

   Then, there would be transformations $\rho_n \in \Lambda$, $n \in \mathbb{N}$ such that

$$\|\rho_n\|_\Lambda \to 0, \ \|f_n \circ \rho_n - g\|_{[0,1]} \to 0.$$

   Then,

   (a) $g(1) = \lim_{n \to +\infty} f_n \circ \rho_n(1) = \lim_{n \to +\infty} f_n(1) = 0$.

   (b) For $0 < t < 1$ there exists $n_0 \in \mathbb{N}$ such that $\rho_n(t) \geq \frac{1}{n}$ for all $n \in \mathbb{N}$, $n \geq n_0$.
   Consequently, $g(t) = \lim_{n \to +\infty} f_n \circ \rho_n(t) = 0$.

   (c) We assume, $g \in \mathsf{cadlag}([0,1])$, therefore, g is continuous at zero from right, and therefore, $g(0) = 0$.

   (d) But hence,

$$\|f_n \circ \rho_n - g\|_{[0,1]} = \|f_n \circ \rho_n\|_{[0,1]} = \|f_n\|_{[0,1]} = 1.$$

   It is a contradiction.

Finally, the sequence possesses no limit in $(\mathsf{cadlag}([0,1]),\mathsf{d})$. It means, $(\mathsf{cadlag}([0,1]),\mathsf{d})$ is incomplete.

Q.E.D.

Theorem 6.27: *Space* $(\mathsf{cadlag}([0,1]),\mathsf{d}_0)$ *is a complete separable metric space.*

Proof: According to Theorem 6.18, $(\mathsf{cadlag}([0,1]),\mathsf{d}_0)$ is a metric space. We have to show its separability and completeness.

1. According to Lemma 6.20, topologies of spaces $(\mathsf{cadlag}([0,1]),\mathsf{d}_0)$ and $(\mathsf{cadlag}([0,1]),\mathsf{d})$ coincide. According to Theorem 6.26, $(\mathsf{cadlag}([0,1]),\mathsf{d})$ is separable, consequently, $(\mathsf{cadlag}([0,1]),\mathsf{d}_0)$ is separable, too.

2. Let f_n, $n\in\mathbb{N}$ be a Cauchy sequence in $(\mathsf{cadlag}([0,1]),\mathsf{d}_0)$. Without any loss of generality we assume $\mathsf{d}_0\left(f_n,f_{n+1}\right)<2^{-n}$ for all $n\in\mathbb{N}$.

   Then for each $n\in\mathbb{N}$, there is a transformation $\lambda_n\in\Lambda_L$ such that

   $$\|f_n-f_{n+1}\circ\lambda_n\|_{[0,1]}<2^{-n},\ \ \langle\!\langle\lambda_n\rangle\!\rangle_\Lambda<2^{-n}.$$

   For $n,m\in\mathbb{N}$, $n<m$, we define a transformation

   $$\rho_n^m=\lambda_m\circ\lambda_{m-1}\circ\cdots\circ\lambda_{n+1}\circ\lambda_n.$$

   Estimate its half-norm

   $$\langle\!\langle\rho_n^m\rangle\!\rangle_\Lambda \ = \ \left\langle\!\!\left\langle\lambda_m\circ\lambda_{m-1}\circ\cdots\circ\lambda_n\right\rangle\!\!\right\rangle_\Lambda\leq\sum_{j=n}^m\left\langle\!\!\left\langle\lambda_j\right\rangle\!\!\right\rangle_\Lambda<\sum_{j=n}^m 2^{-j}<2^{1-n}.$$

   Then for $n,m,k\in\mathbb{N}$, $n<m<k$, we have an estimate

   $$\begin{aligned}\left\|\rho_n^k-\rho_n^m\right\|_{[0,1]} &= \left\|\rho_{m+1}^k-\mathrm{Id}\right\|_{[0,1]}=\left\|\rho_{m+1}^k\right\|_\Lambda\\ &\leq \mathsf{exp}\left\{\left\langle\!\!\left\langle\rho_{m+1}^k\right\rangle\!\!\right\rangle_\Lambda\right\}-1\\ &< \mathsf{exp}\left\{2^{-m}\right\}-1.\end{aligned}$$

   We have shown, that for fixed $n\in\mathbb{N}$ the sequence ρ_n^m, $m\in\mathbb{N}$ is Cauchy in $\mathsf{C}\left([0,1]\right)$. Thus, it possesses a limit in $\mathsf{C}\left([0,1]\right)$. Let us denote the limit by ψ_n. We know, that $\psi_n\in\mathsf{C}\left([0,1]\right)$, $\psi_n(0)=0$. $\psi_n(1)=1$.

   We have to estimate its half-norm.

For $0 \le t < s \le 1$, we observe

$$\begin{aligned}
\frac{\psi_n(s)-\psi_n(t)}{s-t} &= \lim_{m\to+\infty} \frac{\rho_n^m(s)-\rho_n^m(t)}{s-t} \le \liminf_{m\to+\infty} \mathsf{Lip}\left(\rho_n^m\right) \\
&\le \liminf_{m\to+\infty} \exp\left\{\langle\!\langle \rho_n^m \rangle\!\rangle_\Lambda\right\} \le \exp\left\{2^{1-n}\right\}, \\
\frac{\psi_n(s)-\psi_n(t)}{s-t} &\ge \limsup_{m\to+\infty} \left(\mathsf{Lip}\left(\left(\rho_n^m\right)^{-1}\right)\right)^{-1} \\
&\ge \limsup_{m\to+\infty} \exp\left\{-\langle\!\langle \rho_n^m \rangle\!\rangle_\Lambda\right\} \ge \exp\left\{-2^{1-n}\right\}.
\end{aligned}$$

These estimates are giving $\langle\!\langle \psi_n \rangle\!\rangle_\Lambda \le 2^{1-n}$. Hence, $\psi_n \in \Lambda_L$.

The construction is also giving $\psi_n = \psi_{n+1} \circ \lambda_n$. Consequently,

$$\left\| f_n \circ (\psi_n)^{-1} - f_{n+1} \circ \left(\psi_{n+1}\right)^{-1} \right\|_{[0,1]} = \left\| f_n - f_{n+1} \circ \lambda_n \right\|_{[0,1]} < 2^{-n}.$$

The sequence $f_n \circ (\psi_n)^{-1}$, $n \in \mathbb{N}$ is Cauchy in $\left(\mathsf{cadlag}([0,1]), \|\cdot\|_{[0,1]}\right)$, which is Banach, according to Theorem 6.8. Therefore, the sequence possesses a limit in $\left(\mathsf{cadlag}([0,1]), \|\cdot\|_{[0,1]}\right)$, say $g \in \mathsf{cadlag}([0,1])$. Moreover, we have $\mathsf{d}_0\left(f_n, g\right) \to 0$, since

$$\left\| f_n \circ (\psi_n)^{-1} - g \right\|_{[0,1]} \to 0, \ \ \langle\!\langle \psi_n \rangle\!\rangle_\Lambda \to 0.$$

Thus, $(\mathsf{cadlag}([0,1]), \mathsf{d}_0)$ is a complete separable metric space.

Q.E.D.

Convergence is connected with topology, which is the same if metric d or d_0 is used. From now, we will be writing $\mathsf{D}\left([0,1]\right)$ and we will consider it as a topological space with topology induced by metric d. We know, the space is Polish and we know two metrics d, d_0 metrizing the topology.

Now, we introduce two characterizations of compacts.

Theorem 6.28: *Let $A \subset \mathsf{D}\left([0,1]\right)$. Then, A is relatively compact in $\mathsf{D}\left([0,1]\right)$ if and only if*

$$\begin{aligned}
&\sup\left\{ \|f\|_{[0,1]} \, : \, f \in A \right\} < +\infty, \\
&\lim_{\delta\to 0+} \sup\left\{ \mathsf{w}'\left(f,\delta\right) \, : \, f \in A \right\} = 0.
\end{aligned}$$

Proof:

1. **Sufficiency**

   Metric space $(\mathsf{cadlag}([0,1]), \mathsf{d}_0)$ is complete and, therefore, we have to show total boundedness of A in $(\mathsf{cadlag}([0,1]), \mathsf{d}_0)$, only.

