Applied Stochastic Analysis

Lecture Notes

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Introduction

Let us begin with two short prologues which will serve as motivation as well as our starting point. Hopefully, these prologues will shed more light on how the theory of stochastic integration and stochastic differential equations came about and why the objects which we will deal with look the way they do.

Brownian motion

The story of what we now call the Brownian motion began with the Scottish botanist Robert Brown who observed the movement of pollen particles in liquid under a microscope. He noticed a strange, highly irregular motion of these particles; and in 1828, he published the pamphlet [1] where he reports on a series of experiments conducted to study this strange motion.

It was A. Einstein in his article [2] and independently M. von Smoluchowski in [6] who first gave an explanation of the motion observed by Brown. They both assumed that fluids are composed of particles that are in constant motion and, consequently, the motion of the pollen particle is the result of its collisions with the particles of the liquid. By considering a bulk of particles moving on a line, Einstein and Smoluchowski derived an equation for the number of particles per unit volume. More precisely, Einstein assumed that the movements of a single particle in different time intervals are independent (i.e. if $W_t(\omega)$ describes the position of the particle ω on the real line at time t, he assumed that the random process $(W_t)_{t\geq 0}$ has independent increments); that the average displacement of a large number of particles at a given time interval is zero (i.e. $\mathbb{E}(W_t - W_s) = 0$ for $t > s \geq 0$); and that higher moments are negligible on small time intervals (i.e. $\mathbb{E}(W_t - W_s)^n = o(\mathbb{E}(W_t - W_s)^2)$ for $t > s \geq 0$ and integer $n > 2$). Provided that a particle start from point x, Einstein found that the density of the probability distribution of its position at time t, $p(t, x, y)$, has to satisfy

$$\frac{\partial p}{\partial t}(t, x, y) = D\frac{\partial^2 p}{\partial y^2}(t, x, y) \tag{1}$$

where the number D is the mass diffusivity coefficient. In this way, Einstein and Smoluchowski described the movement of the particle by a random process which is determined by the probability density p. The equation (1) has a well-known solution

$$p(t, x, y) = \frac{1}{\sqrt{4\pi Dt}}\mathrm{e}^{-\frac{(y-x)^2}{4Dt}}$$

for $t \geq 0$ and $y \in \mathbb{R}$ which is the density of the normal distribution with mean x and variance $2Dt$.

Stochastic differential equations

Imagine that we wish to describe the time evolution of a certain scalar quantity x. This could be a stock price, the velocity of a moving object, and so on. It is natural to postulate something about the increments of the quantity x after a small time Δt and we assume that the dependence is linear with some constant f which may depend on the current time and current state. In mathematical language, we obtain the equation

$$x(t + \Delta t) - x(t) = f(t, x(t))\Delta t. \tag{2}$$

If we wish to know the difference of the value $x(T)$ at some terminal time T and the value at beginning of the experiment $x(0)$, we need to divide the interval $[0, T]$ into small segments $(\Delta t)_i = t_{i+1} - t_i$ and sum all the increments that satisfy the equation (2) over these small time intervals; that is, we obtain the equation

$$x(T) - x(0) = \sum_i f(t_i, x(t_i))(\Delta t)_i. \tag{3}$$

Of course, ultimately, we wish to know not only the difference of the terminal and the initial value of x, but also the value $x(T)$ itself. Equation (3) tells us that this is possible if we know the value $x(0)$; that is, equation (2) needs to come together with some initial condition $x(0) = x_0$.

Naturally, in many cases, the smaller the increments Δt, the more precise the model (2). Therefore, by dividing equation (2) by Δt and letting $\Delta t \to 0$, we obtain (under some technical assumptions) the ordinary differential equation

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0,$$

or, in the language of the final value $x(T)$, we obtain the integral equation

$$x(T) - x(0) = \int_0^T f(t, x(t))\,\mathrm{d}t \tag{4}$$

as a continuous-time analogue of equation (3).

To account for physical uncertainties, we may wish to add some random error to our model. Similarly as before, we assume that this random error will linearly depend on the time step Δt with some constant g, which may again depend on the current time and current state, and some random noise term N_t; that is, we obtain the equation

$$x(t+\Delta t) - x(t) = f(t, x(t))\Delta t + g(t, x(t)) N_t \Delta t, \quad x(0) = x_0. \tag{5}$$

What do we need to assume about the random variables N_t? Their purpose is to model some random, non-systematic error. Thus, a natural assumption about the collection of N_t's would be that they are independent and they all have the same probability distribution (say, with mean zero and unit variance) so that the character of the error will not change with time. Such a collection of random variables is called the *discreet white noise*. We are, however, interested in the case of continuous time so at the end of the day, the collection (N_t) will have to be indexed by a continuous parameter t. What would such random process need to satisfy? Similarly as in the case of discreet time, we would assume some normalization, that the character of the error does not change with time, and independence; that is we would like to assume that the random process $(N_t)_{t\geq 0}$ satisfies the following conditions:

- $\mathbb{E}\, N_t = 0$ and $\mathbb{E}\, N_t^2 = 1$ for every $t \geq 0$,
- that N_{t_1} is independent of N_{t_2} for every $t_1, t_2 \geq 0$ such that $t_1 \neq t_2$, and
- the process $(N_t)_{t\geq 0}$ is stationary.

However, as it turns out, such a process cannot exist as a measurable stochastic process, see [3]. The main reason is that the last assumption is too strong. Just imagine, we require that the random variables coming from the same stochastic process are independent no matter how close in time they are! This is a very important failure which suggests that it is necessary to significantly change the model (5).

To this end, let us assume that the randomness in the model comes from the *increments* of some other stochastic process W:

$$x(t+\Delta t) - x(t) = f(t, x(t))\Delta t + g(t, x(t))(W_{t+\Delta t} - W_t), \quad x(0) = x_0. \tag{6}$$

In order for equation (6) to model the same situation as we tried to model by equation (5), we need to make the same assumptions about the increments of W as we did about the process N. That is, we assume that the stochastic process $(W_t)_{t\geq 0}$ satisfies the following:

- $W_0 = 0$ and the process W is continuous,
- $\mathbb{E}(W_t - W_s) = 0$ and $\mathbb{E}(W_t - W_s)^2 = |t - s|$ for every $t, s \geq 0$,
- the increments of W are independent, and
- the increments of W are stationary.

It turns out that such a process W does indeed exist and it is the celebrated *Wiener process.* With this process in hand, we will now shift our attention back to equation (6). Diving it by Δt and taking the limit $\Delta t \to 0$ yields, at least formally, the equation

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))\dot{W}_t, \quad x(0) = x_0. \tag{7}$$

Unfortunately, this is where we fail again. It turns out that the derivative of the Wiener process, $\dot{W}$, does not exist, see [5]. This should not be surprising because if it existed, it would be exactly the white noise process N.

Let us look at deterministic ordinary differential equations. Notice that equation (2) is essentially equivalent to the integral equation (4). So what if, instead of considering the differential equation (7) which does not make sense, we considered the problem in its integral form? The intuition why this could work is that the integral is usually a much nicer map than the derivative - where the derivative reduces smoothness, the integral adds it. With this approach, we would obtain the equation

$$X(T) - x(0) = \int_0^T f(t, x(t))\,\mathrm{d}t + \int_0^T g(t, x(t))\,\mathrm{d}W_t$$

where we use the integral symbol to denote that the object

$$\int_0^T g(t, x(t))\,\mathrm{d}W_t$$

arises as some reasonable limit of the sums

$$\sum_i g(t_i, x(t_i))(W_{t_{i+1}} - W_{t_i}).$$

A classical object which makes such a limiting procedure possible is the Lebesgue - Stieltjes integral. The Lebesgue-Stieltjes integral, however, requires that the integrator has bounded variation and, at this point, we should not even be surprised that this is not the case of the Wiener process. Hence, the integral has to be defined in a completely different manner.

1 Preliminaries

In this section, we will recall some preliminary results on the Wiener process, stochastic integrals, and stochastic differential equations. We begin with the definition of a Wiener process.

Definition 1. A stochastic process $W = (W_t)_{t\geq 0}$ on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a *Wiener process*, if the following conditions are satisfied:

- $W_0 = 0$ $\mathbb{P}$-almost surely;
- W has continuous sample paths, i.e. for $\mathbb{P}$-almost every $\omega \in \Omega$, the map $t \mapsto W_t(\omega)$ is continuous;
- W has independent increments, i.e. for every $n \in \mathbb{N}$ and every $0 \leq t_1 < t_2 < \ldots < t_n$, the random variables
$$W_{t_n} - W_{t_{n-1}}, W_{t_{n-1}} - W_{t_{n-2}}, \ldots, W_{t_2} - W_{t_1}$$
are independent;
- for every $0 \leq s < t$, the random variable $W_t - W_s$ is Gaussian with mean zero and variance $t - s$.

The first important fact about the Wiener process is that it does exist (this fact is not trivial - just recall the attempt to construct the continuous-time white noise in the introduction). Some properties of its trajectories are collected in the following proposition.

Proposition 1. *Let $(W_t)_{t\geq 0}$ be a Wiener process defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. It holds that for almost every $\omega \in \Omega$, the sample path $t \mapsto W_t(\omega)$ is*

1. *ε-Hölder continuous for every $\varepsilon \in (0, 1/2)$,*

2. nowhere differentiable, and

3. of infinite variation on every subinterval of $[0, \infty)$.

Of course, the Wiener process is adapted to the filtration it generates. However, sometimes, it can be adapted to other filtrations and in such case, it may be useful to specify this explicitly.

Definition 2. Let W be a Wiener process defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. If $(\mathscr{F}_t)_{t\geq 0}$ is a filtration, W is called an *$(\mathscr{F}_t)$-Wiener process* if W is $(\mathscr{F}_t)$-adapted and $\sigma(W_t - W_s)$ is independent of $\mathscr{F}_s$ for every $t > s$.

1.1 Stochastic integral

In this section, we recall the definition of the stochastic integral with respect to a Wiener process. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and assume that the filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfies the usual conditions[1].

Let W be an $(\mathscr{F}_t)$-Wiener process and let $0 \leq a < b$. Denote by $\mathscr{E}_{\mathscr{F}}(a,b)$ the set of $(\mathscr{F}_t)$-adapted elementary random processes on the interval $[a,b]$. That is, for $Y \in \mathscr{E}_{\mathscr{F}}(a,b)$, there exists a finite partition $a = t_0 < t_1 < \ldots < t_n = b$ of the interval $[a,b]$ and a finite set $\{Y^{(i)}\}_{i=0}^n \subset L^2(\Omega)$ with every Y^i is $\mathscr{F}_{t_i}$-measurable such that the process Y can be written in the form

$$Y_t = \sum_{i=0}^{n-1} Y^{(i)} \mathbf{1}_{[t_i, t_{i+1})}(t) \tag{8}$$

for every $t \in [0,T]$. Let us define the map $I_{a,b} : \mathscr{E}_{\mathscr{F}}(a,b) \to L^2(\Omega)$ by

$$\sum_{i=0}^{n-1} Y^{(i)} \mathbf{1}_{[t_i, t_{i+1})} = Y \quad \longmapsto \quad I_{a,b}(Y) := \sum_{i=0}^{n-1} Y^{(i)} (W_{t_{i+1}} - W_{t_i}).$$

Clearly, the operator $I_{a,b} : (\mathscr{E}_{\mathscr{F}}(a,b), \|\cdot\|_{L^2((0,T)\times\Omega)}) \to L^2(\Omega)$ is linear and it is not difficult to show that it is also an isometry, i.e. for every $Y \in \mathscr{E}_{\mathscr{F}}(a,b)$, we have that the following equality holds:

$$\|I_{a,b}(Y)\|_{L^2(\Omega)} = \mathbb{E}\,|I_{a,b}(Y)|^2 = \int_a^b \mathbb{E}\,|Y_s|^2 \, \mathrm{d}s = \|Y\|_{L^2((a,b)\times\Omega)}.$$

In particular, the operator $I_{a,b}$ is continuous. Furthermore, it can be shown that the space $\mathscr{E}_{\mathscr{F}}(a,b)$ is dense in the subspace of $L^2((a,b)\times\Omega)$ consisting of $(\mathscr{F}_t)$-adapted stochastic processes. This subspace will be denoted by $L^2_{\mathscr{F}}(a,b)$. Hence, there exists a unique extension of $I_{a,b}$ (which will be again denoted by $I_{a,b}$) to the space $L^2_{\mathscr{F}}(a,b)$. In other words, for every $Y \in L^2_{\mathscr{F}}(a,b)$ there exists a sequence $\{Y^n\}_{n\in\mathbb{N}} \subset \mathscr{E}_{\mathscr{F}}(a,b)$ such that $Y^n \to Y$ in $L^2((a,b)\times\Omega)$. Since $I_{a,b}$ is continuous, the sequence $\{I(Y^n)\}_{n\in\mathbb{N}}$ is Cauchy in $L^2(\Omega)$ and since $L^2(\Omega)$ is a complete space, the sequence $\{I(Y^n)\}_{n\in\mathbb{N}}$ has a limit there, i.e. there exists a $Z \in L^2(\Omega)$ such that $I(Y^n)$ tends to Z in $L^2(\Omega)$ and we define $I_{a,b}(Y) := Z$.

Definition 3. For a stochastic process $Y \in L^2_{\mathscr{F}}(a,b)$, the random variable $I_{a,b}(Y)$ constructed above is called the *stochastic integral of Y with respect to the Wiener process W on the interval* (a,b).

[1]The usual conditions are that $\mathbb{P}$-null sets are contained in $\mathscr{F}_0$ and that the filtration is right continuous. In most cases, this is only a technical assumption that simplifies proofs.

If $a = 0$, then we will simply use $I_b(Y)$ instead of $I_{0,b}(Y)$. We will also use the notation

$$\int_a^b Y_s \, \mathrm{d}W_s := I_{a,b}(Y).$$

whenever necessary. The stochastic integral has many useful properties.

Proposition 2 (Integral as a random variable)**.** *Let $T > 0$ and let Y be a stochastic process that belongs to the space $L^2_{\mathscr{F}}(a, b)$.*

1. *It holds that*

$$\mathbb{E} \int_0^T Y_s \, \mathrm{d}W_s = 0 \quad \text{and} \quad \mathbb{E} \left| \int_0^T Y_s \, \mathrm{d}W_s \right|^2 = \int_0^T \mathbb{E} \, |Y_s|^2 \, \mathrm{d}s.$$

2. *The stochastic integral is additive, i.e. for $u \in [0, T]$, we have that*

$$\int_0^T Y_s \, \mathrm{d}W_s = \int_0^u Y_s \, \mathrm{d}W_s + \int_u^T Y_s \, \mathrm{d}W_s$$

   holds $\mathbb{P}$-almost surely.

3. *The stochastic integral is linear, i.e. for $c_1, c_2 \in \mathbb{R}$ and stochastic processes $Y^{(1)}, Y^{(2)}$ that belong to the space $L^2_{\mathscr{F}}(a, b)$, we have that*

$$\int_0^T (c_1 Y_s^{(1)} + c_2 Y_s^{(2)}) \, \mathrm{d}W_s = c_1 \int_0^T Y_s^{(1)} \, \mathrm{d}W_s + c_2 \int_0^T Y_s^{(2)} \, \mathrm{d}W_s$$

   holds $\mathbb{P}$-almost surely.

Proposition 3 (Integral as a process)**.** *Let $T > 0$ and Y be a stochastic process that belongs to the space $L^2_{\mathscr{F}}(a, b)$. The process $(I_t(Y))_{t \in [0,T]}$ is*

1. *an $(\mathscr{F}_t)$-martingale, i.e. it is an $(\mathscr{F}_t)$-adapted process such that $\mathbb{E} \, |X_t| < \infty$ for every $t \in [0, T]$ and such that the equality*

$$\mathbb{E}[I_t(Y) | \mathscr{F}_s] = I_s(Y)$$

   holds $\mathbb{P}$-almost surely for every $s, t \in [0, T]$, $s \leq t$; and

2. *it has a version with continuous sample paths.*

All the notions defined above, can be generalized into higher dimensions. Let $n \in \mathbb{N}$. An *n-dimensional Wiener process* is the $\mathbb{R}^n$-valued stochastic process $(\boldsymbol{W}_t)_{t \geq 0}$ where $\boldsymbol{W}_t = (W_t^1, W_t^2, \ldots, W_t^n)^*$ where $(W_t^i)_{t \geq 0}$ are mutually independent one-dimensional Wiener processes. The *n-dimensional $(\mathscr{F}_t)$-Wiener process* is defined similarly as before. Now, once can generalize the space $L^2_{\mathscr{F}}(a, b)$ as follows. For a Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$, the space $L^2_{\mathscr{F}}(a, b; H)$ is space of stochastic processes $Y \in L^2((a, b) \times \Omega; H)$ such that Y is $(\mathscr{F}_t)$-adapted. Then for $\boldsymbol{Y} = (Y^{i,j})_{i,j=1}^{m,n} \in L^2_{\mathscr{F}}(a, b; \mathbb{R}^{m \times n})$, the stochastic integral is defined by

$$\int_a^b \boldsymbol{Y}_s \, \mathrm{d}\boldsymbol{W}_s := \left(\sum_{i=1}^n \int_a^b Y_s^{1,j} \, \mathrm{d}W_s^j, \sum_{i=1}^n \int_a^b Y_s^{2,j} \, \mathrm{d}W_s^j, \ldots, \sum_{i=1}^n \int_a^b Y_s^{m,j} \, \mathrm{d}W_s^j \right).$$

Such a random variable, clearly belongs to the space $L^2(\Omega;\mathbb{R}^m)$, and Proposition 2 and Proposition 3 still hold. We remark, that the norm that we consider on the space of matrices $\mathbb{R}^{m\times n}$ is *the Hilbert-Schmidt norm* defined by

$$\|\boldsymbol{A}\|_{\mathbb{R}^{m\times n}} := \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{i,j}|^2\right)^{\frac{1}{2}}$$

for $A=(a_{i,j})\in\mathbb{R}^{m\times n}$.

In what follows, we define a stochastic differential of a random process and formulate a stochastic version of the chain rule, the Itô formula.

Definition 4. Let $\boldsymbol{W}$ is an n-dimensional $(\mathscr{F}_t)$-Wiener process. We say that an $(\mathscr{F}_t)$-adapted $\mathbb{R}^m$-valued stochastic process $(\boldsymbol{X}_t)_{t\in[0,T]}$ has *a stochastic differential* if there exist $(\mathscr{F}_t)$-progressively measurable stochastic processes $\boldsymbol{a}:\mathbb{R}_+\times\Omega\to\mathbb{R}^m$ and $\boldsymbol{b}:\mathbb{R}_+\times\Omega\to\mathbb{R}^{m\times n}$ such that

$$\mathbb{E}\int_0^t \left(\|\boldsymbol{a}_s\|+\|\boldsymbol{b}_s\|^2\right)\,\mathrm{d}s<\infty$$

holds for every $t\geq 0$ and the equality

$$\boldsymbol{X}_t=\boldsymbol{X}_0+\int_0^t \boldsymbol{a}_s\,\mathrm{d}s+\int_0^t \boldsymbol{b}_s\,\mathrm{d}\boldsymbol{W}_s \tag{9}$$

holds $\mathbb{P}$-almost surely for every $t\geq 0$.

Remark 1. Formally, equality (9) is usually written as

$$\mathrm{d}\boldsymbol{X}_t=\boldsymbol{a}_t\,\mathrm{d}t+\boldsymbol{b}_t\,\mathrm{d}\boldsymbol{W}_t.$$

However, it should be stressed that the above equation is purely symbolic and it should be interpreted as the integral equation (9).

Before we state the Itô formula, denote by $\mathscr{C}^{1,2}(\mathbb{R}_+\times\mathbb{R}^m)$ the space of all functions φ such that $\varphi\in\mathscr{C}^1(\mathbb{R}\times\mathbb{R}^m)$ and $\varphi(t,\cdot)\in\mathscr{C}^2(\mathbb{R}^m)$ for every $t\geq 0$.

Theorem 1 (Itô formula)**.** *Let $\boldsymbol{X}$ be an $\mathbb{R}^m$-valued stochastic process that has the stochastic differential*

$$\boldsymbol{X}_t=\boldsymbol{X}_0+\int_0^t \boldsymbol{a}_s\,\mathrm{d}s+\int_0^t \boldsymbol{b}_s\,\mathrm{d}\boldsymbol{W}_s$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy the conditions in Definition 4. Let $f\in\mathscr{C}^{1,2}(\mathbb{R}_+\times\mathbb{R}^m)$. Then the formula

$$\begin{aligned} f(t,\boldsymbol{X}_t)=f(0,\boldsymbol{X}_0)+\int_0^t\Bigg[\frac{\partial f}{\partial s}(s,\boldsymbol{X}_s)+\sum_{i=1}^{m}\frac{\partial f}{\partial x_i}(s,\boldsymbol{X}_s)a_s^i+\\ +\frac{1}{2}\sum_{r=1}^{n}\sum_{i,j=1}^{m}\frac{\partial^2 f}{\partial x_i\partial x_j}(s,\boldsymbol{X}_s)b_s^{i,r}b_s^{j,r}\Bigg]\,\mathrm{d}s\\ +\int_0^t\sum_{r=1}^{n}\sum_{i=1}^{m}\frac{\partial f}{\partial x_i}(s,\boldsymbol{X}_s)b_s^{i,r}\,\mathrm{d}\boldsymbol{W}_s \end{aligned} \tag{10}$$

holds $\mathbb{P}$-almost surely for every $t\geq 0$.

Remark 2. In the special case when $m = n = 1$, we have that formula (10) reads as follows:

$$f(t, X_t) = f(0, X_0) + \int_0^t \left[\frac{\partial f}{\partial s}(s, X_s) + a_s \frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, X_s) \right] \mathrm{d}s$$
$$+ \int_0^t b_s \frac{\partial f}{\partial x}(s, X_s) \, \mathrm{d}W_s.$$

1.2 Stochastic differential equations

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a filtered probability space with an n-dimensional $(\mathscr{F}_t)$-Wiener process $(\boldsymbol{W}_t)_{t \geq 0}$ defined on it. Let $T > 0$ and consider the following *stochastic differential equation*:

$$\mathrm{d}\boldsymbol{X}_t = b(t, \boldsymbol{X}_t) \, \mathrm{d}t + \boldsymbol{\sigma}(t, \boldsymbol{X}_t) \, \mathrm{d}\boldsymbol{W}_t \tag{11}$$
$$\boldsymbol{X}_0 = \boldsymbol{\psi} \tag{12}$$

where $\boldsymbol{b} : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\boldsymbol{\sigma} : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m \times n}$ are some Borel measurable functions and $\boldsymbol{\psi} : \Omega \to \mathbb{R}^m$ is a random variable.

Definition 5. An $\mathbb{R}^m$-valued $(\mathscr{F}_t)$-progressively measurable stochastic process $(\boldsymbol{X}_t)$ is called the *solution* to the stochastic differential equation (11) on the interval $[0, T]$ with the initial condition (12) if it holds that

$$\mathbb{E} \int_0^T \left(\|\boldsymbol{b}(s, \boldsymbol{X}_s)\| + \|\boldsymbol{\sigma}(s, \boldsymbol{X}_s)\|^2 \right) \mathrm{d}s < \infty$$

$\mathbb{P}$-almost surely; and if the process $\boldsymbol{X}$ satisfies

$$\boldsymbol{X}_t = \boldsymbol{\psi} + \int_0^t \boldsymbol{b}(s, \boldsymbol{X}_s) \, \mathrm{d}s + \int_0^t \boldsymbol{\sigma}(s, \boldsymbol{X}_s) \, \mathrm{d}\boldsymbol{W}_s$$

for every $t \in [0, T]$ $\mathbb{P}$-almost surely.

Definition 6. We say that a solution $\boldsymbol{X}$ to the stochastic differential equation (11) with the initial condition (12) is *unique* if for another such solution $\tilde{\boldsymbol{X}}$ it holds that

$$\mathbb{P} \left(\boldsymbol{X}_t = \tilde{\boldsymbol{X}}_t \text{ for all } t \in [0, T] \right) = 1.$$

It turns out that a solution to the problem (11), (12) exists under some (rather strong) conditions on the *drift* $\boldsymbol{b}$ and *diffusion* $\boldsymbol{\sigma}$ coefficients.

Theorem 2. *Let $\boldsymbol{b} : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\boldsymbol{\sigma} : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m \times n}$ be Borel measurable functions that satisfy the following growth conditions:*

1. *There exists a finite constant K_1 such that for every $t \in [0, T]$ and every $\boldsymbol{x} \in \mathbb{R}^m$ it holds that*
$$\max \{ \|\boldsymbol{b}(t, \boldsymbol{x})\|, \|\boldsymbol{\sigma}(t, \boldsymbol{x})\| \} \leq K \left(1 + \|\boldsymbol{x}\| \right).$$

2. *There exists a finite constant K_2 such that for every $t \in [0, T]$ and every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ it holds that*
$$\max \{ \|\boldsymbol{b}(t, \boldsymbol{x}) - \boldsymbol{b}(t, \boldsymbol{y})\|, \|\boldsymbol{\sigma}(t, \boldsymbol{x}) - \boldsymbol{\sigma}(t, \boldsymbol{y})\| \} \leq K_2 \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Then for every $\mathscr{F}_0$-measurable initial condition $\boldsymbol{\psi}$ such that $\mathbb{E} \|\boldsymbol{\psi}\|^2 < \infty$ there exists a unique solution $\boldsymbol{X}^{\boldsymbol{\psi}}$ to the problem (11), (12). Moreover, for every $p \geq 0$, there exists a finite constant $C \equiv C_{p,T,K_1}$ such that

$$\mathbb{E} \sup_{t \in [0,T]} \|\boldsymbol{X}_t^{\boldsymbol{\psi}}\|^p \leq C \left(1 + \mathbb{E}\|\boldsymbol{\psi}\|^p \right).$$

1.2.1 Intermezzo: Linear ODEs

Before we consider linear stochastic differential equations, let us make a small detour and recall some facts about deterministic linear differential equations. Consider the (inhomogeneous) linear differential equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{b}(t) \tag{13}$$

where $\boldsymbol{A} \in L^1(0, T; \mathbb{R}^{m\times m})$ and $\boldsymbol{b} \in L^1(0, T; \mathbb{R}^m)$.

Definition 7. A *(Carathéodory) solution* to equation (13) is a measurable function $\boldsymbol{x} : [0, T] \to \mathbb{R}^m$ such that $\boldsymbol{A}(\cdot)\boldsymbol{x}(\cdot) \in L^1(0, T; \mathbb{R}^m)$ and such that it satisfies the equation

$$\boldsymbol{x}(t) = \boldsymbol{x}(s) + \int_s^t \boldsymbol{A}(u)\boldsymbol{x}(u)\,\mathrm{d}u + \int_s^t \boldsymbol{b}(u)\,\mathrm{d}u$$

for every $s, t \in [0, T]$. The integrals in the above equality are defined in the sense of Lebesgue. Moreover, if a solution $\boldsymbol{x}$ to equation (13) satisfies

$$\boldsymbol{x}(s) = \boldsymbol{y} \tag{14}$$

for some fixed $s \in [0, T]$ and $\boldsymbol{y} \in \mathbb{R}^m$, then we say that $\boldsymbol{x}$ is a solution of (13) *with the initial condition* (14).

As far as the existence (and uniqueness) of the solution to the problem (13), (14) is concerned, we have the following well-known result.

Theorem 3. *For every $s \in [0, T]$ and every $\boldsymbol{y} \in \mathbb{R}^m$ there exists a unique solution $\boldsymbol{x}^{\boldsymbol{y}}$ to equation* (13) *with initial condition* (14).