   Start with total boundedness of A in $(\mathsf{cadlag}([0,1]), \mathsf{d})$.

   Take $\varepsilon > 0$.

   Then, there is $k \in \mathbb{N}$ such that $k > \frac{1}{\varepsilon}$ and $\sup\left\{\mathsf{w}'\left(f, \frac{1}{k}\right) : f \in A\right\} < \varepsilon$. Let us denote $M = \left\lceil k \sup\left\{\|f\|_{[0,1]} : f \in A\right\}\right\rceil$ and consider a set of functions

$$E = \left\{\sum_{j=1}^{k} \frac{\alpha_j}{k} \mathbb{I}\left[\left[\frac{j-1}{k}, \frac{j}{k}\right)\right] + \frac{\beta}{k}\mathbb{I}\left[\{1\}\right] : \right. \\ \left. : \alpha_i,\ \beta \in \{-M, \dots, -1, 0, 1, \dots, M\}\right\}.$$

   For $f \in A$, there exists a partition $\mathcal{D} \in \Delta(\frac{1}{k})$ such that for all $i = 1, 2, \dots, K(\mathcal{D})$ we have $\widetilde{\mathsf{w}}\left(f, [t_{i-1}(\mathcal{D}), t_i(\mathcal{D}))\right) < \varepsilon$.

   Let us denote $j_0 = 0$, $j_{K(\mathcal{D})} = k$ and for $i = 1, 2, \dots, K(\mathcal{D}) - 1$ we find j_i such that $\frac{j_i}{k} \le t_i(\mathcal{D}) < \frac{j_{i+1}}{k}$. This correspondence is uniquely defined and none of the points is repeated, since distance between neighbors of the partition is larger than $\frac{1}{k}$.

   Take a transformation $\lambda \in \Lambda$ such that $\lambda(\frac{j_i}{k}) = t_i(\mathcal{D})$, and, it is continuous linear between neighboring points. Then,

$$\|\lambda\|_\Lambda = \max\left\{\left|\frac{j_i}{k} - t_i(\mathcal{D})\right| : i = 1, 2, \dots, K(\mathcal{D}) - 1\right\} < \frac{1}{k} < \varepsilon .$$

   Set

$$g = \sum_{j=1}^{k} \frac{\alpha_j}{k} \mathbb{I}\left[\left[\frac{j-1}{k}, \frac{j}{k}\right)\right] + \frac{\beta}{k}\mathbb{I}\left[\{1\}\right],$$
$$\text{where} \qquad \alpha_i = \left\lfloor k \cdot f \circ \lambda\left(\frac{j-1}{k}\right)\right\rfloor, \ \beta = \lfloor k \cdot f(1) \rfloor .$$

   Hence,

$$\|f \circ \lambda - g\|_{[0,1]} < \frac{1}{k} + \varepsilon < 2\varepsilon .$$

   Thus, $\mathsf{d}(f, g) < 2\varepsilon$ and E is 2ε-net for A in $(\mathsf{cadlag}([0,1]), \mathsf{d})$.

If $0 < \varepsilon < \frac{1}{8}$ we can set $\delta = \sqrt{2\varepsilon}$. Then $0 < \delta < \frac{1}{2}$ and $\mathsf{d}(f,g) < \delta^2$. We can apply Lemma 6.19 and receive

$$\begin{aligned} \mathsf{d}_0(f,g) &< \mathsf{w}'(f,\delta) + |\log(1-2\delta)| \\ &\leq \sup_{h\in A} \mathsf{w}'(h,\delta) + |\log(1-2\delta)| \,. \end{aligned}$$

Denoting $\eta = \sup_{h\in A} \mathsf{w}'(h,\delta) + |\log(1-2\delta)|$, we see E is an η-net for A in $(\mathsf{cadlag}([0,1]), \mathsf{d}_0)$.

Metric space $(\mathsf{cadlag}([0,1]), \mathsf{d}_0)$ is complete and η can be adjusted to be arbitrary small. It means, A is relatively compact in $(\mathsf{cadlag}([0,1]), \mathsf{d}_0)$.

2. **Necessity**

   Let A be relatively compact in $(\mathsf{cadlag}([0,1]), \mathsf{d}_0)$.

   (a) Assume $f_n \in A$, $n \in \mathbb{N}$ with a property $\lim_{n\to+\infty} \|f_n\|_{[0,1]} = +\infty$.
   A is relatively compact in $(\mathsf{cadlag}([0,1]), \mathsf{d}_0)$, therefore, the sequence possesses a cluster point, say $g \in \mathsf{D}([0,1])$.
   Thus for a subsequence and convenient transformations $\lambda_k \in \Lambda_L$, we have $\lim_{k\to+\infty} \|f_{n_k} \circ \lambda_k - g\|_{[0,1]} = 0$ and $\lim_{k\to+\infty} \langle\!\langle \lambda_k \rangle\!\rangle_\Lambda = 0$.
   Consequently, $\|f_{n_k}\|_{[0,1]} \leq \|f_{n_k} \circ \lambda_k - g\|_{[0,1]} + \|g\|_{[0,1]}$.
   We received, functions of the subsequence are uniquely bounded.
   It is a contradiction. Finally,

$$\sup \left\{ \|f\|_{[0,1]} \,:\, f \in A \right\} < +\infty \,.$$

   (b) Assume

$$\Delta = \lim_{\delta\to 0+} \sup \{\mathsf{w}'(f,\delta) \,:\, f \in A\} > 0.$$

   Then, there exists a sequence of functions $f_n \in A$ such that

$$\lim_{n\to+\infty} \mathsf{w}'\left(f_n, \frac{1}{n}\right) = \Delta.$$

   The set A is relatively compact and, therefore, there exists at least one cluster point of the sequence, say $g \in \mathsf{D}([0,1])$.
   Hence according to Lemma 6.23, $\mathsf{w}'(g,\delta) \geq \Delta > 0$ for all $\delta > 0$.
   Accordingly to Lemma 6.6, $g \notin \mathsf{D}([0,1])$.
   It is a contradiction.

Therefore, the condition is necessary.

Q.E.D.

Theorem 6.29: *Let $A \subset \mathsf{D}([0,1])$. Then, A is relatively compact in $\mathsf{D}([0,1])$ if and only if*

$$\sup\left\{\|f\|_{[0,1]} \,:\, f \in A\right\} < +\infty,$$
$$\lim_{\delta \to 0+} \sup\left\{\mathsf{w}''(f,\delta) \,:\, f \in A\right\} = 0,$$
$$\lim_{\delta \to 0+} \sup\left\{\widetilde{\mathsf{w}}(f,[0,\delta)) \,:\, f \in A\right\} = 0,$$
$$\lim_{\delta \to 0+} \sup\left\{\widetilde{\mathsf{w}}(f,[1-\delta,1)) \,:\, f \in A\right\} = 0.$$

Proof: For a proof see [2], Theorem 14.4, pp.166-8.

Q.E.D.

6.3.4 Relation between topologies of $\mathsf{D}([0,1])$ and $\mathbb{R}^{[0,1]}$

This section is taken from [2], chapter 3. Let us begin with Theorem 15.1 from [2], p.174.