In order to find an explicit formula to the solution, we will now briefly shift our attention to the corresponding *homogeneous* equation. That is, consider now the equation

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}(t)\boldsymbol{z}(t) \tag{15}$$

and fix an arbitrary $s \in [0, T]$. The *fundamental matrix* $\boldsymbol{\Phi}_s(t)$ is an absolutely continuous $\mathbb{R}^{m\times m}$-valued function defined on the interval $[0, T]$ that satisfies the equation

$$\dot{\boldsymbol{\Phi}}_s(t) = \boldsymbol{A}(t)\boldsymbol{\Phi}_s(t)$$

for almost every $t \in [0, T]$ and $\boldsymbol{\Phi}_s(s) = \boldsymbol{I}_m$.

Remark 3. For example, in dimension one ($m = 1$), we have that Φ_s takes the form

$$\Phi_s(t) = \mathrm{e}^{A(t-s)}$$

if $A \in \mathbb{R}$ is constant and, more generally,

$$\Phi_s(t) = \mathrm{e}^{\int_s^t A(u)\,\mathrm{d}u}$$

if A is an (essentially) bounded function. In higher dimensions ($m > 1$), the situation is not entirely analogous. In particular, if $\boldsymbol{A}$ is a constant matrix, then the fundamental matrix is formed by a matrix exponential as in the one-dimensional case, i.e.

$$\boldsymbol{\Phi}_s(t) = \mathrm{e}^{\boldsymbol{A}(t-s)}.$$

However, if $\boldsymbol{A}$ is an (essentially) bounded matrix-valued function, then the formula

$$\boldsymbol{\Phi}_s(t) = \mathrm{e}^{\int_s^t \boldsymbol{A}(u)\,\mathrm{d}u} \tag{16}$$

only holds if $\boldsymbol{A}(u)\boldsymbol{A}(v) = \boldsymbol{A}(v)\boldsymbol{A}(u)$ for every $u, v \in [0, T]$. If the system $(\boldsymbol{A}(t), t \in [0, T])$ is not commutative in this sense, then formula (16) need not hold.

In any case, however, if $\mathbf{\Phi}_s$ is a fundamental matrix of equation (15), then its inverse function $\mathbf{\Phi}_s^{-1}$ exists and is also absolutely continuous on $[0,T]$. Using this fact, we can now give the explicit solution to the problem (13), (14). The formula is generally known as the *variation of constants* formula.

Theorem 4. *The unique solution to equation* (13) *with initial condition* (14) *takes the form*

$$\boldsymbol{x}^{\boldsymbol{y}}(t) = \mathbf{\Phi}_s(t)\boldsymbol{y} + \int_s^t \mathbf{\Phi}_s(t)\mathbf{\Phi}_s^{-1}(r)\boldsymbol{b}(r)\,\mathrm{d}r, \quad t \in [0,T],$$

where $\mathbf{\Phi}_s$ is the fundamental matrix to the corresponding homogeneous equation.

1.2.2 Linear SDEs

Apart from a few cases (see e.g. Exercise 6), we will not be able to find explicit solutions to stochastic differential equations. However, in the case of linear stochastic differential equations, it turns out that the solution can be found using a variation of constants formula not unlike the one in Theorem 4.

On the interval $[0,T]$, consider the *linear* stochastic differential equation

$$\begin{cases} \mathrm{d}\boldsymbol{X} &= [\boldsymbol{A}(t)\boldsymbol{X}_t + \boldsymbol{b}(t)]\,\mathrm{d}t + \boldsymbol{\sigma}(t)\,\mathrm{d}\boldsymbol{W}_t, \\ \boldsymbol{X}_0 &= \boldsymbol{\psi} \end{cases} \tag{17}$$

where $\boldsymbol{A} \in L^1(0,T;\mathbb{R}^{m\times m})$, $\boldsymbol{b} \in L^1(0,T;\mathbb{R})$ and $\boldsymbol{\sigma} \in L^2(0,T;\mathbb{R}^{m\times m})$ and where the initial condition $\boldsymbol{\psi}$ is $\mathscr{F}_0$-measurable.

Clearly, equation (17) is a particular case of the general problem (11), (12). We have the following explicit representation of the solution.

Theorem 5. *The stochastic process $(\boldsymbol{X}_t^{\boldsymbol{\psi}})_{t\in[0,T]}$ defined by*

$$\boldsymbol{X}_t^{\boldsymbol{\psi}} = \mathbf{\Phi}(t)\boldsymbol{\psi} + \mathbf{\Phi}(t)\int_0^t \mathbf{\Phi}^{-1}(s)\boldsymbol{b}(s)\,\mathrm{d}s + \mathbf{\Phi}(t)\int_0^t \mathbf{\Phi}^{-1}(s)\boldsymbol{\sigma}(s)\,\mathrm{d}\boldsymbol{W}_s,$$

where $\mathbf{\Phi} \equiv \mathbf{\Phi}_0$ is the fundamental solution to the homogeneous equation (15)*, is the unique solution to equation* (17)*.*

1.3 Exercises

Exercise 1. Compute the integral $\int_0^T W_s\,\mathrm{d}W_s$. Compare the result with the formula for $\int_0^T \varphi(s)\,\mathrm{d}\varphi(s)$ for a smooth function φ such that $\varphi(0) = 0$.

Exercise 2. Let $(X_t)_{t\geq 0}$, $X_0 \neq 0$, and $(Y_t)_{t\geq 0}$ have the stochastic differentials

$$\begin{aligned} \mathrm{d}X_t &= -\sin\Phi_t\,\mathrm{d}W_t \\ \mathrm{d}Y_t &= \cos\Phi_t\,\mathrm{d}W_t \end{aligned}$$

for $t > 0$ where the process $(\Phi_t)_{t\geq 0}$ is defined by $\Phi_t = \arctan(Y_t/X_t)$. Prove that the formula

$$X_t^2 + Y_t^2 = X_0^2 + Y_0^2 + t$$

holds for every $t \geq 0$.

Exercise 3 (Ornstein-Uhlenbeck process)**.** Let $(X_t)_{t\geq 0}$ have the following stochastic differential

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + B\,\mathrm{d}W_t$$

for $t > 0$ where $A, B \in \mathbb{R}$ are constants and x is a random variable.

1. By using the Itô formula on the process X and the function f defined by $f(t,x) = \mathrm{e}^{-At}x$, show that the process X admits the explicit formula

$$X_t = \mathrm{e}^{At}x + \int_0^t \mathrm{e}^{A(t-s)}\,\mathrm{d}W_s, \quad t \geq 0.$$

2. Show the same claim by using the stochastic Fubini theorem: Let $T > 0$ and let $h : [0,T]^2 \to \mathbb{R}$ be a Borel measurable function such that

$$\int_0^T \int_0^T h(x,y)^2\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Then the following holds almost surely:

$$\int_0^T \left(\int_0^T h(s,t)\,\mathrm{d}W_s \right) \mathrm{d}t = \int_0^T \left(\int_0^T h(s,t)\,\mathrm{d}t \right) \mathrm{d}W_s.$$

3. Show that $\mathbb{E}\,X_t = \mathrm{e}^{At}$ for every $t \geq 0$. Moreover, show that if the initial condition x is uncorrelated with the increments $(W_t - W_s)$ for $0 \leq s < t$, then

$$\operatorname{Var} X_t = \mathrm{e}^{2At}\left(\operatorname{Var} x + \frac{B^2}{2A}\right) - \frac{B^2}{2A}, \quad t \geq 0.$$

4. Show that if $x \sim \mathrm{N}(0, \sigma_\infty)$ where $\sigma_\infty = -B^2/(2A)$, then $X_t \overset{D}{=} X_0$ for every $t \geq 0$. Show that in this case, $(X_t)_{t\geq 0}$ is a strictly stationary stochastic process.

5. Show that if $A < \infty$ and $x \sim \mathrm{N}(\mu, \sigma^2)$ with some $\mu > 0$ and $\sigma^2 > 0$, then

$$X_t \xrightarrow[n\to\infty]{D} X_0.$$

Exercise 4 (Geometric Brownian motion)**.** Let $(X_t)_{t\geq 0}$ have the following stochastic differential

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + BX_t\,\mathrm{d}W_t$$

for $t > 0$ and $X_0 = x$ with $x > 0$ deterministic.

1. By using the Itô formula on the process X and the function f defined by $f(t,x) = g(t)\mathrm{e}^{Bx}$ where g is some smooth function, show that the process $(X_t)_{t\geq 0}$ admits the explicit formula

$$X_t = x\mathrm{e}^{(A-\frac{1}{2}B^2)t + BW_t}, \quad t \geq 0.$$

2. Let $p \geq 1$. By appealing to the probability distribution of W_t, show that the moments of X_t satisfy

$$\mathbb{E}\,X_t^p = x^p \mathrm{e}^{p(A+\frac{p-1}{2}B^2)t}$$

3. Show the same claim by using the Itô formula.

4. Show that if $A - B^2/2 < 0$, then $\lim_{t\to\infty} X_t = 0$ almost surely.

Exercise 5. Let $(X_t)_{t\geq 0}$, $X_0 = 0$, and $(Y_t)_{t\geq 0}$, $Y_0 = 1$, have the following stochastic differentials

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2} X_t \,\mathrm{d}t + Y_t \,\mathrm{d}W_t \\ \mathrm{d}Y_t &= -\frac{1}{2} Y_t \,\mathrm{d}t - X_t \,\mathrm{d}W_t \end{aligned}$$

for $t > 0$. Show that $X_t = \sin W_t$ and $Y_t = \cos W_t$ for every $t \geq 0$.

Hint: If $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{2\times 2}$ *commute, then the solution* $(\boldsymbol{X}_t)_{t\geq 0}$ *to*

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{A}\boldsymbol{X}_t \,\mathrm{d}t + \boldsymbol{B}\boldsymbol{X}_t \,\mathrm{d}W_t, \quad t > 0,$$

with $\boldsymbol{X}_0 = \boldsymbol{x_0} \in \mathbb{R}^2$ *takes the form*

$$\boldsymbol{X}_t = \mathrm{e}^{\boldsymbol{B}W_t} \mathrm{e}^{(\boldsymbol{A} - \frac{1}{2}\boldsymbol{B}^2)t} \boldsymbol{x_0}, \quad t \geq 0,$$

where $\mathrm{e}^{\boldsymbol{M}}$ *denotes the matrix exponential of the matrix* $\boldsymbol{M}$ *defined by*

$$\mathrm{e}^{\boldsymbol{M}} := \sum_{n=0}^{\infty} \frac{\boldsymbol{M}^n}{n!}.$$

Exercise 6. Let $a, b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be such that $b \in \mathscr{C}^{1,1}(\mathbb{R}_+ \times \mathbb{R})$ and let $u \in \mathscr{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ satisfy

$$\begin{aligned} &\frac{\partial u}{\partial t}(t, y) = a(t, u) - \frac{1}{2} b(t, u) \frac{\partial b}{\partial x}(t, u) \\ &\frac{\partial u}{\partial x}(t, y) = b(t, u) \end{aligned}$$

with the initial condition $u(0, 0) = x$, $x \in \mathbb{R}$. Show that the process $(X_t)_{t\geq 0}$ defined by $X_t = u(t, W_t)$ satisfies the stochastic differential equation

$$\mathrm{d}X_t = a(t, X_t) \,\mathrm{d}t + b(t, X_t) \,\mathrm{d}W_t$$

for $t > 0$ with $X_0 = x$ almost surely.

Exercise 7. Let $(X_t)_{t\geq 0}$ be the solution of the stochastic differential equation

$$\mathrm{d}X_t = \frac{1}{4} \,\mathrm{d}t + \sqrt{X_t} \,\mathrm{d}W_t$$

for every $t > 0$ with $X_0 = x$ for some deterministic $x \geq 0$. By means of Exercise 6, show that the formula

$$X_t = \frac{1}{4}(W_t + \sqrt{x})^2$$

holds for every $t \geq 0$.

2 Stochastic optimal control

In this section, the so-called *stochastic optimal control* problems will be considered. Suppose that we are interested in modelling the output $\boldsymbol{X}$ of a system whose dynamics is governed by a stochastic differential equation (as discussed in the introduction, stochastic differential equations can be, for example, used to model the dynamics of physical systems that are subject to some random perturbations). Suppose however, that we are able to *control* the equation, i.e. we are able to influence the output by modifying the drift or diffusion coefficient of the equation at each time t by choosing some parameter $\boldsymbol{u}_t$. The process $\boldsymbol{u}$ is called the *control*. There may be many possibilities how to control the equation and our task is to select the best possible control from all the controls that we are permitted to use. That is, each control $\boldsymbol{u}$ is assigned a number $J(\boldsymbol{u})$, called the *cost*, that penalizes undesirable controls by being large. The goal is then to find such control among all *admissible controls* (the set admissible controls will be denoted by $\mathscr{U}$ in the sequel) whose cost is minimal.

Crucial parts part of this Section follow analogous results in [7] and [8].

More precisely, let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with an n-dimensional $(\mathscr{F}_t)$-Wiener process $(\boldsymbol{W}_t)_{t\geq 0}$ defined on it. Let $T > 0$ and $U \subseteq \mathbb{R}^k$, and consider the following (controlled) stochastic differential equation:

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{b}(t, \boldsymbol{X}_t, \boldsymbol{u}_t)\,\mathrm{d}t + \boldsymbol{\sigma}(t, \boldsymbol{X}_t, \boldsymbol{u}_t)\,\mathrm{d}\boldsymbol{W}_t, \quad \boldsymbol{X}_0 = \boldsymbol{y}, \tag{18}$$

where $\boldsymbol{b} : [0, T] \times \mathbb{R}^m \times U \to \mathbb{R}^m$ and $\boldsymbol{\sigma} : [0, T] \times \mathbb{R}^m \times U \to \mathbb{R}^{m\times n}$ are measurable functions, and $\boldsymbol{y} : \Omega \to \mathbb{R}^m$ is an $\mathscr{F}_0$-measurable random variable. The process $(\boldsymbol{u}_t)_{t\in[0,T]}$ is an admissible control, that is $\boldsymbol{u} \in \mathscr{U}$ where the set $\mathscr{U}$ is a subset[2] of U-valued $(\mathscr{F}_t)$-progressively measurable stochastic processes for which the equation (18) admits a unique solution.

Remark 4. The controlled equation (18) is does not belong to the class of stochastic differential equations of the form (11) because the coefficients $\boldsymbol{b}$ and $\boldsymbol{\sigma}$ depend on $\omega \in \Omega$ not only through the solution $\boldsymbol{X}(\omega)$ but also through $\boldsymbol{u}(\omega)$. As a result, Theorem 2 cannot be applied to equation (18) in general[3] and stronger conditions are needed.

Finally, we are given a cost functional that is, in general a mapping $J : \mathscr{U} \to \mathbb{R}$. There are many possibilities what such a cost functional can look like and it depends on the particular problem that is being modelled. We will restrict our attention to two-types of optimal control problems: optimal control on the *finite time horizon* $[s, T]$ and optimal control on the *infinite time horizon* $[s, \infty)$.

In a finite time horizon optimal control problem, we often wish to find a control $\boldsymbol{u} \in \mathscr{U}$ that minimizes the cost functional

$$J(\boldsymbol{u}) = \mathbb{E}\left[\int_s^T L(r, \boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{u}_r)\,\mathrm{d}r + l(\boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}})\right]$$

where $L : [s, T] \times \mathbb{R}^m \times U \to \mathbb{R}$ and $l : \mathbb{R}^m \to \mathbb{R}$ are measurable functions, $0 \leq s < T < \infty$, and where $\boldsymbol{X}^{\boldsymbol{y},\boldsymbol{u}}$ is the solution to equation (18) where we use the superscript to stress the dependence of the solution on the initial condition $\boldsymbol{y}$ and on the admissible control $\boldsymbol{u}$.

[2] In order for the equation to be well-posed, we need to assume that all the elements of the set of admissible controls $\mathscr{U}$ are $(\mathscr{F}_t)$-measurable stochastic processes for which the equation has a solution; however, sometimes the particular problem requires that we restrict our attention to only a subset of such controls that satisfy some additional conditions.

[3] Clearly, we could apply Theorem 2 to equation (18), for example, if the conditions of Theorem 2 are satisfied by $\boldsymbol{b}$ and $\boldsymbol{\sigma}$ uniformly in the third variable.

In an infinite time horizon optimal control, two of the most common cost functionlal are the discounted cost functional and the ergodic cost functional. A discounted cost is a cost that takes the form

$$J(\boldsymbol{u}) = \mathbb{E} \int_s^\infty \mathrm{e}^{-\alpha r} L(r, \boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{u}_r) \, \mathrm{d}r$$

where $L : [s, \infty) \times \mathbb{R}^m \times U \to \mathbb{R}$ is a measurable function and $\alpha > 0$. This is a very common choice of a cost function in financial applications (e.g. today's value of a dollar is smaller than its value tomorrow due to inflation). On the other hand, the ergodic cost

$$J(\boldsymbol{u}) = \limsup_{T\to\infty} \mathbb{E} \frac{1}{T-s} \int_s^T L(r, \boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{u}_r) \, \mathrm{d}r$$

is commonly used when the control is required to perform consistently over long time periods.

Remark 5. The above list of possible cost functionals $J(\boldsymbol{u})$ is by no means exhaustive and there exist many other possible choices of the cost functionals that are suitable for different practical problems.

2.1 Finite horizon stochastic LQ optimal control

Throughout this section, we fix a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with a one-dimensional $(\mathscr{F}_t)$-Wiener process defined on it. We will be interested in the following linear-quadratic (LQ) optimal control problem.

2.1.1 Problem formulation

Let $T > 0$ and $\boldsymbol{y} \in \mathbb{R}^m$. Assume that the dynamics of the state of some physical system is described by an m-dimensional stochastic process $(\boldsymbol{X}_t^{\boldsymbol{y},\boldsymbol{u}})_{t\in[0,T]}$ that satisfies the (linear) stochastic differential equation

$$\mathrm{d}\boldsymbol{X}_t = [\boldsymbol{A}(t)\boldsymbol{X}_t + \boldsymbol{B}(t)\boldsymbol{u}_t + \boldsymbol{b}(t)] \, \mathrm{d}t + \boldsymbol{\sigma}(t) \, \mathrm{d}W_t \tag{19}$$

on $(0, T)$ with the initial condition $\boldsymbol{X}_0 = \boldsymbol{y}$. Here, $\boldsymbol{A} \in L^\infty(0, T; \mathbb{R}^{m\times m})$, $\boldsymbol{B} \in L^\infty(0, T; \mathbb{R}^{m\times k})$, and $\boldsymbol{b}, \boldsymbol{\sigma} \in L^2(0, T; \mathbb{R}^m)$. The system is influenced by means of a control $(\boldsymbol{u}_t)_{t\in[0,T]}$ which belongs to the following set (of admissible controls):

$$\mathscr{U} := L^2_{\mathscr{F}}(0, T; \mathbb{R}^k). \tag{20}$$

Consider the (quadratic) cost functional

$$\begin{aligned} J_T(\boldsymbol{y}, \boldsymbol{u}) := \frac{1}{2} \mathbb{E} \int_0^T \Big[\langle \boldsymbol{Q}(s)\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}} \rangle_{\mathbb{R}^m} + 2\langle \boldsymbol{S}(s)\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{u}_s \rangle_{\mathbb{R}^k} + \langle \boldsymbol{R}(s)\boldsymbol{u}_s, \boldsymbol{u}_s \rangle_{\mathbb{R}^k} \Big] \, \mathrm{d}s \\ + \frac{1}{2} \mathbb{E} \langle \boldsymbol{G}\boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}} \rangle_{\mathbb{R}^m} \end{aligned} \tag{21}$$

where $\boldsymbol{Q} \in L^\infty(0, T; \mathbb{R}^{m\times m})$ is such that $[\boldsymbol{Q}(t)]^* = \boldsymbol{Q}(t)$ for every $t \in [0, T]$, $\boldsymbol{S} \in L^\infty(0, T; \mathbb{R}^{m\times k})$, $\boldsymbol{R} \in L^\infty(0, T; \mathbb{R}^{k\times k})$ is such that $[\boldsymbol{R}(t)]^* = \boldsymbol{R}(t)$ for every $t \in [0, T]$, and $\boldsymbol{G} \in \mathbb{R}^{m\times m}$ is such that $\boldsymbol{G}^* = \boldsymbol{G}$. We wish to find such admissible control that minimizes the above cost functional; that is, we wish to find

$$\bar{\boldsymbol{u}} := \arg \min_{\boldsymbol{u}\in\mathscr{U}} J_T(\boldsymbol{y}, \boldsymbol{u}).$$

2.1.2 Solvability of the control problem

In this section, we shall find conditions under which the problem (19) - (21) has a solution. Let us state precisely what we mean by a solution.

Definition 8. We say that the linear-quadratic control problem (19) - (21) is *solvable* if for every $\boldsymbol{y} \in \mathbb{R}^m$, there exists $\bar{\boldsymbol{u}} \in \mathscr{U}$ such that

$$J_T(\boldsymbol{y}, \bar{\boldsymbol{u}}) = \inf_{\boldsymbol{u} \in \mathscr{U}} J_T(\boldsymbol{y}, \boldsymbol{u}).$$

Moreover, such $\bar{\boldsymbol{u}}$ is called an *optimal control* for the problem (19) - (21) and the value $J_T(\boldsymbol{y}, \bar{\boldsymbol{u}})$ is called the *optimal cost*.

Let us denote $L^2_{\mathscr{F}} := L^2_{\mathscr{F}}(0, T; \mathbb{R}^m)$ for simplicity. By Theorem 5, the solution to equation (19) is given by

$$\begin{aligned} \boldsymbol{X}^{\boldsymbol{y},\boldsymbol{u}}_t = \boldsymbol{\Phi}(t)\boldsymbol{y} \\ + \boldsymbol{\Phi}(t) \int_0^t \boldsymbol{\Phi}^{-1}(s)\boldsymbol{B}(s)\boldsymbol{u}_s \, \mathrm{d}s \\ + \boldsymbol{\Phi}(t) \int_0^t \boldsymbol{\Phi}^{-1}(s)\boldsymbol{b}(s) \, \mathrm{d}s + \boldsymbol{\Phi}(t) \int_0^t \boldsymbol{\Phi}^{-1}(s)\boldsymbol{\sigma}(s) \, \mathrm{d}W_s. \end{aligned}$$

Therefore, we can define the operators $\boldsymbol{\Gamma} : \mathbb{R}^m \to L^2_{\mathscr{F}}$, $\boldsymbol{L} : \mathscr{U} \to L^2_{\mathscr{F}}$ and a stochastic process $\boldsymbol{f} \in L^2_{\mathscr{F}}$ by

$$\begin{aligned} (\boldsymbol{\Gamma}\boldsymbol{y})(t) &:= \boldsymbol{\Phi}(t)\boldsymbol{y}, \\ (\boldsymbol{L}\boldsymbol{u})(t) &:= \boldsymbol{\Phi}(t) \int_0^t \boldsymbol{\Phi}^{-1}(s)\boldsymbol{B}(s)\boldsymbol{u}_s \, \mathrm{d}s, \\ \boldsymbol{f}(t) &:= \boldsymbol{\Phi}(t) \int_0^t \boldsymbol{\Phi}^{-1}(s)\boldsymbol{b}(s) \, \mathrm{d}s + \boldsymbol{\Phi}(t) \int_0^t \boldsymbol{\Phi}^{-1}(s)\boldsymbol{\sigma}(s) \, \mathrm{d}W_s, \end{aligned}$$

so that we can simply write

$$\boldsymbol{X}^{\boldsymbol{y},\boldsymbol{u}}_t = (\boldsymbol{\Gamma}\boldsymbol{y})(t) + (\boldsymbol{L}\boldsymbol{u})(t) + \boldsymbol{f}(t).$$

Similarly, let us define the operators $\hat{\boldsymbol{\Gamma}} : \mathbb{R}^m \to L^2(\Omega; \mathbb{R}^m)$, $\hat{\boldsymbol{L}} : \mathscr{U} \to L^2(\Omega; \mathbb{R}^m)$, and a random variable $\hat{\boldsymbol{f}} \in L^2(\Omega; \mathbb{R}^m)$ by

$$\begin{aligned} \hat{\boldsymbol{\Gamma}}\boldsymbol{y} &:= \boldsymbol{\Phi}(T)\boldsymbol{y}, \\ \hat{\boldsymbol{L}}\boldsymbol{u} &:= (\boldsymbol{L}\boldsymbol{u})(T), \\ \hat{\boldsymbol{f}} &:= \boldsymbol{f}(T), \end{aligned}$$

so that we can write

$$\boldsymbol{X}^{\boldsymbol{y},\boldsymbol{u}}_T = \hat{\boldsymbol{\Gamma}}\boldsymbol{y} + \hat{\boldsymbol{L}}\boldsymbol{u} + \hat{\boldsymbol{f}}.$$

We will start by finding a new expression for the cost.

Theorem 6. *Assume that for some $\boldsymbol{y} \in \mathbb{R}^m$ it holds that*

$$V(\boldsymbol{y}) := \inf_{\boldsymbol{u} \in \mathscr{U}} J_T(\boldsymbol{y}, \boldsymbol{u}) > -\infty.$$

Then the equality

$$J_T(\boldsymbol{y},\boldsymbol{u}) = \frac{1}{2}\Big(\langle \boldsymbol{N}\boldsymbol{u},\boldsymbol{u}\rangle_{\mathscr{U}} + 2\langle \boldsymbol{H}\boldsymbol{y},\boldsymbol{u}\rangle_{\mathscr{U}} + M(\boldsymbol{y})\Big) \tag{22}$$

holds for every $\boldsymbol{u} \in \mathscr{U}$ *where* $\boldsymbol{N} : \mathscr{U} \to \mathscr{U}$ *is a self-adjoint*[4]*, non-negative*[5] *bounded linear operator given by*

$$\boldsymbol{N} := \boldsymbol{R} + \boldsymbol{L}^*\boldsymbol{Q}\boldsymbol{L} + \boldsymbol{S}\boldsymbol{L} + \boldsymbol{L}^*\boldsymbol{S}^* + \hat{\boldsymbol{L}}^*\boldsymbol{G}\hat{\boldsymbol{L}}$$

and where $\boldsymbol{H}\boldsymbol{y}$ *and* $M(\boldsymbol{y})$ *are given by*

$$\begin{aligned}\boldsymbol{H}\boldsymbol{y} &:= (\boldsymbol{L}^*\boldsymbol{Q} + \boldsymbol{S})\big(\boldsymbol{\Gamma}\boldsymbol{y} + \boldsymbol{f}\big) + \hat{\boldsymbol{L}}^*\boldsymbol{G}\big(\hat{\boldsymbol{\Gamma}}\boldsymbol{y} + \hat{\boldsymbol{f}}\big),\\ M(\boldsymbol{y}) &:= \mathbb{E}\int_0^T \Big\langle \boldsymbol{Q}(s)\big[(\boldsymbol{\Gamma}\boldsymbol{y})(s) + \boldsymbol{f}(s)\big], (\boldsymbol{\Gamma}\boldsymbol{y})(s) + \boldsymbol{f}(s)\Big\rangle_{\mathbb{R}^k}\,\mathrm{d}s\\ &\quad + \mathbb{E}\Big\langle \boldsymbol{G}(\hat{\boldsymbol{\Gamma}}\boldsymbol{y} + \hat{\boldsymbol{f}}), \hat{\boldsymbol{\Gamma}}\boldsymbol{y} + \hat{\boldsymbol{f}}\Big\rangle_{\mathbb{R}^k}.\end{aligned}$$