Theorem 6.30: *Let $T = \{t_k, k \in \mathbb{N}\} \subset [0,1]$ be countable dense in $[0,1]$ and $1 \in T$. Then, for all $x \in \mathsf{D}([0,1])$ and $\varepsilon > 0$ we have*

$$\mathcal{U}(x;\varepsilon|\,\mathsf{d}_0) = \tag{6.16}$$
$$= \mathsf{D}([0,1]) \cap \bigcup_{k=1}^{+\infty} \bigcap_{m=1}^{+\infty} \bigcup_{\langle\!\langle\lambda\rangle\!\rangle_\Lambda < \varepsilon - \frac{1}{k}} \bigcap_{i=1}^{m}$$
$$\left\{ y \in \mathbb{R}^{[0,1]} \,:\, x(\lambda(t_i)) - \varepsilon + \frac{1}{k} < y(t_i) < x(\lambda(t_i)) + \varepsilon - \frac{1}{k} \right\}.$$

Proof: Expression (6.16) is evident, since T is dense in $[0,1]$.

$$\forall i = 1,2,\ldots,m \qquad x(\lambda_m(t_i)) - \varepsilon + \frac{1}{k} < y(t_i) < x(\lambda_m(t_i)) + \varepsilon - \frac{1}{k}.$$

Then, there exists a subsequence such that

$$\forall i \in \mathbb{N} \qquad \lim_{n \to +\infty} \lambda_{m_n}(t_i) = \lambda(t_i).$$

Function λ is non-decreasing on T, therefore, we can enlarge its definition

$$\lambda(t) = \inf\{\lambda(s) \,:\, t \le s,\; s \in T\} \quad \text{for all} \quad 0 \le t \le 1.$$

Then, $\lim_{n\to+\infty} \langle\!\langle\lambda_{m_n}\rangle\!\rangle_\Lambda = \langle\!\langle\lambda\rangle\!\rangle_\Lambda$, and consequently, $\lambda \in \Lambda_L$.

Q.E.D.

Theorem 6.31: *Let $T \subset [0,1]$ be dense in $[0,1]$ and $1 \in T$. Then for all $G \in \mathcal{G}(\mathsf{D}([0,1]))$, there are $G_{i,j} \in \mathcal{G}_T^{[0,1]} \cap \mathsf{D}([0,1])$ such that $G = \bigcup_{i=1}^{+\infty} \bigcap_{j=1}^{+\infty} G_{i,j}$.*

Proof: The space $\mathsf{D}([0,1])$ is separable, therefore, each $G \in \mathcal{G}(\mathsf{D}([0,1]))$ is a countable union of open balls. Accordingly to Theorem 6.30, each of these balls possesses a representation (6.16).
Consequently, proposition of the theorem is shown.

Q.E.D.

Theorem 6.32: *If $T \subset [0,1]$ is dense in $[0,1]$ and $1 \in T$, then we have*

$$\sigma\left(\mathcal{F}_T^{[0,1]} \cap \mathsf{D}([0,1])\right) = \mathcal{B}(\mathsf{D}([0,1])). \tag{6.17}$$

Proof: The statement is a consequence of Theorem 6.31.

Q.E.D.

6.4 $\mathsf{D}([0,1])$ and randomness

Definition 6.33 *We define for each $0 < t \leq 1$*

$$J_t = \{x \in \mathsf{cadlag}([0,1]) \,:\, x(t) \neq x(t-)\} \tag{6.18}$$

and for $T \subset (0,1]$

$$\mathsf{cadlag}_T = \mathsf{cadlag}([0,1]) \setminus \bigcup_{t \in T} J_t. \tag{6.19}$$

Particularly, $\mathsf{cadlag}_{(0,1]} = \mathsf{C}([0,1])$.

Definition 6.34 *For X random process in $\mathsf{D}([0,1])$ we introduce*

$$S_X = \{t \in (0,1) \,:\, \mathsf{P}(X(t) = X(t-)) = 1\}, \tag{6.20}$$

$$\bar{S}_X = S_X \cup \{0,1\}, \tag{6.21}$$

$$T_X = \{t \in (0,1) \,:\, \mathsf{P}(X(t) \neq X(t-)) > 0\}. \tag{6.22}$$

S_X is a set of all points from $(0,1)$ in which X is stochasticaly continuous, T_X is a set of all points from $(0,1)$ in which X is stochasticaly discontinuous.

Lemma 6.35 *For X random process in $\mathsf{D}([0,1])$ the set T_X is at most countable.*

Proof: For $\varepsilon > 0$ and $t \in (0,1)$, we define sets

$$J_t(\varepsilon) = \{\omega \in \Omega \,:\, |X(t;\omega) - X(t-;\omega)| \geq \varepsilon\}.$$

For $\varepsilon > 0$ and $\eta > 0$, we define sets

$$S(\varepsilon,\eta) = \{t \in (0,1) \,:\, \mathsf{P}(J_t(\varepsilon)) \geq \eta\}.$$

Assume $S(\varepsilon,\eta)$ is infinite.
Then, it possesses a cluster point $s \in [0,1]$, i.e. there is $t_n \to s$ and $t_n \neq s$.
Take $\omega \in \limsup_{n\to+\infty} J_{t_n}(\varepsilon)$.
Then, we have a subsequence with property $\omega \in J_{t_{n_k}}(\varepsilon)$, $k \in \mathbb{N}$.
Now, we have two possibilities.

1. There is a subsequence $t_{n_{k_j}}$, $j \in \mathbb{N}$ with $t_{n_{k_j}} > s$ and $\omega \in J_{t_{n_{k_j}}}(\varepsilon)$.

   Then, we have

   $$\varepsilon \leq \lim_{j\to+\infty} \left|X\left(t_{n_{k_j}};\omega\right) - X\left(t_{n_{k_j}}-;\omega\right)\right| = |X(s;\omega) - X(s;\omega)| = 0.$$

   It is a contradiction.

2. There is a subsequence $t_{n_{k_j}}$, $j \in \mathbb{N}$ with $t_{n_{k_j}} < s$ and $\omega \in J_{t_{n_{k_j}}}(\varepsilon)$.

   Then, we have

   $$\varepsilon \leq \lim_{j\to+\infty} \left|X\left(t_{n_{k_j}};\omega\right) - X\left(t_{n_{k_j}}-;\omega\right)\right| = |X(s-;\omega) - X(s-;\omega)| = 0.$$

   It is a contradiction.

We have shown, that $\limsup_{n\to+\infty} J_{t_n}(\varepsilon) = \emptyset$, but, continuity of probability measure is giving $\mathsf{P}\left(\limsup_{n\to+\infty} J_{t_n}(\varepsilon)\right) \geq \eta$.
It is a contradiction. Therefore, $S(\varepsilon,\eta)$ must be finite.
Consequently, $T_X = \bigcup_{\varepsilon>0,\eta>0} S(\varepsilon,\eta) = \bigcup_{m,n=0}^{+\infty} S(2^{-m},2^{-n})$ is at most countable.

Q.E.D.

Lemma 6.36 *Let $T \subset [0,1]$ then ${}_{[0,1]}\Pi_T \,:\, \mathsf{D}([0,1]) \to \mathbb{R}^T$ is continuous at* $\mathsf{cadlag}_{T\cap(0,1)}$.

Proof: Consider a sequence $f_n \in \mathsf{cadlag}([0,1])$, $n \in \mathbb{N}$ and $f \in \mathsf{cadlag}_{T\cap(0,1)}$ such that $f_n \xrightarrow[n\to+\infty]{} f$ in $\mathsf{D}([0,1])$. Then, there are λ_n, $n \in \mathbb{N}$ such that $\|f_n - f\circ\lambda_n\|_{[0,1]} \to 0$ and $\|\lambda_n\|_\Lambda \to 0$. Take $t \in T$ and consider three cases:

1. $t = 0$

$$\begin{aligned}\left|{}_{[0,1]}\Pi_T(f_n)(0) - {}_{[0,1]}\Pi_T(f)(0)\right| &= |f_n(0) - f(0)| \\ &= |f_n(0) - f\circ\lambda_n(0)| \le \|f_n - f\circ\lambda_n\|_{[0,1]} \to 0.\end{aligned}$$

2. $0 < t < 1$

$$\begin{aligned}\left|{}_{[0,1]}\Pi_T(f_n)(t) - {}_{[0,1]}\Pi_T(f)(t)\right| &= |f_n(t) - f(t)| \\ &\le |f_n(t) - f\circ\lambda_n(t)| + |f\circ\lambda_n(t) - f(t)| \\ &\le \|f_n - f\circ\lambda_n\|_{[0,1]} + |f\circ\lambda_n(t) - f(t)| \to 0,\end{aligned}$$

since $\lambda_n(t) \to t$ and f is continuous at t.

3. $t = 1$

$$\begin{aligned}\left|{}_{[0,1]}\Pi_T(f_n)(1) - {}_{[0,1]}\Pi_T(f)(1)\right| &= |f_n(1) - f(1)| \\ &= |f_n(1) - f\circ\lambda_n(1)| \le \|f_n - f\circ\lambda_n\|_{[0,1]} \to 0.\end{aligned}$$

It means ${}_{[0,1]}\Pi_T : \mathsf{D}([0,1]) \to \mathbb{R}^T$ is continuous at $\mathsf{cadlag}_{T\cap(0,1)}$.

Q.E.D.

Theorem 6.37: *Let X_n, $n \in \mathbb{N}$ and X be random processes in $\mathsf{D}([0,1])$. If $X_n \xrightarrow[n\to+\infty]{\mathcal{D}} X$ in $\mathsf{D}([0,1])$, then*

$$\left(X_n(t), t \in \bar{S}_X\right) \xrightarrow[n\to+\infty]{fidi} \left(X(t), t \in \bar{S}_X\right). \tag{6.23}$$

Proof: The statement immediately follows from Theorem 1.97, since for each $I \in \mathsf{Fin}([0,1])$ projection ${}_{[0,1]}\Pi_I : \mathsf{D}([0,1]) \to \mathbb{R}^I$ is continuous at $\mathsf{cadlag}_{I\cap(0,1)}$ and $\mathsf{P}\left(X \in \mathsf{cadlag}_{S_X}\right) = 1$.

Q.E.D.

Theorem 6.38: *Let X_n, $n \in \mathbb{N}$ be random processes in $\mathsf{D}([0,1])$ and X be a random processes in $\mathsf{D}([0,1])$. If*

i) $\left(X_n(t), t \in \bar{S}_X\right) \xrightarrow[n\to+\infty]{fidi} \left(X(t), t \in \bar{S}_X\right)$;

ii) the sequence X_n, $n \in \mathbb{N}$ is tight in $\mathsf{D}([0,1])$;

hence $X_n \xrightarrow[n\to+\infty]{\mathcal{D}} X$ *in* $\mathsf{D}([0,1])$.

Proof: See Theorem 15.1, p.174 in [2].

Q.E.D.

Theorem 6.39: *The sequence $X_n = (X_n(t), t \in [0,1])$ random processes in $\mathsf{D}([0,1])$ is tight in $\mathsf{D}([0,1])$ if and only if for all $\varepsilon > 0$, $\eta > 0$, there exist $\delta > 0$, $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$ we have*

i) $\mathsf{P}\left(\left|\sup\left\{X_n(t) \,:\, 0 \leq t \leq 1\right\}\right| > \alpha\right) < \varepsilon$;

ii) $\mathsf{P}\left(\mathsf{w}'(X_n, \delta) > \eta\right) < \varepsilon$.

Proof: See Theorem 15.2, p.174 in [2].

Q.E.D.

Theorem 6.40: *The sequence $X_n = (X_n(t), t \in [0,1])$ random processes in $\mathsf{D}([0,1])$ is tight in $\mathsf{D}([0,1])$ if and only if for all $\varepsilon > 0$, $\eta > 0$, there exist $\delta > 0$, $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$ we have*

i) $\mathsf{P}\left(\left|\sup\left\{X_n(t) \,:\, 0 \leq t \leq 1\right\}\right| > \alpha\right) < \varepsilon$;

ii) $\mathsf{P}\left(\mathsf{w}''(X_n, \delta) > \eta\right) < \varepsilon$.

iii) $\mathsf{P}\left(\widetilde{\mathsf{w}}(X_n, [0, \delta)) > \eta\right) < \varepsilon$.

iv) $\mathsf{P}\left(\widetilde{\mathsf{w}}(X_n, [1-\delta, 1)) > \eta\right) < \varepsilon$.

Proof: See Theorem 15.3, p.175 in [2].

Q.E.D.

Theorem 6.41: *Let X_n, $n \in \mathbb{N}$ and X be random processes in* $\mathsf{D}([0,1])$. *If*

i) $(X_n(t), t \in \bar{S}_X) \xrightarrow[n \to +\infty]{fidi} (X(t), t \in \bar{S}_X)$;

ii) $\mathsf{P}(X \in J_1) = 0$;

iii) For each $\varepsilon > 0$, $\eta > 0$, *there exist* $\delta > 0$ *and* $n_0 \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$, $n \geq n_0$ *we have* $\mathsf{P}(\mathsf{w}''(X_n, \delta) > \eta) < \varepsilon$.

Then, $X_n \xrightarrow[n \to +\infty]{\mathcal{D}} X$ *in* $\mathsf{D}([0,1])$.

Proof: See Theorem 15.4, p.175 in [2].

Q.E.D.

Theorem 6.42: *If a sequence* $X_n = (X_n(t), t \in [0,1])$ *of random processes in* $\mathsf{D}([0,1])$ *fulfills:*

i) The sequence $\|X_n\|_{[0,1]}$, $n \in \mathbb{N}$ *is tight.*

ii) There are $\alpha > 0$, $\beta \geq 0$ *and a non-decreasing continuous functions* $F : [0,1] \to \mathbb{R}$ *such that for all* $n \in \mathbb{N}$, $0 \leq t < s \leq 1$, $\lambda > 0$ *we have*

$$\mathsf{P}\left(\widetilde{\mathsf{w}''}(X_n, [t,s]) \geq \lambda\right) \leq \lambda^{-\beta}(F(s) - F(t))^{1+\alpha}. \tag{6.24}$$

Then the sequence is tight in $\mathsf{D}([0,1])$.

Proof: It is a part of Theorem 15.6, p.179 in [2].

Q.E.D.

Now, we can introduce a condition for tightness of random processes in $\mathsf{D}([0,1])$, which is easily verifiable.

Theorem 6.43: *If a sequence* $X_n = (X_n(t), t \in [0,1])$ *random processes in* $\mathsf{D}([0,1])$ *fulfills:*

i) The sequence $\|X_n\|_{[0,1]}$, $n \in \mathbb{N}$ *is tight.*

ii) $(X_n(t), t \in \bar{S}_X) \xrightarrow[n \to +\infty]{fidi} (X(t), t \in \bar{S}_X)$;

iii) $\mathsf{P}(X \in J_1) = 0$;

iv) *There are* $\alpha > 0$, $\beta \geq 0$ *and non-decreasing continuous functions* $F : [0,1] \to \mathbb{R}$ *such that for all* $n \in \mathbb{N}$, $0 \leq s < u < t \leq 1$, $\lambda > 0$ *we have*

$$\mathsf{P}(|X_n(t) - X_n(u)| \geq \lambda, |X_n(u) - X_n(s)| \geq \lambda) \leq \lambda^{-\beta} |F(t) - F(s)|^{1+\alpha} \quad (6.25)$$

Then, $X_n \xrightarrow[n\to+\infty]{\mathcal{D}} X$ *in* $\mathsf{D}([0,1])$.

Proof: See Theorem 15.6, p.179 in [2].

Q.E.D.