Sketch of proof. We have that

$$\begin{aligned}\frac{1}{2}\mathbb{E}\int_0^T &\langle \boldsymbol{Q}(s)\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m}\,\mathrm{d}s =\\ &= \frac{1}{2}\Big\langle \boldsymbol{Q}(\boldsymbol{\Gamma}\boldsymbol{y} + \boldsymbol{L}\boldsymbol{u} + \boldsymbol{f}), \boldsymbol{\Gamma}\boldsymbol{y} + \boldsymbol{L}\boldsymbol{u} + \boldsymbol{f}\Big\rangle_{L^2_{\mathscr{F}}}\\ &= \frac{1}{2}\Big[\langle \boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{y}, \boldsymbol{\Gamma}\boldsymbol{y}\rangle_{L^2_{\mathscr{F}}} + \langle \boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{y}, \boldsymbol{L}\boldsymbol{u}\rangle_{L^2_{\mathscr{F}}} + \langle \boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{y}, \boldsymbol{f}\rangle_{L^2_{\mathscr{F}}}\\ &\qquad + \langle \boldsymbol{Q}\boldsymbol{L}\boldsymbol{u}, \boldsymbol{\Gamma}\boldsymbol{y}\rangle_{L^2_{\mathscr{F}}} + \langle \boldsymbol{Q}\boldsymbol{L}\boldsymbol{u}, \boldsymbol{L}\boldsymbol{u}\rangle_{L^2_{\mathscr{F}}} + \langle \boldsymbol{Q}\boldsymbol{L}\boldsymbol{u}, \boldsymbol{f}\rangle_{L^2_{\mathscr{F}}}\\ &\qquad + \langle \boldsymbol{Q}\boldsymbol{f}, \boldsymbol{\Gamma}\boldsymbol{y}\rangle_{L^2_{\mathscr{F}}} + \langle \boldsymbol{Q}\boldsymbol{f}, \boldsymbol{L}\boldsymbol{u}\rangle_{L^2_{\mathscr{F}}} + \langle \boldsymbol{Q}\boldsymbol{f}, \boldsymbol{f}\rangle_{L^2_{\mathscr{F}}}\Big].\end{aligned}$$

By using duality, we obtain that

$$\langle \boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{y}, \boldsymbol{\Gamma}\boldsymbol{y}\rangle_{L^2_{\mathscr{F}}} = \langle \boldsymbol{\Gamma}^*\boldsymbol{Q}\boldsymbol{L}\boldsymbol{u}, \boldsymbol{y}\rangle_{\mathbb{R}^m} \quad \text{and} \quad \langle \boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{y}, \boldsymbol{L}\boldsymbol{u}\rangle_{L^2_{\mathscr{F}}} = \langle \boldsymbol{L}^*\boldsymbol{Q}\boldsymbol{L}\boldsymbol{u}, \boldsymbol{u}\rangle_{\mathscr{U}}$$

and similarly for the remaining terms. The rearrangement (22) is then easily obtained. The operator $\boldsymbol{N}$ is clearly self-adjoint (this follows from its definition); however, it is non-trivial to show that $\boldsymbol{N}$ is non-negative. □

Theorem 7. *The linear-quadratic control problem* (19) - (21) *is solvable if and only if there exists a solution* $\hat{\mathbf{u}} \in \mathscr{U}$ *to the equation*

$$\boldsymbol{N}\hat{\mathbf{u}} + \boldsymbol{H}\boldsymbol{y} = 0. \tag{23}$$

In this case, the solution $\hat{\mathbf{u}}$ *is an optimal control for the problem* (19) - (21)*. The optimal control is unique if and only if equation* (23) *has only one solution in the set of admissible controls* $\mathscr{U}$*.*

Corollary 1. *If it holds for the operator* $\boldsymbol{N}$ *that* $\boldsymbol{N} \geq \alpha\boldsymbol{I}$ *for some* $\alpha > 0$[6]*, then there is a unique optimal control to the problem* (19) - (21) *that is given by*

$$\bar{\boldsymbol{u}} = -\boldsymbol{N}^{-1}\boldsymbol{H}\boldsymbol{y}$$

and the optimal cost is

$$V(\boldsymbol{y}) = \frac{1}{2}\Big(- \langle \boldsymbol{N}^{-1}\boldsymbol{H}\boldsymbol{y}, \boldsymbol{H}\boldsymbol{y}\rangle_{\mathscr{U}} + M(\boldsymbol{y})\Big).$$

[4]That is, $\boldsymbol{N} = \boldsymbol{N}^*$.

[5]That is, it holds for every $\boldsymbol{u} \in \mathscr{U}$ that $\langle \boldsymbol{N}\boldsymbol{u}, \boldsymbol{u}\rangle_{\mathscr{U}} \geq 0$.

[6]More precisely, the condition reads as follows: There exists $\alpha > 0$ such that the inequality $\langle \boldsymbol{N}\boldsymbol{u}, \boldsymbol{u}\rangle_{\mathscr{U}} \geq \alpha\|\boldsymbol{u}\|^2_{\mathscr{U}}$ holds for every $u \in \mathscr{U}$. This conditions ensures that the operator $\boldsymbol{N}$ has a bounded inverse.

Proof of Theorem 7. Assume that the problem (19) - (21) is solvable. Let us denote

$$\varphi(\boldsymbol{u}) := \frac{1}{2}\langle \boldsymbol{N}\boldsymbol{u}, \boldsymbol{u}\rangle_{\mathscr{U}} + \langle \boldsymbol{H}\boldsymbol{y}, \boldsymbol{u}\rangle_{\mathscr{U}}$$

for $\boldsymbol{u} \in \mathscr{U}$. Let $\boldsymbol{h}$ be another control from the set of admissible controls $\mathscr{U}$. We shall compute the directional derivative of φ in the direction $\boldsymbol{h}$, $D_{\boldsymbol{h}}f$. We obtain the equality

$$\begin{aligned}(D_{\boldsymbol{h}}\varphi)(\boldsymbol{u}) &= \frac{1}{2}\Big(\langle \boldsymbol{N}\boldsymbol{u}, \boldsymbol{h}\rangle_{\mathscr{U}} + \langle \boldsymbol{u}, \boldsymbol{N}\boldsymbol{h}\rangle_{\mathscr{U}}\Big) + \langle \boldsymbol{H}\boldsymbol{y}, \boldsymbol{h}\rangle_{\mathscr{U}} \\ &= \langle \boldsymbol{N}\boldsymbol{u}, \boldsymbol{h}\rangle_{\mathscr{U}} + \langle \boldsymbol{H}\boldsymbol{y}, \boldsymbol{h}\rangle_{\mathscr{U}} \\ &= \langle \boldsymbol{N}\boldsymbol{u} + \boldsymbol{H}\boldsymbol{y}, \boldsymbol{h}\rangle_{\mathscr{U}}\end{aligned}$$

where we used that the operator $\boldsymbol{N}$ is self-adjoint. Since the problem (19) - (21) is solvable, we have that there is an optimal control $\bar{\boldsymbol{u}}$. It follows that

$$0 = D_{\boldsymbol{h}}\varphi(\bar{\boldsymbol{u}}) = \langle \boldsymbol{N}\bar{\boldsymbol{u}} + \boldsymbol{H}\boldsymbol{y}, \boldsymbol{h}\rangle_{\mathscr{U}}.$$

The last equality is satisfied for every $\boldsymbol{h} \in \mathscr{U}$ and hence, it must hold that

$$\boldsymbol{N}\bar{\boldsymbol{u}} + \boldsymbol{H}\boldsymbol{y} = 0.$$

Let us now assume that the equation (23) has a solution $\hat{\boldsymbol{u}}$ and let $\boldsymbol{u} \in \mathscr{U}$, $\boldsymbol{u} \neq \hat{\boldsymbol{u}}$. and $\boldsymbol{y} \in \mathbb{R}^m$ be arbitrary. Then we have that

$$\begin{aligned}J_T(\boldsymbol{y}, \boldsymbol{u}) - J_T(\boldsymbol{y}, \hat{\boldsymbol{u}}) &= J_T(\boldsymbol{y}, \hat{\boldsymbol{u}} + \boldsymbol{u} - \hat{\boldsymbol{u}}) - J_T(\boldsymbol{y}, \hat{\boldsymbol{u}}) \\ &= \frac{1}{2}\langle \boldsymbol{N}(\hat{\boldsymbol{u}} + \boldsymbol{u} - \hat{\boldsymbol{u}}), \hat{\boldsymbol{u}} + \boldsymbol{u} - \hat{\boldsymbol{u}}\rangle_{\mathscr{U}} + \langle \boldsymbol{H}\boldsymbol{y}, \hat{\boldsymbol{u}} + \boldsymbol{u} - \hat{\boldsymbol{u}}\rangle_{\mathscr{U}} \\ &\qquad - \frac{1}{2}\langle \boldsymbol{N}\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}\rangle_{\mathscr{U}} - \langle \boldsymbol{H}\boldsymbol{y}, \hat{\boldsymbol{u}}\rangle_{\mathscr{U}} \\ &= \langle \boldsymbol{N}\hat{\boldsymbol{u}} + \boldsymbol{H}\boldsymbol{y}, \boldsymbol{u} - \hat{\boldsymbol{u}}\rangle_{\mathscr{U}} + \frac{1}{2}\langle \boldsymbol{N}(\boldsymbol{u} - \hat{\boldsymbol{u}}), \boldsymbol{u} - \hat{\boldsymbol{u}}\rangle_{\mathscr{U}}.\end{aligned}$$

Since $\hat{\boldsymbol{u}}$ is the solution to (23), the first inner product in the last expression is zero. Since $\boldsymbol{N}$ is a non-negative operator, the second inner product is non-negative and hence, we can conclude that the inequality

$$J_T(\boldsymbol{y}, \boldsymbol{u}) \geq J_T(\boldsymbol{y}, \hat{\boldsymbol{u}})$$

holds. □

In Theorem 7, we have found a formula that allows us to find a solution to the linear-quadratic optimal control problem. This formula, however, requires that we solve equation (23). Let us assume from now on that $\boldsymbol{S} \equiv 0$ and $\boldsymbol{b} \equiv 0$ for simplicity. That is, we assume that the state of the controlled system $\boldsymbol{X}^{\boldsymbol{y},\boldsymbol{u}}$ is governed by the stochastic differential equation

$$\mathrm{d}\boldsymbol{X}_t = [\boldsymbol{A}(t)\boldsymbol{X}_t + \boldsymbol{B}(t)\boldsymbol{u}_t]\,\mathrm{d}t + \boldsymbol{\sigma}(t)\,\mathrm{d}W_t, \quad \boldsymbol{X}_0 = \boldsymbol{y}, \tag{24}$$

where the control $\boldsymbol{u}$ belongs to the set of admissible controls $\mathscr{U}$, and the cost functional is given by

$$\begin{aligned}J_T(\boldsymbol{y}, \boldsymbol{u}) = \frac{1}{2}\,\mathbb{E}\int_0^T \Big[\langle \boldsymbol{Q}(s)\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} + \langle \boldsymbol{R}(s)\boldsymbol{u}_s, \boldsymbol{u}_s\rangle_{\mathbb{R}^k}\Big]\,\mathrm{d}s \\ + \frac{1}{2}\,\mathbb{E}\langle \boldsymbol{G}\boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m}.\end{aligned} \tag{25}$$

In this situation, the operators $\boldsymbol{N}$ and $\boldsymbol{H}\boldsymbol{y}$ take the following form:

$$\boldsymbol{N} = \boldsymbol{R} + \boldsymbol{L}^*\boldsymbol{Q}\boldsymbol{L} + \hat{\boldsymbol{L}}^*\boldsymbol{G}\hat{\boldsymbol{L}},$$
$$\boldsymbol{H}\boldsymbol{y} = \boldsymbol{L}^*\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{y} + \boldsymbol{L}^*\boldsymbol{Q}\boldsymbol{f} + \hat{\boldsymbol{L}}^*\boldsymbol{G}\hat{\boldsymbol{\Gamma}}\boldsymbol{y} + \hat{\boldsymbol{L}}^*\boldsymbol{G}\hat{\boldsymbol{f}}.$$

Recall also that

$$\boldsymbol{X}_t^{\boldsymbol{y},\boldsymbol{u}} = (\boldsymbol{\Gamma}\boldsymbol{y})(t) + (\boldsymbol{L}\boldsymbol{u})(t) + \boldsymbol{f}(t), \tag{26}$$
$$\boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}} = \hat{\boldsymbol{\Gamma}}\boldsymbol{y} + \hat{\boldsymbol{L}}\boldsymbol{u} + \hat{\boldsymbol{f}}. \tag{27}$$

By Theorem 7, we obtain that, if it exists, the optimal control $\bar{\boldsymbol{u}}$ to the problem (24) - (25) satisfies

$$\begin{aligned} 0 &= \boldsymbol{N}\bar{\boldsymbol{u}} + \boldsymbol{H}\boldsymbol{y} \\ &= (\boldsymbol{R} + \boldsymbol{L}^*\boldsymbol{Q}\boldsymbol{L} + \hat{\boldsymbol{L}}^*\boldsymbol{G}\hat{\boldsymbol{L}})\bar{\boldsymbol{u}} + \boldsymbol{L}^*\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{y} + \boldsymbol{L}^*\boldsymbol{Q}\boldsymbol{f} + \hat{\boldsymbol{L}}^*\boldsymbol{G}\hat{\boldsymbol{\Gamma}}\boldsymbol{y} + \hat{\boldsymbol{L}}^*\boldsymbol{G}\hat{\boldsymbol{f}} \\ &= \boldsymbol{R}\bar{\boldsymbol{u}} + \boldsymbol{L}^*\boldsymbol{Q}\bar{\boldsymbol{X}}^{\boldsymbol{y}} + \hat{\boldsymbol{L}}^*\boldsymbol{G}\bar{\boldsymbol{X}}^{\boldsymbol{y}}_T \end{aligned} \tag{28}$$

where we denoted by $\bar{\boldsymbol{X}}^{\boldsymbol{y}}$ the solution to the equation

$$\mathrm{d}\boldsymbol{X}_t = [\boldsymbol{A}(t)\boldsymbol{X}_t + \boldsymbol{B}(t)\bar{\boldsymbol{u}}_t]\,\mathrm{d}t + \boldsymbol{\sigma}(t)\,\mathrm{d}W_t, \quad \boldsymbol{X}_0 = \boldsymbol{y},$$

that is, to equation (24) controlled by the optimal control $\bar{\boldsymbol{u}}$. From (28), we obtain that

$$\boldsymbol{R}(t)\bar{\boldsymbol{u}}_t = -(\boldsymbol{L}^*[\boldsymbol{Q}(\cdot)\bar{\boldsymbol{X}}^{\boldsymbol{y}}_\cdot])(t) - (\hat{\boldsymbol{L}}^*\boldsymbol{G}\bar{\boldsymbol{X}}^{\boldsymbol{y}}_T)(t).$$