6.4.1 Donsker invariance principle

Theorem 6.44: *If* ξ_i, $i \in \mathbb{N}$ *are i.i.d. random variables with* $\mathsf{E}[\xi_1] = 0$ *and* $\mathsf{var}(\xi_1) = \sigma^2 \in \mathbb{R}_+$, *then* $V^{(n)} \xrightarrow[n\to+\infty]{\mathcal{D}} W$ *in* $\mathsf{D}([0,1])$, *where* $W = (W_t, t \in [0,1])$ *is a Wiener process.*

Proof: In Theorem 5.26, we have proved $V^{(n)} \xrightarrow[n\to+\infty]{fidi} W$.
We intend to verify assumptions of Theorem 6.43.
For $n \in \mathbb{N}$, $0 \leq s < u < t \leq 1$ and $\lambda > 0$ we have:

$$\begin{aligned}
&\mathsf{P}(|V_n(t) - V_n(u)| \geq \lambda,\ |V_n(u) - V_n(s)| \geq \lambda) = \\
&= \mathsf{P}\left(\left|\mathsf{S}_{\lfloor nt \rfloor} - \mathsf{S}_{\lfloor nu \rfloor}\right| \geq \sigma\lambda\sqrt{n},\ \left|\mathsf{S}_{\lfloor nu \rfloor} - \mathsf{S}_{\lfloor ns \rfloor}\right| \geq \sigma\lambda\sqrt{n}\right) = \\
&= \mathsf{P}\left(\left|\mathsf{S}_{\lfloor nt \rfloor} - \mathsf{S}_{\lfloor nu \rfloor}\right| \geq \sigma\lambda\sqrt{n}\right) \mathsf{P}\left(\left|\mathsf{S}_{\lfloor nu \rfloor} - \mathsf{S}_{\lfloor ns \rfloor}\right| \geq \sigma\lambda\sqrt{n}\right) \leq \\
&\leq \frac{1}{\sigma^4\lambda^4 n^2} \mathsf{E}\left[\left|\mathsf{S}_{\lfloor nt \rfloor} - \mathsf{S}_{\lfloor nu \rfloor}\right|^2\right] \mathsf{E}\left[\left|\mathsf{S}_{\lfloor nu \rfloor} - \mathsf{S}_{\lfloor ns \rfloor}\right|^2\right] = \\
&= \frac{1}{\sigma^4\lambda^4 n^2} \sigma^2(\lfloor nt \rfloor - \lfloor nu \rfloor)\sigma^2(\lfloor nu \rfloor - \lfloor ns \rfloor) \leq \\
&\leq \frac{4}{\lambda^4}(t-s)^2.
\end{aligned}$$

Final estimation employes following observations:

- If $nt - ns < 1$, then either $\lfloor nt \rfloor = \lfloor nu \rfloor$ or $\lfloor ns \rfloor = \lfloor nu \rfloor$.
- If $nt - ns \geq 1$, then

$$\begin{aligned}
(\lfloor nt \rfloor - \lfloor nu \rfloor)(\lfloor nu \rfloor - \lfloor ns \rfloor) &\leq (\lfloor nt \rfloor - \lfloor ns \rfloor)^2 \leq (nt - ns + 1)^2 \\
&\leq (2(nt - ns))^2 \leq 4\,n^2(t-s)^2.
\end{aligned}$$

Assumptions of Theorem 6.43 are in power. Therefore, required weak convergence is proved.

Q.E.D.

6.5 Space $\mathsf{D}([a,b])$

Consider a compact interval given by a couple of points $-\infty \leq a < b \leq +\infty$ and $\mathsf{cadlag}([a,b])$ the set of all càdlàg functions defined on a segment $[a,b]$. Since $[a,b]$ is a compact, we are able to develop a metric $\tilde{d}$ on $\mathsf{cadlag}([a,b])$ emulating construction of metric d on $\mathsf{cadlag}([0,1])$. Metric $\tilde{d}$ is giving a topology on $\mathsf{cadlag}([a,b])$, and, arisen topological space is denoted by $\mathsf{D}([a,b])$.

Theorem 6.45: *$\mathsf{D}([a,b])$ is a Polish space, moreover, if $\iota : [0,1] \to [a,b]$ is an increasing bijection then a mapping $\zeta : \mathsf{cadlag}([a,b]) \to \mathsf{cadlag}([0,1]) : f \in \mathsf{cadlag}([a,b]) \to f \circ \iota$ is an isomorphism between $\mathsf{D}([a,b])$ and $\mathsf{D}([0,1])$.*

Proof: Both maps $\zeta : \mathsf{D}([a,b]) \to \mathsf{D}([0,1])$ and $\zeta^{-1} : \mathsf{D}([0,1]) \to \mathsf{D}([a,b])$ are continuous, hence, topological spaces $\mathsf{D}([a,b])$, $\mathsf{D}([0,1])$ are isomorphic. Therefore, $\mathsf{D}([a,b])$ is a Polish space, as $\mathsf{D}([0,1])$ is.

Q.E.D.

6.6 Space $\mathsf{D}([0,+\infty))$

Construction of a Skorokhod topology on $\mathsf{cadlag}([0,+\infty))$ is a bit delicate.

Consider a metric ρ_1 defined for $x,y \in \mathsf{cadlag}([0,+\infty))$ by

$$\rho_1(x,y) = \sum_{k=0}^{+\infty} 2^{-k} \frac{\mathsf{d}_0(y,x|[k,k+1])}{1+\mathsf{d}_0(y,x|[k,k+1])}, \qquad (6.26)$$

where $\mathsf{d}_0(y,x|[k,k+1]) = \mathsf{d}_0((y(t+k), 0 \leq t \leq 1), (x(t+k), 0 \leq t \leq 1))$.

Theorem 6.46: *$(\mathsf{cadlag}([0,+\infty)), \rho_1)$ is a complete separable metric space.*

Definition 6.47 *A topological space of $\mathsf{cadlag}([0,+\infty))$ equipped with topology induced by the metric ρ_1 will be denote by $\mathsf{D}([0,+\infty))$.*

Lemma 6.48 *Topology of* $\mathsf{D}([0,+\infty))$ *is determined by a subbasis*

$$\mathcal{G} = \{\mathcal{U}(x,k,\varepsilon) \,:\, x \in \mathsf{cadlag}([0,+\infty)), k \in \mathbb{N}, \varepsilon > 0\}, \tag{6.27}$$
$$\text{where } \mathcal{U}(x,k,\varepsilon) = \{y \in \mathsf{cadlag}([0,+\infty)) \,:\, \mathsf{d}(y,x|[k,k+1]) < \varepsilon\},$$
$$\mathsf{d}(y,x|[k,k+1]) = \mathsf{d}((y(t+k), 0 \le t \le 1), (x(t+k), 0 \le t \le 1)).$$

Unfortunately, this topology fixes jumps in natural numbers. If a limit function possesses a jump in a natural number whole sequence must jump in the same point eventually. That is inconvenient for applications.

A possibility to overcome the obstacle is to reduce our consideration to $\mathsf{cadlag}_1([0,1])$ the set of all càdlàg functions continuous at 1.

Definition 6.49 *We define a topological space* $\mathsf{D}_1([0,1])$ *as* $\mathsf{cadlag}_1([0,1])$ *equipped with a relative topology induced by* $\mathsf{D}([0,1])$.

Theorem 6.50: *Topological space* $\mathsf{D}_1([0,1])$ *is a Polish space.*

Proof: For example metric d_0 is making $\mathsf{D}_1([0,1])$ to be a complete separable metric space.

Q.E.D.

We will consider $\mathsf{cadlag}_{\mathbb{N}}([0,+\infty))$ the set of all càdlàg functions continuous at each natural number.

Definition 6.51 *We define a topological space* $\mathsf{D}_{\mathbb{N}}([0,+\infty))$ *as* $\mathsf{cadlag}_{\mathbb{N}}([0,+\infty))$ *equipped with a relative topology induced by* $\mathsf{D}([0,+\infty))$.

Lemma 6.52 *Topology of* $\mathsf{D}_{\mathbb{N}}([0,+\infty))$ *is determined by a subbasis*

$$\mathcal{G} = \{\mathcal{U}(x,k,\varepsilon) \,:\, x \in \mathsf{cadlag}_{\mathbb{N}}([0,+\infty)), k \in \mathbb{N}, \varepsilon > 0\}, \tag{6.28}$$
$$\text{where } \mathcal{U}(x,k,\varepsilon) = \{y \in \mathsf{cadlag}_{\mathbb{N}}([0,+\infty)) \,:\, \mathsf{d}(y,x|[k,k+1]) < \varepsilon\}.$$

Theorem 6.53: *Topological space* $\mathsf{D}_{\mathbb{N}}([0,+\infty))$ *is a Polish space.*

Proof: For example metric ρ_1 is making $\mathsf{D}_{\mathbb{N}}([0,+\infty))$ to be a complete separable metric space.

Q.E.D.

Chapter 7

General maximal inequalities

Dealing with weak convergence of random processes requires some convenient maximal inequalities. We must be able to estimate tails of distributions of maxims of partial sums of real random variables. Propositions in this chapter together with proofs are taken from the monograph [2].

Consider $\mathsf{N} \in \mathbb{N}$, real random variables $\xi_1, \xi_2, \ldots, \xi_{\mathsf{N}}$ and their partial sums $\mathsf{S}_1, \mathsf{S}_2, \ldots, \mathsf{S}_{\mathsf{N}}$, where $\mathsf{S}_k = \sum_{i=1}^{k} \xi_i$. For more clear formulations, we set $\mathsf{S}_0 = 0$.

We will investigate tails of distribution of maxims

$$\mathsf{M}_{\mathsf{N}} = \max\{|\mathsf{S}_1|, |\mathsf{S}_2|, \ldots, |\mathsf{S}_{\mathsf{N}}|\}, \tag{7.1}$$
$$\mathsf{M}'_{\mathsf{N}} = \max\left\{\min\{|\mathsf{S}_k|, |\mathsf{S}_{\mathsf{N}} - \mathsf{S}_k|\} \,:\, k = 0, 1, 2, \ldots, \mathsf{N}-1\right\}, \tag{7.2}$$
$$\mathsf{M}''_{\mathsf{N}} = \max\left\{\min\{|\mathsf{S}_j - \mathsf{S}_i|, |\mathsf{S}_k - \mathsf{S}_j|\} \,:\, i, j, k = 0, 1, 2, \ldots, \mathsf{N},\ i < j < k\right\}. \tag{7.3}$$

To abbreviate formulations, we are introducing auxiliary variables

$$\mathsf{M}_i^j = \max\left\{|\mathsf{S}_k - \mathsf{S}_i| \,:\, k = i, i+1, \ldots, j\right\}, \tag{7.4}$$
$$\mathsf{N}_{\mathsf{N}} = \max\left\{\min\{\mathsf{M}_0^k, \mathsf{M}_k^{\mathsf{N}}\} \,:\, k = 0, 1, 2, \ldots, \mathsf{N}-1\right\}. \tag{7.5}$$

We begin with simple basic relations introduced as (12.3) and (12.6) in [2], pp.126-127.

Lemma 7.1 *Without any additional assumption we have*

$$\mathsf{M}'_{\mathsf{N}} \le \mathsf{M}''_{\mathsf{N}} \le 2\mathsf{N}_{\mathsf{N}}. \tag{7.6}$$

Proof:

Q.E.D.

Lemma 7.2 *Without any additional assumption we have*

$$\mathsf{M}'_\mathsf{N} \leq \mathsf{M}_\mathsf{N} \leq \mathsf{M}'_\mathsf{N} + |\mathsf{S}_\mathsf{N}|. \tag{7.7}$$

Proof: For each $k = 1, 2, \ldots, \mathsf{N}-1$ we have a simple estimate

$$\min\{|\mathsf{S}_k|, |\mathsf{S}_\mathsf{N} - \mathsf{S}_k|\} \leq |\mathsf{S}_k|,$$

which implies immediately

$$\mathsf{M}'_\mathsf{N} \leq \mathsf{M}_\mathsf{N}. \tag{7.8}$$

Moreover for all $k = 1, 2, \ldots, \mathsf{N}-1$, we have an estimate

$$|\mathsf{S}_k| \leq \min\{|\mathsf{S}_\mathsf{N}| + |\mathsf{S}_k|, |\mathsf{S}_\mathsf{N}| + |\mathsf{S}_\mathsf{N} - \mathsf{S}_k|\} \leq |\mathsf{S}_\mathsf{N}| + \min\{|\mathsf{S}_k|, |\mathsf{S}_\mathsf{N} - \mathsf{S}_k|\},$$

which implies

$$\mathsf{M}_\mathsf{N} \leq \mathsf{M}'_\mathsf{N} + |\mathsf{S}_\mathsf{N}|. \tag{7.9}$$

Q.E.D.

Lemma 7.3 *Without any additional assumption we have an estimate*

$$|\mathsf{S}_\mathsf{N}| \leq 2\mathsf{M}'_\mathsf{N} + \max\{|\xi_k| \,:\, k = 1, 2, \ldots, \mathsf{N}\}. \tag{7.10}$$

For $\mathsf{N} = 5$ *and* $\xi_1 = \xi_2 = \xi_3 = \xi_4 = -\xi_5$ *the estimate is an equality.*

Proof: Fix $\omega \in \Omega$.

1. If $|\mathsf{S}_1(\omega)| \geq |\mathsf{S}_\mathsf{N}(\omega) - \mathsf{S}_1(\omega)|$, then

$$\begin{aligned} |\mathsf{S}_\mathsf{N}(\omega)| &\leq |\mathsf{S}_1(\omega)| + |\mathsf{S}_\mathsf{N}(\omega) - \mathsf{S}_1(\omega)| \leq |\xi_1(\omega)| + \mathsf{M}'_\mathsf{N}(\omega) \leq \\ &\leq \mathsf{M}'_\mathsf{N}(\omega) + \max\{|\xi_k(\omega)| \,:\, k = 1, 2, \ldots, \mathsf{N}\}. \end{aligned}$$

2. If $|\mathsf{S}_{\mathsf{N}-1}(\omega)| < |\mathsf{S}_\mathsf{N}(\omega) - \mathsf{S}_{\mathsf{N}-1}(\omega)|$, then

$$|\mathsf{S}_\mathsf{N}(\omega)| \leq |\mathsf{S}_{\mathsf{N}-1}(\omega)| + |\xi_\mathsf{N}(\omega)| \leq \mathsf{M}'_\mathsf{N}(\omega) + \max\{|\xi_k(\omega)| \,:\, k = 1, 2, \ldots, \mathsf{N}\}.$$

3. If $|\mathsf{S}_1(\omega)| < |\mathsf{S}_\mathsf{N}(\omega) - \mathsf{S}_1(\omega)|$ and $|\mathsf{S}_{\mathsf{N}-1}(\omega)| \geq |\mathsf{S}_\mathsf{N}(\omega) - \mathsf{S}_{\mathsf{N}-1}(\omega)|$, then there is $k = 2, 3, \ldots, \mathsf{N}-1$ such that $|\mathsf{S}_{k-1}(\omega)| < |\mathsf{S}_\mathsf{N}(\omega) - \mathsf{S}_{k-1}(\omega)|$ and $|\mathsf{S}_k(\omega)| \geq |\mathsf{S}_\mathsf{N}(\omega) - \mathsf{S}_k(\omega)|$. After that we receive

$$\begin{aligned} |\mathsf{S}_\mathsf{N}(\omega)| &\leq |\mathsf{S}_{k-1}(\omega)| + |\xi_k(\omega)| + |\mathsf{S}_\mathsf{N}(\omega) - \mathsf{S}_k(\omega)| \leq \\ &\leq 2\mathsf{M}'_\mathsf{N}(\omega) + \max\{|\xi_k(\omega)| \,:\, k = 1, 2, \ldots, \mathsf{N}\}. \end{aligned}$$

Q.E.D.

Lemma 7.4 *Without any additional assumption we have an estimate*

$$\mathsf{M}_\mathsf{N} \le 3\mathsf{M}'_\mathsf{N} + \max\left\{|\xi_k| \,:\, k = 1, 2, \ldots, \mathsf{N}\right\}. \tag{7.11}$$

For $\mathsf{N} = 5$ *and* $\xi_1 = \xi_2 = \xi_3 = \xi_4 = -\xi_5$ *the estimate is an equality.*

Proof: Estimate follows previous lemmas 7.2 and 7.3.