If there exists $\alpha > 0$ such that $\boldsymbol{R}(t) \geq \alpha\boldsymbol{I}$ holds for every $t \in [0, T]$, then the values of $\boldsymbol{R}$ are invertible matrices and we can extract $\bar{\boldsymbol{u}}$ as follows:

$$\bar{\boldsymbol{u}}_t = [\boldsymbol{R}(t)]^{-1}\left[-(\boldsymbol{L}^*[\boldsymbol{Q}(\cdot)\bar{\boldsymbol{X}}^{\boldsymbol{y}}_\cdot])(t) - (\hat{\boldsymbol{L}}^*\boldsymbol{G}\bar{\boldsymbol{X}}^{\boldsymbol{y}}_T)(t)\right]. \tag{29}$$

Remark 6. If $\boldsymbol{G} \geq 0$ and if it holds for every $t \in [0, T]$ that $\boldsymbol{Q}(t) \geq 0$ and that $\boldsymbol{R}(t) \geq \alpha\boldsymbol{I}$ with some $\alpha > 0$, then it holds that $\boldsymbol{N} \geq \alpha\boldsymbol{I}$ and, consequently, there is a unique optimal control $\bar{\boldsymbol{u}}$ to the problem (24) - (25). Indeed, we have that

$$\begin{aligned} \langle \boldsymbol{N}\boldsymbol{u}, \boldsymbol{u}\rangle_{\mathscr{U}} &= \mathbb{E}\int_0^T \left(\langle \boldsymbol{R}(s)\boldsymbol{u}_s, \boldsymbol{u}_s\rangle_{\mathbb{R}^k} + \langle \boldsymbol{Q}(s)\boldsymbol{L}\boldsymbol{u}_s, \boldsymbol{u}_s\rangle_{\mathbb{R}^k}\right)\mathrm{d}s + \underbrace{(\dots)}_{\geq 0} \\ &\geq \mathbb{E}\int_0^T \left(\alpha\|\boldsymbol{u}_s\|^2_{\mathbb{R}^k} + \|\boldsymbol{Q}^{\frac{1}{2}}(s)\boldsymbol{L}\boldsymbol{u}_s\|^2_{\mathbb{R}^k}\right)\mathrm{d}s + \underbrace{(\dots)}_{\geq 0} \\ &\geq \alpha\|\boldsymbol{u}\|^2_{\mathscr{U}} + \underbrace{(\dots)}_{\geq 0} + \underbrace{(\dots)}_{\geq 0} \\ &\geq \alpha\|\boldsymbol{u}\|^2_{\mathscr{U}}. \end{aligned}$$

Of course, in order to be able to compute $\bar{\boldsymbol{u}}$ directly using formula (29), we need explicit formulas for the adjoints $\boldsymbol{L}^*$ and $\hat{\boldsymbol{L}}^*$. These formulas are given in the next proposition.

Proposition 4. *The following two claims are true:*

1. *The operator* $\boldsymbol{L}^*$ *is a bounded linear operator from the space* $L^2_{\mathscr{F}}$ *to the space* $\mathscr{U}$ *and it is given by*
$$(\boldsymbol{L}^*\boldsymbol{\psi})(t) = \boldsymbol{B}^*(t)[\boldsymbol{\Phi}^*(t)]^{-1}\int_t^T \boldsymbol{\Phi}^*(r)\mathbb{E}[\boldsymbol{\psi}_r|\mathscr{F}_r]\,\mathrm{d}r$$
for $\boldsymbol{\psi} \in L^2_{\mathscr{F}}$ *and* $t \in [0,T]$.
2. *The operator* $\hat{\boldsymbol{L}}^*$ *is a bounded linear operator from* $L^2(\Omega;\mathbb{R}^m)$ *to* $\mathscr{U}$ *and it is given by*
$$(\hat{\boldsymbol{L}}^*\boldsymbol{\psi})(t) = \boldsymbol{B}^*(t)[\boldsymbol{\Phi}^{-1}(t)]^*\boldsymbol{\Phi}(T)^*\mathbb{E}[\boldsymbol{\psi}|\mathscr{F}_t].$$
for $\boldsymbol{\psi} \in L^2(\Omega;\mathbb{R}^m)$ *and* $t \in [0,T]$.

Proof. The proof is left as an exercise. □

2.1.3 Intermezzo: BSDEs

In order to further understand the structure of the LQ control problem deeper, let us make a brief detour. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space with a Wiener process $(W_t)_{t\in[0,T]}$ defined on it. Without loss of generality, we assume that $\mathscr{F} = \mathscr{F}^W_T$. Let us try to find a process a solution to the following stochastic differential equation:

$$\begin{aligned} \mathrm{d}Y_t &= 0, \quad t \in (0,T), \\ Y_T &= \xi, \end{aligned} \tag{30}$$

where $\xi \in L^2(\Omega)$ is $\mathscr{F}^W_T$-measurable random variable. Our requirement is that the solution to (30) is an $(\mathscr{F}^W_t)$-adapted process. Since the solution to (30) should have zero differential and it should be ξ at time T, the solution could be a constant process, i.e. a stochastic process $(\tilde{Y}_t)_{t\in[0,T]}$ defined by
$$\tilde{Y}_t := \xi.$$
The problem is that the process $(\tilde{Y}_t)$ is not an $(\mathscr{F}^W_t)$-adapted process. Let us therefore find the closest[7] process to $\tilde{Y}$ that is adapted. This is the process $(Y_t)_{t\in[0,T]}$ defined by

$$Y_t := \mathbb{E}[\,\xi\,|\mathscr{F}^W_t].$$

Clearly, we have that $Y_T = \xi$ almost surely. However, the problem is that the process $(Y_t)_{t\in[0,T]}$ does not satisfy the stochastic differential equation (30). We can ask whether it satisfies a different stochastic differential equation. It turns out that it indeed does. By the Martingale Representation Theorem, we have that there exists an $(\mathscr{F}^W_t)$-adapted square integrable process $(Z_t)_{t\in[0,T]}$ such that it holds that

$$Y_t = Y_0 + \int_0^t Z_s\,\mathrm{d}W_s$$

for every $t \in [0,T]$ almost surely. We do not know the random variable Y_0 but we do know that

$$Y_T = Y_0 + \int_0^T Z_s\,\mathrm{d}W_s$$

[7] Recall that the conditional expectation $\mathbb{E}[\cdots|\mathscr{F}^W_t]$ is the orthogonal projection from $L^2([0,T]\times\Omega)$ to its subspace whose elements are square-integrable $(\mathscr{F}^W_t)$-adapted processes.

holds almost surely and therefore, it must hold that

$$Y_t = \xi - \int_t^T Z_s \, \mathrm{d}W_s$$

for every $t \in [0, T]$ almost surely. Hence, the process Y satisfies the stochastic differential equation

$$\begin{aligned} \mathrm{d}Y_t &= Z_t \, \mathrm{d}W_t, \quad t \in (0, T), \\ Y_T &= \xi. \end{aligned}$$

Unfortunately, the process Z is rather difficult to obtain (in fact, Z can be expressed in terms of Malliavin derivatives in the so-called Clark-Ocone formula).

The above reasoning gives rise to the notion of the *backward stochastic differential equation (BSDE)*. In general, a BSDE is an equation of the following form

$$\begin{cases} \mathrm{d}Y_t &= f(t, Y_t, Z_t) \, \mathrm{d}t + Z_t \, \mathrm{d}W_t, \quad t \in (0, T), \\ Y_T &= \xi. \end{cases}$$

Its solution is an $\mathscr{F}_t^W$-adapted process $(Y_t, Z_t)_{t \in [0,T]}$ such that the equality

$$Y_t = \xi - \int_t^T f(s, Y_s, Z_s) \, \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s$$

holds for every $t \in [0, T]$ almost surely.

2.1.4 Solution of the control problem and BSDEs

Let us now consider the following backward stochastic differential equation:

$$\begin{cases} \mathrm{d}\boldsymbol{p}(t) &= (-\boldsymbol{A}(t)^* \boldsymbol{p}(t) - \boldsymbol{\xi}(t)) \, \mathrm{d}t + \boldsymbol{q}(t) \, \mathrm{d}W_t, \quad t \in (0, T), \\ \boldsymbol{p}(T) &= \boldsymbol{\eta} \end{cases} \tag{31}$$

where $\boldsymbol{\xi} \in L^2_{\mathscr{F}}$ and $\boldsymbol{\eta} \in L^2(\Omega; \mathbb{R}^m)$ is an $\mathscr{F}_T$-measurable random variable. The solution to this equation is the process $(\boldsymbol{p}(t), \boldsymbol{q}(t))_{t \in [0,T]}$. We have the following result.

Theorem 8. *The following two claims are true:*

1. *Let $\boldsymbol{\xi} \in L^2_{\mathscr{F}}$ and denote the solution to equation (31) with $\boldsymbol{\eta} = 0$ by $(\boldsymbol{p_0}(t), \boldsymbol{q_0}(t))_{t \in [0,T]}$. Then it holds that*
$$(\boldsymbol{L}^* \boldsymbol{\xi})(t) = \boldsymbol{B}(t)^* \boldsymbol{p_0}(t) \quad \textit{and} \quad \hat{\boldsymbol{\Gamma}}^* \boldsymbol{\xi} = \boldsymbol{p_0}(0).$$

2. *Let $\boldsymbol{\eta} \in L^2(\Omega; \mathbb{R}^m)$ is an $\mathscr{F}_T$-measurable random variable and denote the solution to (31) with $\boldsymbol{\xi} = 0$ by $(\boldsymbol{p_1}(t), \boldsymbol{q_1}(t))_{t \in [0,T]}$. Then it holds that*
$$(\hat{\boldsymbol{L}}^* \boldsymbol{\eta})(t) = \boldsymbol{B}(t)^* \boldsymbol{p_1}(t) \quad \textit{and} \quad \hat{\boldsymbol{\Gamma}}^* \boldsymbol{\eta} = \boldsymbol{p_1}(0).$$

Proof. Set $\boldsymbol{\sigma} \equiv 0$ in equation (24) so that we obtain that its solution $\boldsymbol{X}^{\boldsymbol{y},\boldsymbol{u}}$ satisfies the deterministic differential equation

$$\boldsymbol{X}_t = \boldsymbol{y} + \int_0^t [\boldsymbol{A}(s)\boldsymbol{X}_s + \boldsymbol{B}(s)\boldsymbol{u}_s] \, \mathrm{d}s, \quad t \in [0, T].$$

Let $\boldsymbol{\xi} \in L^2_{\mathscr{F}}$ and $\boldsymbol{\eta} \in L^2(\Omega; \mathbb{R}^m)$ be arbitrary and let $(\boldsymbol{p}, \boldsymbol{q})$ be the solution to the BSDE (31). By applying the Itô formula from Theorem 1 to the (scalar) function $(t, \boldsymbol{x}) \mapsto \langle \boldsymbol{X}_t^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{x}\rangle_{\mathbb{R}^m}$, we obtain that

$$\begin{aligned}
&\mathbb{E}\Big[\langle \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{\eta}\rangle_{\mathbb{R}^m} - \langle \boldsymbol{y}, \boldsymbol{p}(0)\rangle_{\mathbb{R}^m}\Big] = \\
&\quad = \mathbb{E}\int_0^T \Big[\langle \boldsymbol{A}(s)\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}} + \boldsymbol{B}(s)\boldsymbol{u}_s, \boldsymbol{p}(s)\rangle_{\mathbb{R}^m} - \langle \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{A}(s)^*\boldsymbol{p}(s) - \boldsymbol{\xi}(s)\rangle_{\mathbb{R}^m}\Big]\,\mathrm{d}s \\
&\quad = \mathbb{E}\int_0^T \Big[\langle \boldsymbol{u}_s, \boldsymbol{B}(s)^*\boldsymbol{p}(s)\rangle_{\mathbb{R}^m} - \langle \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{\xi}(s)\rangle_{\mathbb{R}^m}\Big]\,\mathrm{d}s
\end{aligned}$$

holds. By using to expressions (26) and (27), we obtain from the last equality that

$$\begin{aligned}
&\mathbb{E}\Big[\langle \hat{\boldsymbol{\Gamma}}\boldsymbol{y} + \hat{\boldsymbol{L}}\boldsymbol{u}, \boldsymbol{\eta}\rangle_{\mathbb{R}^m} - \langle \boldsymbol{y}, \boldsymbol{p}(0)\rangle_{\mathbb{R}^m}\Big] = \\
&\qquad = \mathbb{E}\int_0^T \Big[\langle \boldsymbol{u}_s, \boldsymbol{B}(s)^*\boldsymbol{p}(s)\rangle_{\mathbb{R}^m} - \langle (\boldsymbol{\Gamma}\boldsymbol{y})(s) + (\boldsymbol{L}\boldsymbol{u})(s), \boldsymbol{\xi}(s)\rangle_{\mathbb{R}^m}\Big]\,\mathrm{d}s.
\end{aligned}$$