Q.E.D.

Now we introduce a theorem for distribution tails of M'_N due to [2], Theorem 12.1., pp.128-134.

Theorem 7.5: *Let* $u_1, u_2, \ldots, u_\mathsf{N}$ *be nonnegative real numbers,* $\alpha > 0$ *and* $\beta > 0$ *such that for all* $1 \le i \le h < j \le \mathsf{N}$ *and* $\lambda > 0$ *we have*

$$\mathsf{P}\left(\left|\sum_{k=i}^{h} \xi_k\right| \ge \lambda,\ \left|\sum_{k=h+1}^{j} \xi_k\right| \ge \lambda\right) \le \lambda^{-\beta}\left(\sum_{k=i}^{j} u_k\right)^{1+\alpha}, \tag{7.12}$$

hence for all $\lambda > 0$ *we have*

$$\mathsf{P}\left(\mathsf{M}'_\mathsf{N} \ge \lambda\right) \le \mathsf{K}(\alpha,\beta)\ \lambda^{-\beta}\left(\sum_{k=1}^{\mathsf{N}} u_k\right)^{1+\alpha}, \tag{7.13}$$

where

$$\mathsf{K}(\alpha,\beta) = \left[\frac{1}{2^{\frac{1}{\beta+1}}} - \left(\frac{1}{2^{\frac{1}{\beta+1}}}\right)^{\alpha+1}\right]^{-(\beta+1)}. \tag{7.14}$$

For example $\mathsf{K}(1,4) \doteq 55021.088$.

Proof: Let us denote $u = \sum_{k=1}^{\mathsf{N}} u_k$, $\delta = \frac{1}{\beta+1}$ and take $K \ge 1$ such that

$$\frac{1}{2^{\delta\alpha}} + \left(\frac{2}{K}\right)^{\delta} \le 1. \tag{7.15}$$

1. Let $u = 0$, then condition (7.12) implies $\min\{|\mathsf{S}_k|,\ |\mathsf{S}_\mathsf{N} - \mathsf{S}_k|\} = 0$ a.s. for all $k = 1, 2, \ldots, \mathsf{N}$. Therefore, $\mathsf{M}'_\mathsf{N} = 0$ a.s. and proposition is in power.

2. Let $u > 0$.

   Proposition will be shown by induction over N.

   (a) For $\mathsf{N} = 1$ the statement is trivial.

   (b) For $\mathsf{N} = 2$ we have

$$\begin{aligned} \mathsf{P}\left(\mathsf{M}'_{\mathsf{N}} \geq \lambda\right) &= \mathsf{P}\left(\min\{|\mathsf{S}_1|,\ |\mathsf{S}_2 - \mathsf{S}_1|\} \geq \lambda\right) \leq \lambda^{-\beta}\left(u_1 + u_2\right)^{1+\alpha} \leq \\ &\leq K\lambda^{-\beta}\left(u_1 + u_2\right)^{1+\alpha} \end{aligned}$$

   (c) Let statement is in power for all $\mathsf{N} = 1, 2, \ldots, m$. We will show it for $\mathsf{N} = m + 1$.
   Fix $h = \min\left\{i \in \{1, 2, \ldots, m+1\} \,:\, u_1 + u_2 + \cdots + u_i \geq \frac{u}{2}\right\}$.
   We define

$$\begin{aligned} U_1 &= \max\left\{\min\{|\mathsf{S}_k|,\ |\mathsf{S}_{h-1} - \mathsf{S}_k|\} \,:\, k = 0, 1, \ldots, h-1\right\}, \\ U_2 &= \max\left\{\min\{|\mathsf{S}_k - \mathsf{S}_h|,\ |\mathsf{S}_{m+1} - \mathsf{S}_k|\} \,:\, k = h, h+1, \ldots, m+1\right\}, \\ D_1 &= \min\{|\mathsf{S}_{h-1}|,\ |\mathsf{S}_{m+1} - \mathsf{S}_{h-1}|\}, \\ D_2 &= \min\{|\mathsf{S}_h|,\ |\mathsf{S}_{m+1} - \mathsf{S}_h|\}. \end{aligned}$$

   i. Fix $\omega \in \Omega$ and $i = 1, 2, \ldots, h-1$.

   A. If $|\mathsf{S}_i(\omega)| \leq U_1(\omega)$, then

$$\begin{aligned} &\min\{|\mathsf{S}_i(\omega)|,\ |\mathsf{S}_{m+1}(\omega) - \mathsf{S}_i(\omega)|\} \leq |\mathsf{S}_i(\omega)| \leq \\ &\quad \leq U_1(\omega) \leq U_1(\omega) + D_1(\omega). \end{aligned}$$

   B. If $|\mathsf{S}_{h-1}(\omega) - \mathsf{S}_i(\omega)| \leq U_1(\omega)$ and $|\mathsf{S}_{h-1}(\omega)| = D_1(\omega)$, then

$$\begin{aligned} &\min\{|\mathsf{S}_i(\omega)|,\ |\mathsf{S}_{m+1}(\omega) - \mathsf{S}_i(\omega)|\} \leq |\mathsf{S}_i(\omega)| \leq \\ &\quad \leq |\mathsf{S}_{h-1}(\omega) - \mathsf{S}_i(\omega)| + |\mathsf{S}_{h-1}(\omega)| \leq U_1(\omega) + D_1(\omega). \end{aligned}$$

   C. If $|\mathsf{S}_{h-1}(\omega) - \mathsf{S}_i(\omega)| \leq U_1(\omega)$ and $|\mathsf{S}_{m+1}(\omega) - \mathsf{S}_{h-1}(\omega)| = D_1(\omega)$, then

$$\begin{aligned} &\min\{|\mathsf{S}_i(\omega)|,\ |\mathsf{S}_{m+1}(\omega) - \mathsf{S}_i(\omega)|\} \leq |\mathsf{S}_{m+1}(\omega) - \mathsf{S}_i(\omega)| \leq \\ &\quad \leq |\mathsf{S}_{m+1}(\omega) - \mathsf{S}_{h-1}(\omega)| + |\mathsf{S}_{h-1}(\omega) - \mathsf{S}_i(\omega)| \leq \\ &\quad \leq U_1(\omega) + D_1(\omega). \end{aligned}$$

   For all $i = 1, 2, \ldots, h-1$ we have shown

$$\min\{|\mathsf{S}_i|,\ |\mathsf{S}_{m+1} - \mathsf{S}_i|\} \leq U_1 + D_1.$$

ii. In a similar way for all $i = h, h+1, \ldots, m+1$ one can show

$$\min\{|\mathsf{S}_i|,\ |\mathsf{S}_{m+1} - \mathsf{S}_i|\} \leq U_2 + D_2.$$

iii. Sums U_1, U_2 possess at most m non-vanishing summands. Therefore according to induction assumption for all $\lambda > 0$ we have

$$\begin{aligned} \mathsf{P}(U_1 \geq \lambda) &\leq K\lambda^{-\beta}\left(\sum_{k=1}^{h-1} u_k\right)^{1+\alpha} \\ &\leq K\lambda^{-\beta}\left(\frac{u}{2}\right)^{1+\alpha} = \frac{K}{2^{1+\alpha}}\lambda^{-\beta}u^{1+\alpha}, \\ \mathsf{P}(U_2 \geq \lambda) &\leq K\lambda^{-\beta}\left(\sum_{k=h+1}^{m+1} u_k\right)^{1+\alpha} \\ &\leq K\lambda^{-\beta}\left(\frac{u}{2}\right)^{1+\alpha} = \frac{K}{2^{1+\alpha}}\lambda^{-\beta}u^{1+\alpha}. \end{aligned}$$

iv. According to the assumption (7.12) for all $\lambda > 0$ we have

$$\mathsf{P}(D_1 \geq \lambda) \leq \lambda^{-\beta}u^{1+\alpha},$$
$$\mathsf{P}(D_2 \geq \lambda) \leq \lambda^{-\beta}u^{1+\alpha}.$$