All the four expressions in the claim of the theorem are obtained from the last equality by an appropriate choice of $\boldsymbol{u}, \boldsymbol{y}, \boldsymbol{\eta}$, or $\boldsymbol{\xi}$. For example, by setting $\boldsymbol{y} = 0$ and $\boldsymbol{\eta} = 0$, we obtain that

$$0 = \mathbb{E}\int_0^T \langle \boldsymbol{u}_s, \boldsymbol{B}(s)^*\boldsymbol{p}(s)\rangle_{\mathbb{R}^k}\,\mathrm{d}s - \mathbb{E}\int_0^T \langle \boldsymbol{u}_s, (\boldsymbol{L}^*\boldsymbol{\xi})(s)\rangle_{\mathbb{R}^k}\,\mathrm{d}s$$

holds for every $\boldsymbol{u} \in \mathscr{U}$ and, therefore, we have that

$$(\boldsymbol{L}^*\boldsymbol{\xi})(s) = \boldsymbol{B}(s)^*\boldsymbol{p}(s).$$

Similarly, by setting $\boldsymbol{u} = 0$ and $\boldsymbol{\eta} = 0$, we obtain that

$$-\mathbb{E}\langle \boldsymbol{y}, \boldsymbol{p}(0)\rangle_{\mathbb{R}^m} = -\mathbb{E}\int_0^T \langle (\boldsymbol{\Gamma}\boldsymbol{y})(s), \boldsymbol{\xi}(s)\rangle_{\mathbb{R}^m}\,\mathrm{d}s = -\mathbb{E}\langle \boldsymbol{y}, \boldsymbol{\Gamma}^*\boldsymbol{\xi}\rangle_{\mathbb{R}^m}$$

holds for every $\boldsymbol{y} \in \mathbb{R}^m$. Therefore, we have that

$$\boldsymbol{\Gamma}^*\boldsymbol{\xi} = \boldsymbol{p}(0).$$

The claims of the second part of the theorem are proved similarly. □

Theorem 9. *Let the triple $(\bar{\boldsymbol{X}}^{\boldsymbol{y}}, \boldsymbol{p}, \boldsymbol{q})$ be the solution to the following forward-backward stochastic differential equation:*

$$\begin{cases} \mathrm{d}\bar{\boldsymbol{X}}_t &= \big[\boldsymbol{A}(t)\bar{\boldsymbol{X}}_t + \boldsymbol{B}(t)\boldsymbol{R}^{-1}\boldsymbol{B}(t)^*\boldsymbol{p}(t)\big]\,\mathrm{d}t + \boldsymbol{\sigma}(t)\,\mathrm{d}W_t, \quad \bar{\boldsymbol{X}}_0 = \boldsymbol{y}, \\ \mathrm{d}\boldsymbol{p}(t) &= \big[-\boldsymbol{A}(t)^*\boldsymbol{p}(t) + \boldsymbol{Q}(t)\bar{\boldsymbol{X}}_t\big]\,\mathrm{d}t + \boldsymbol{q}(t)\,\mathrm{d}W_t, \quad \boldsymbol{p}(T) = -\boldsymbol{G}\bar{\boldsymbol{X}}_T. \end{cases} \tag{32}$$

Then it holds that

$$\boldsymbol{R}(t)\bar{\boldsymbol{u}}_t = \boldsymbol{B}(t)^*\boldsymbol{p}(t)$$

for every $t \in [0, T]$. Consequently, if there exists $\alpha > 0$ such that $\boldsymbol{R}(t) \geq \alpha\boldsymbol{I}$ for every $t \in [0, T]$, then we have that

$$\bar{\boldsymbol{u}}_t = [\boldsymbol{R}(t)]^{-1}\boldsymbol{B}^*(t)\boldsymbol{p}(t).$$

holds for every $t \in [0, T]$.

Theorem 9 shows that the solution to the linear-quadratic optimal control problem is closely connected to the solution of a forward-backward stochastic differential equation (FBSDE). Let us now assume that the solution to the FBSDE (32) takes the form

$$\boldsymbol{p}(t) = -\boldsymbol{P}(t)\bar{\boldsymbol{X}}_t^{\boldsymbol{y}}$$

where $\boldsymbol{P}$ is some non-negative function that belongs to the space $AC([0,T];\mathbb{R}^{m\times m})$ and such that it satisfies the equality $\boldsymbol{P}(t)^* = \boldsymbol{P}(t)$ for every $t \in [0,T]$. From the second equation in (32), we have that

$$\mathrm{d}\boldsymbol{p}(t) = \left[-\boldsymbol{A}(t)^*\boldsymbol{p}(t) + \boldsymbol{Q}(t)\bar{\boldsymbol{X}}_t^{\boldsymbol{y}}\right]\,\mathrm{d}t + \boldsymbol{q}(t)\,\mathrm{d}W_t.$$

On the other hand, the process $\bar{\boldsymbol{X}}^{\boldsymbol{y}}$ satisfies the first equation in (32), and, therefore, by using the Itô formula, we obtain

$$\mathrm{d}\boldsymbol{p}(t) = \left\{-\dot{\boldsymbol{P}}(t)\bar{\boldsymbol{X}}_t^{\boldsymbol{y}} - \boldsymbol{P}(t)[\boldsymbol{A}(t)\bar{\boldsymbol{X}}_t^{\boldsymbol{y}} + \boldsymbol{B}(t)\bar{\boldsymbol{u}}_t]\right\}\,\mathrm{d}t - \boldsymbol{P}(t)\boldsymbol{\sigma}(t)\,\mathrm{d}W_t.$$

Solutions to stochastic differential equations are semimartingales, that is processes of the form $BV + M$ where BV is a bounded variation process and M is a martingale. Such decomposition is unique and therefore, it must hold that

$$\boldsymbol{q}(t) = -\boldsymbol{P}(t)\boldsymbol{\sigma}(t)$$

and

$$-\boldsymbol{A}(t)^*\boldsymbol{p}(t) + \boldsymbol{Q}(t)\bar{\boldsymbol{X}}_t^{\boldsymbol{y}} = -\dot{\boldsymbol{P}}(t)\bar{\boldsymbol{X}}_t^{\boldsymbol{y}} - \boldsymbol{P}(t)[\boldsymbol{A}(t)\bar{\boldsymbol{X}}_t^{\boldsymbol{y}} + \boldsymbol{B}(t)\bar{\boldsymbol{u}}_t]. \tag{33}$$

Equation (33) is easily rewritten as

$$\dot{\boldsymbol{P}}(t) + \boldsymbol{P}(t)\boldsymbol{A}(t) + \boldsymbol{A}(t)^*\boldsymbol{P}(t) + \boldsymbol{Q}(t) - \boldsymbol{P}(t)\boldsymbol{B}(t)[\boldsymbol{R}(t)]^{-1}\boldsymbol{B}(t)^*\boldsymbol{P}(t) = 0$$

where we used that

$$\bar{\boldsymbol{u}}_t = [\boldsymbol{R}(t)]^{-1}\boldsymbol{B}(t)^*\boldsymbol{p}(t) = -[\boldsymbol{R}(t)]^{-1}\boldsymbol{B}(t)^*\boldsymbol{P}(t)\bar{\boldsymbol{X}}_t^{\boldsymbol{y}}.$$

Moreover, we have that

$$-\boldsymbol{G}\bar{\boldsymbol{X}}_T^{\boldsymbol{y}} = \boldsymbol{p}(T) = -\boldsymbol{P}(T)\bar{\boldsymbol{X}}_T^{\boldsymbol{y}}$$

from which the equality $\boldsymbol{P}(T) = \boldsymbol{G}$ is obtained. Therefore, we arrive at the following differential equation with a terminal condition that is usually called the *Riccati differential equation*:

$$\dot{\boldsymbol{P}} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^*\boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{P} = 0, \quad \boldsymbol{P}(T) = \boldsymbol{G}.$$

2.1.5 Intermezzo: Solutions to ODEs

The Riccati equation is a non-linear first-order ordinary differential equation that is solved backwards in time. As such, its analysis may be non-trivial. Let us therefore recall some facts about solutions of ordinary differential equations. Let $J \subseteq \mathbb{R}$ be an interval. Consider the equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)) \tag{34}$$

where $\boldsymbol{f}$ is a measurable function on $J \times \mathbb{R}^m$ with values in $\mathbb{R}^m$.

Definition 9. Let $I \subseteq J$ be an interval. A *(Carathéodory) solution* to equation (34) *defined on the interval* I is a measurable function $\boldsymbol{x} : I \to \mathbb{R}^m$ such that the function $\boldsymbol{f}(\cdot, \boldsymbol{x}(\cdot))$ belongs to the space $L^1(I; \mathbb{R}^m)$ and such that the equality

$$\boldsymbol{x}(t) = \boldsymbol{x}(s) + \int_s^t \boldsymbol{f}(r, \boldsymbol{x}(r))\, \mathrm{d}r$$

holds for every $s, t \in I$. The integral is defined here in the sense of Lebesgue. Moreover, if a solution $\boldsymbol{x}$ to equation (34) satisfies

$$\boldsymbol{x}(s) = \boldsymbol{y} \tag{35}$$

for some given $s \in I$ and $\boldsymbol{y} \in \mathbb{R}^m$, the we say that $\boldsymbol{x}$ is a solution to equation (34) *with the initial condition* (35).

Remark 7. A measurable function $\boldsymbol{x} : I \to \mathbb{R}^m$ is a Carathéodory solution to equation (34) defined on the interval I if and only if $\boldsymbol{x}$ is *absolutely continuous* and equation (34) holds for *almost every* $t \in I$. On the other hand, a classical solution to equation (34) defined on the interval I is a *continuously differentiable* function $\boldsymbol{x} : I \to \mathbb{R}^m$ such that equation (34) is satisfied for *every* $t \in I$. Clearly, every classical solution is also a Carathéodory solution. On the other hand, a classical solution might not exist while Carathéodory solution does. Consider, for example, the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} 1, & x > 0, \\ z, & x = 0, \\ -1, & x < 0, \end{cases}$$

where $z \in (0, 1)$, and the differential equation

$$\begin{aligned} &\dot{x}(t) = f(x(t)), \quad t \geq 0, \\ &x(0) = 0. \end{aligned}$$

Let us suppose, for a moment that the equation has a classical solution x defined on some interval $[0, T]$. Then we have that $\dot{x}(0) = f(x(0)) = f(0) = z > 0$ and therefore, x is increasing on some right neighbourhood of zero and hence, for all sufficiently small $t > 0$, we have that $x(t) - x(0) > 0$ which is equivalent to $x(t) > 0$. Therefore, we have that $\dot{x}(t) = f(x(t)) = 1$ for all sufficiently small $t > 0$. On the other hand, we already know that $\dot{x}(0) = z < 1$ and we obtain a contradiction with the assumption that $\dot{x}$ is continuous. Hence, the equation does not admit a classical solution. On the other hand, the functions $x_1(t) := t$ and $x_2(t) := -t$ are both Carathéodory solutions to the problem on the interval $[0, \infty)$ because it holds that

$$x_i(s) + \int_s^t f(x_i(r))\, \mathrm{d}r = \pm s + \int_s^t \pm 1\, \mathrm{d}r = x_i(t)$$

for every $s, t \in [0, \infty)$ where the choice of signs corresponds to $i = 1$ and $i = 2$.

The next theorem gives us sufficient conditions for the existence of a (local) solution to the initial value problem (34), (35). These sufficient conditions are called Carathéodory's conditions.

Definition 10. Let $\mathcal{O} \subseteq \mathbb{R} \times \mathbb{R}^m$ be an open set. A function $\boldsymbol{f} : \mathcal{O} \to \mathbb{R}^m$ *satisfies Carathéodory's conditions on* $\mathcal{O}$ if for every $(t_0, \boldsymbol{x}_0) \in \mathcal{O}$, there exists a cylinder $U_\delta(t_0) \times U_{\delta'}(\boldsymbol{x_0}) \subset \mathcal{O}$ and a function $m \in L^1(U_\delta(t_0))$ such that the following conditions are satisfied:

(C1) For every $\boldsymbol{y} \in U_{\delta'}(\boldsymbol{x_0})$, the function $\boldsymbol{f}(\cdot, \boldsymbol{y})$ is measurable in $U_\delta(t_0)$;

(C2) for almost every $t \in U_\delta(t_0)$, the function $\boldsymbol{f}(t, \cdot)$ is continuous in $U_{\delta'}(\boldsymbol{x_0})$; and

(C3) for almost every $t \in U_\delta(t_0)$ and every $\boldsymbol{y} \in U_{\delta'}(\boldsymbol{x_0})$, we have the estimate

$$\|\boldsymbol{f}(t, \boldsymbol{y})\|_{\mathbb{R}^m} \leq m(t).$$

Theorem 10. *Assume that $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is a function that satisfies Carathéodory's conditions on $\mathbb{R} \times \mathbb{R}^m$. Then for every $(t_0, \boldsymbol{x}_0) \in \mathbb{R} \times \mathbb{R}^m$, there exists an interval $I \subseteq \mathbb{R}$ such that $t_0 \in I$ and such that there exists a solution $\boldsymbol{x}$ to the equation*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)),$$

defined on the interval I that satisfies $\boldsymbol{x}(t_0) = \boldsymbol{x_0}$.

Note that Theorem 10 only provides us with a solution that is defined (and solves the equation) on some, possibly very small, interval I. It would, of course, be desirable to find a *global* solution, i.e. solution to the problem (34), (35) that is defined on the whole interval J. This is, however, not always possible and thus, we consider the so-called *maximal* solution instead.

Definition 11. A solution $\boldsymbol{x}$ to the initial value problem (34), (35) on the interval I is called *maximal* if it cannot be extended to a larger interval, i.e. if there exists no interval I' such that $I \subsetneq I'$ and such that there exists a solution $\boldsymbol{x}'$ to the problem (34), (35) on I'.