Hence for all $\lambda > 0$ we have

$$\begin{aligned} &\mathsf{P}\left(\mathsf{M}'_{m+1} \geq \lambda\right) \leq \mathsf{P}\left(\max\{U_1 + D_1, U_2 + D_2\} \geq \lambda\right) \leq \\ &\quad \leq \mathsf{P}(U_1 \geq \lambda_U) + \mathsf{P}(U_2 \geq \lambda_U) + \mathsf{P}(D_1 \geq \lambda_D) + \mathsf{P}(D_2 \geq \lambda_D) \\ &\quad \leq 2\left(\frac{K}{2^{1+\alpha}}\lambda_U^{-\beta} + \lambda_D^{-\beta}\right)u^{1+\alpha}, \end{aligned}$$

where $\lambda_U + \lambda_D = \lambda$, $\lambda_U > 0$, $\lambda_D > 0$ are arbitrary chosen. Their optimal choice is a solution of optimization program

$$\begin{aligned} &\min\left\{\frac{K}{2^{1+\alpha}}\lambda_U^{-\beta} + \lambda_D^{-\beta} \,:\, \lambda_U + \lambda_D = \lambda,\ \lambda_U > 0,\ \lambda_D > 0\right\} = \\ &\quad = \lambda^{-\beta}\left(\left(\frac{K}{2^{1+\alpha}}\right)^{\delta} + 1\right)^{\frac{1}{\delta}}. \end{aligned}$$

Accordingly to (7.15) we know that for all $\lambda > 0$ we have

$$\begin{aligned} &\mathsf{P}\left(\mathsf{M}'_{m+1} \geq \lambda\right) \leq 2\left(\left(\frac{K}{2^{1+\alpha}}\right)^{\delta} + 1\right)^{\frac{1}{\delta}}\lambda^{-\beta}u^{1+\alpha} = \\ &\quad = K\left(\frac{1}{2^{\delta\alpha}} + \left(\frac{2}{K}\right)^{\delta}\right)^{\frac{1}{\delta}}\lambda^{-\beta}u^{1+\alpha} = K\lambda^{-\beta}u^{1+\alpha}. \end{aligned}$$

Optimal choice of K is a solution of (7.15) considered as an equality. It is given by the formula (7.14).

Q.E.D.

Now we introduce a theorem for distribution tails of $\mathsf{M}''_{\mathsf{N}}$ due to [2], Theorem 12.5., pp.140-143.

Theorem 7.6: *Let $u_1, u_2, \ldots, u_{\mathsf{N}}$ be nonnegative real numbers, $\alpha > 0$ and $\beta > 0$ such that for all $1 \le i \le h < j \le \mathsf{N}$ and $\lambda > 0$ we have*

$$\mathsf{P}\left(\left|\sum_{k=i}^{h} \xi_k\right| \ge \lambda, \ \left|\sum_{k=h+1}^{j} \xi_k\right| \ge \lambda\right) \le \lambda^{-\beta}\left(\sum_{k=i}^{j} u_k\right)^{1+\alpha}, \tag{7.16}$$

hence for all $\lambda > 0$ we have

$$\mathsf{P}\left(\mathsf{M}''_{\mathsf{N}} \ge \lambda\right) \le \mathsf{K}''(\alpha, \beta)\ \lambda^{-\beta}\left(\sum_{k=1}^{\mathsf{N}} u_k\right)^{1+\alpha}, \tag{7.17}$$

where $\mathsf{K}''(\alpha, \beta)$ is a convenient constant depending only on α and β.

Proof: For a proof see [2], Theorem 12.5., pp.140-143.

Q.E.D.

Now we introduce a theorem for distribution tails of M_{N} due to [2], Theorem 12.2., pp.134-135.

Theorem 7.7: *Let $u_1, u_2, \ldots, u_{\mathsf{N}}$ be nonnegative real numbers, $\alpha > 0$ and $\beta > 0$ such that for all $1 \le i < j \le \mathsf{N}$ and $\lambda > 0$ we have*

$$\mathsf{P}\left(\left|\sum_{k=i}^{j} \xi_k\right| \ge \lambda\right) \le \lambda^{-\beta}\left(\sum_{k=i}^{j} u_k\right)^{1+\alpha}, \tag{7.18}$$

hence for all $\lambda > 0$ we have

$$\mathsf{P}\left(\mathsf{M}_{\mathsf{N}} \ge \lambda\right) \le \mathsf{C}(\alpha, \beta)\lambda^{-\beta}\left(\sum_{k=1}^{\mathsf{N}} u_k\right)^{1+\alpha}, \tag{7.19}$$

where

$$\mathsf{C}(\alpha, \beta) = 2^{\beta}\left(1 + \mathsf{K}(\alpha, \beta)\right). \tag{7.20}$$

For example $\mathsf{C}(1, 4) \doteq 880353.402$.

Proof: Applying Schwartz's inequality and assumption (7.18) we are receiving

$$\mathsf{P}\left(\left|\sum_{k=i}^{h}\xi_k\right| \geq \lambda,\ \left|\sum_{k=h+1}^{j}\xi_k\right| \geq \lambda\right) \leq \sqrt{\mathsf{P}\left(\left|\sum_{k=i}^{h}\xi_k\right| \geq \lambda\right)\mathsf{P}\left(\left|\sum_{k=h+1}^{j}\xi_k\right| \geq \lambda\right)} \leq$$
$$\leq \sqrt{\lambda^{-\beta}\left(\sum_{k=i}^{h}u_k\right)^{1+\alpha}\lambda^{-\beta}\left(\sum_{k=h+1}^{j}u_k\right)^{1+\alpha}} \leq \lambda^{-\beta}\left(\sum_{k=i}^{j}u_k\right)^{1+\alpha}.$$

Assumptions of Theorem 7.5 are fulfilled, therefore, for all $\lambda > 0$ we have

$$\mathsf{P}\left(\mathsf{M}'_{\mathsf{N}} \geq \lambda\right) \leq \mathsf{K}(\alpha,\beta)\ \lambda^{-\beta}\left(\sum_{k=1}^{\mathsf{N}}u_k\right)^{1+\alpha}.$$

Moreover according to assumption (7.18), we have

$$\mathsf{P}\left(|\mathsf{S}_{\mathsf{N}}| \geq \lambda\right) \leq \lambda^{-\beta}\left(\sum_{k=1}^{\mathsf{N}}u_k\right)^{1+\alpha}.$$

Hence we are receiving according to Lemma 7.2 an estimate

$$\begin{aligned}
&\mathsf{P}\left(\mathsf{M}_{\mathsf{N}} \geq \lambda\right) \leq \mathsf{P}\left(\mathsf{M}'_{\mathsf{N}} + |\mathsf{S}_{\mathsf{N}}| \geq \lambda\right) \leq \\
&\quad \leq \mathsf{P}\left(\mathsf{M}'_{\mathsf{N}} \geq \frac{\lambda}{2}\right) + \mathsf{P}\left(|\mathsf{S}_{\mathsf{N}}| \geq \frac{\lambda}{2}\right) \leq \\
&\quad \leq \mathsf{K}(\alpha,\beta)\ \left(\frac{\lambda}{2}\right)^{-\beta}\left(\sum_{k=1}^{\mathsf{N}}u_k\right)^{1+\alpha} + \left(\frac{\lambda}{2}\right)^{-\beta}\left(\sum_{k=1}^{\mathsf{N}}u_k\right)^{1+\alpha} = \\
&\quad = 2^{\beta}(\mathsf{K}(\alpha,\beta)+1)\lambda^{-\beta}\left(\sum_{k=1}^{\mathsf{N}}u_k\right)^{1+\alpha}.
\end{aligned}$$

Q.E.D.

Having independent real random variables, more accurate and efficient maximal inequalities are known. We already presented one of them in Theorem 5.27.

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