2.1.6 Solution of the control problem and the Riccati equation

We continue with the linear-quadratic problem (24), (25). Recall that in Theorem 9 we obtained the optimal control $\bar{\boldsymbol{u}}$ in terms of a solution to a forward-backward stochastic differential equation. Subsequently, using an ansatz, we have heuristically obtained that the optimal control $\bar{\boldsymbol{u}}$ can be expressed as

$$\bar{\boldsymbol{u}}_t = -[\boldsymbol{R}(t)]^{-1} \boldsymbol{B}(t)^* \boldsymbol{P}(t) \bar{\boldsymbol{X}}_t^{\boldsymbol{y}}$$

where the matrix-valued function $\boldsymbol{P}$ satisfies the non-linear differential equation with a terminal condition

$$\dot{\boldsymbol{P}} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^*\boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{P} = 0, \quad \boldsymbol{P}(T) = \boldsymbol{G}.$$

This Riccati equation satisfies Carathéodory's conditions (and hence admits a local solution by Theorem 10) and is locally Lipschitz (and hence this local solution is unique). The precise relationship between the Riccati equation and the LQ control problem is given in the following theorem that is proved using the *completion of squares method.*

Theorem 11. *Let us assume that $\boldsymbol{G} \geq 0$, $\boldsymbol{Q}(t) \geq 0$ for every $t \in [0, T]$ and that there exists $\alpha > 0$ such that $\boldsymbol{R}(t) \geq \alpha \boldsymbol{I}$ holds for every $t \in [0, T]$. Let $\boldsymbol{P}$ be the non-negative local solution to the Riccati equation*

$$\dot{\boldsymbol{P}} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^*\boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{P} = 0, \quad \boldsymbol{P}(T) = \boldsymbol{G}, \tag{RE}$$

that is, $\boldsymbol{P}$ is the solution to equation (RE) defined on the interval $(s, T]$ with some $s \in [0, T)$. Then the control

$$\bar{\boldsymbol{u}}_t := -[\boldsymbol{R}(t)]^{-1} \boldsymbol{B}(t)^* \boldsymbol{P}(t) \boldsymbol{X}_t, \quad t \in (s, T],$$

is the optimal control for the linear system

$$\begin{cases} \mathrm{d}\boldsymbol{X}_t = [\boldsymbol{A}(t)\boldsymbol{X}_t + \boldsymbol{B}(t)\boldsymbol{u}_t]\,\mathrm{d}t + \boldsymbol{\sigma}(t)\,\mathrm{d}W_t, & t \in (s, T], \\ \boldsymbol{X}_s = \boldsymbol{y} \end{cases} \tag{36}$$

in the set of admissible control $\mathscr{U}$ minimizing the quadratic cost

$$J_{s,T}(\boldsymbol{y},\boldsymbol{u}) := \frac{1}{2}\,\mathbb{E}\int_s^T [\langle \boldsymbol{Q}(r)\boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} + \langle \boldsymbol{R}(r)\boldsymbol{u}_r, \boldsymbol{u}_r\rangle_{\mathbb{R}^k}]\,\mathrm{d}r + \frac{1}{2}\,\mathbb{E}\langle \boldsymbol{G}\boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m}.$$

where $\boldsymbol{X}^{\boldsymbol{y},\boldsymbol{u}}$ is the solution to equation (36). The optimal cost is

$$V_s(\boldsymbol{y}) := J_{s,T}(\boldsymbol{y},\bar{\boldsymbol{u}}) = \frac{1}{2}\langle \boldsymbol{P}(s)\boldsymbol{y},\boldsymbol{y}\rangle_{\mathbb{R}^m} + \frac{1}{2}\int_s^T \langle \boldsymbol{P}(r)\boldsymbol{\sigma}(r),\boldsymbol{\sigma}(r)\rangle_{\mathbb{R}^m}\,\mathrm{d}r.$$

Proof. Let $\boldsymbol{u}\in\mathscr{U}$ and let $\boldsymbol{X}^{\boldsymbol{y},\boldsymbol{u}}$ be a solution to equation (36) with this control. By the Itô formula applied to the function $(t,\boldsymbol{x})\mapsto\langle \boldsymbol{P}(t)\boldsymbol{x},\boldsymbol{x}\rangle_{\mathbb{R}^m}$ on the interval $(s,T]$, we obtain that

$$\begin{aligned}\mathbb{E}\langle \boldsymbol{G}\boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} - \langle \boldsymbol{P}(s)\boldsymbol{y},\boldsymbol{y}\rangle_{\mathbb{R}^m} &= \\ = \mathbb{E}\int_s^T \Big\{\Big\langle [\dot{\boldsymbol{P}}(r) + \boldsymbol{P}(r)\boldsymbol{A}(r) + \boldsymbol{A}(r)^*\boldsymbol{P}(r)]\boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}\Big\rangle_{\mathbb{R}^m} &\\ + 2\Big\langle \boldsymbol{B}(r)^*\boldsymbol{P}(r)\boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{u}_r\Big\rangle_{\mathbb{R}^k} + \Big\langle \boldsymbol{P}(r)\boldsymbol{\sigma}(r),\boldsymbol{\sigma}(r)\Big\rangle_{\mathbb{R}^m}\Big\}\,\mathrm{d}r.\end{aligned}$$

By multiplying the above equation by $1/2$, adding the term

$$\frac{1}{2}\,\mathbb{E}\int_s^T \Big\{\Big\langle \boldsymbol{Q}(r)\boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}\Big\rangle_{\mathbb{R}^m} + \Big\langle \boldsymbol{R}(r)\boldsymbol{u}_r,\boldsymbol{u}_r\Big\rangle_{\mathbb{R}^k}\Big\}\,\mathrm{d}r$$

to both its sides, and by using the Riccati equation (RE) to express $\dot{\boldsymbol{P}}$, we obtain

$$\begin{aligned}J_{s,T}(\boldsymbol{y},\boldsymbol{u}) &= \langle \boldsymbol{P}(s)\boldsymbol{y},\boldsymbol{y}\rangle_{\mathbb{R}^m} + \frac{1}{2}\,\mathbb{E}\int_s^T \Big\langle \boldsymbol{P}(r)\boldsymbol{\sigma}(r),\boldsymbol{\sigma}(r)\Big\rangle_{\mathbb{R}^m}\mathrm{d}r \\ &\quad + \frac{1}{2}\,\mathbb{E}\int_s^T \Big\{\Big\langle \boldsymbol{P}(r)\boldsymbol{B}(r)[\boldsymbol{R}(r)]^{-1}\boldsymbol{B}(r)^*\boldsymbol{P}(r)\boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}\Big\rangle_{\mathbb{R}^m} \\ &\qquad + 2\Big\langle \boldsymbol{B}(r)^*\boldsymbol{P}(r)\boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}},\boldsymbol{u}_r\Big\rangle_{\mathbb{R}^k} + \Big\langle \boldsymbol{R}(r)\boldsymbol{u}_r,\boldsymbol{u}_r\Big\rangle_{\mathbb{R}^k}\Big\}\,\mathrm{d}r \\ &= \langle \boldsymbol{P}(s)\boldsymbol{y},\boldsymbol{y}\rangle_{\mathbb{R}^m} + \frac{1}{2}\,\mathbb{E}\int_s^T \Big\langle \boldsymbol{P}(r)\boldsymbol{\sigma}(r),\boldsymbol{\sigma}(r)\Big\rangle_{\mathbb{R}^m}\mathrm{d}r \\ &\quad + \frac{1}{2}\,\mathbb{E}\int_s^T \Big\{\Big\|[\boldsymbol{R}(r)]^{-\frac{1}{2}}\boldsymbol{B}(r)^*\boldsymbol{P}(r)\boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}\Big\|_{\mathbb{R}^k}^2 \\ &\qquad + 2\Big\langle [\boldsymbol{R}(r)]^{-\frac{1}{2}}\boldsymbol{B}(r)^*\boldsymbol{P}(r)\boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}, [\boldsymbol{R}(r)]^{\frac{1}{2}}\boldsymbol{u}_r\Big\rangle_{\mathbb{R}^k} + \Big\|[\boldsymbol{R}(r)]^{\frac{1}{2}}\boldsymbol{u}_r\Big\|_{\mathbb{R}^k}^2\Big\}\,\mathrm{d}r \\ &= \langle \boldsymbol{P}(s)\boldsymbol{y},\boldsymbol{y}\rangle_{\mathbb{R}^m} + \frac{1}{2}\,\mathbb{E}\int_s^T \Big\langle \boldsymbol{P}(r)\boldsymbol{\sigma}(r),\boldsymbol{\sigma}(r)\Big\rangle_{\mathbb{R}^m}\mathrm{d}r \\ &\quad + \frac{1}{2}\,\mathbb{E}\int_s^T \Big\|[\boldsymbol{R}(r)]^{-\frac{1}{2}}\boldsymbol{B}(r)^*\boldsymbol{P}(r)\boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}} + [\boldsymbol{R}(r)]^{\frac{1}{2}}\boldsymbol{u}_r\Big\|_{\mathbb{R}^k}^2\,\mathrm{d}r\end{aligned}$$

and we see that the cost $J_{s,T}(\boldsymbol{y},\boldsymbol{u})$ attains its minimum at

$$\bar{\boldsymbol{u}}_t = -[\boldsymbol{R}(t)]^{-1}\boldsymbol{B}(t)^*\boldsymbol{P}(t)\bar{\boldsymbol{X}}_t^{\boldsymbol{y}}$$

where the optimal process satisfies the stochastic differential equation

$$\mathrm{d}\bar{\boldsymbol{X}}_t = \{\boldsymbol{A}(t) - \boldsymbol{B}(t)[\boldsymbol{R}(t)]^{-1}\boldsymbol{B}(t)^*\boldsymbol{P}(t)\}\bar{\boldsymbol{X}}_t + \boldsymbol{\sigma}(t)\,\mathrm{d}W_t$$

on the interval $(s,T]$ with the initial condition $\bar{\boldsymbol{X}}_s = \boldsymbol{y}$. □

We can obtain the existence of a unique global solution to the Riccati differential equation as a consequence of Theorem 11. This is an important example of a stochastic control argument used to obtain a result for a purely deterministic system.

Corollary 2. *The Riccati equation* (RE) *admits a unique solution on the interval* $[0, T]$.

Proof. Let $\tau \in [0, T)$ be the explosion time of the local solution $\boldsymbol{P}$ to the Riccati equation (RE). Consider the deterministic control problem where the state of a system is governed by the differential equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t), \quad \boldsymbol{x}(s) = \boldsymbol{y}$$

on the interval $(s, T]$ where $\boldsymbol{u} \in L^2(0, T; \mathbb{R}^k)$ is subject to the quadratic cost

$$\begin{aligned}\tilde{J}_{s,T}(\boldsymbol{y}, \boldsymbol{u}) := \frac{1}{2}\int_s^T \Big[\Big\langle \boldsymbol{Q}(r)\boldsymbol{x}^{\boldsymbol{y},\boldsymbol{u}}(r), \boldsymbol{x}^{\boldsymbol{y},\boldsymbol{u}}(r)\Big\rangle_{\mathbb{R}^m} + \Big\langle \boldsymbol{R}(r)\boldsymbol{u}(r), \boldsymbol{u}(r)\Big\rangle_{\mathbb{R}^k}\Big] \,\mathrm{d}r \\ + \Big\langle \boldsymbol{G}\boldsymbol{x}^{\boldsymbol{y},\boldsymbol{u}}(T), \boldsymbol{x}^{\boldsymbol{y},\boldsymbol{u}}(T)\Big\rangle_{\mathbb{R}^m}.\end{aligned}$$

By Theorem 11, we have that

$$\inf_{\boldsymbol{u} \in L^2(s,T;\mathbb{R}^k)} \tilde{J}_{s,T}(\boldsymbol{y}, \boldsymbol{u}) = \frac{1}{2}\langle \boldsymbol{P}(s)\boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m}.$$

It follows immediately that

$$\langle \boldsymbol{P}(s)\boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m} \geq 0 \tag{37}$$

for every $s \in (\tau, T)$. Moreover, Theorem 11 implies that for every $s \in (\tau, T)$, the inequality

$$\begin{aligned}\langle \boldsymbol{P}(s)\boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m} &\leq \tilde{J}_{s,T}(\boldsymbol{y}, 0) \\ &= \frac{1}{2}\int_s^T \langle \boldsymbol{Q}(r)\boldsymbol{\Phi}(r)\boldsymbol{y}, \boldsymbol{\Phi}(r)\boldsymbol{y}\rangle_{\mathbb{R}^m} \,\mathrm{d}r + \langle \boldsymbol{G}\boldsymbol{\Phi}(T)\boldsymbol{y}, \boldsymbol{\Phi}(T)\boldsymbol{y}\rangle_{\mathbb{R}^m} \\ &\leq \frac{1}{2}\int_s^T K_1\|y\|^2_{\mathbb{R}^m} \,\mathrm{d}r + \frac{1}{2}K_2\|\boldsymbol{y}\|_{\mathbb{R}^m}\end{aligned}$$

holds with some positive constants K_1, K_2 that do not depend on s. Hence, we obtain that

$$\langle \boldsymbol{P}(s)\boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m} \leq K_3\|\boldsymbol{y}\|^2_{\mathbb{R}^m} \tag{38}$$

with some positive constant K_3 that does not depend on s. The inequalities (37) and (38) imply that for every $\boldsymbol{y} \in \mathbb{R}^m$ the function

$$s \mapsto \langle \boldsymbol{P}(s)\boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m}$$

is bounded on (τ, T). Moreover, since $\boldsymbol{P}(s)$ is a symmetric matrix, the equality

$$\langle \boldsymbol{P}(s)\boldsymbol{x}, \boldsymbol{y}\rangle_{\mathbb{R}^m} = \frac{1}{2}\Big(\langle \boldsymbol{P}(s)(\boldsymbol{x}+\boldsymbol{y}), \boldsymbol{x}+\boldsymbol{y}\rangle_{\mathbb{R}^m} - \langle \boldsymbol{P}(s)(\boldsymbol{x}-\boldsymbol{y}), \boldsymbol{x}-\boldsymbol{y}\rangle_{\mathbb{R}^m}\Big)$$

holds for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ and every $s \in (\tau, T)$ and hence, the map $s \mapsto \boldsymbol{P}(s)$ is bounded on the interval (τ, T). This is, however, a contradiction with τ being the time of explosion. □

2.2 Infinite horizon LQ optimal control

In the previous section, we dealt with a linear-quadratic control problem on a finite time horizon. This section will concern some linear-quadratic control problems on infinite time horizon.

Let us assume that the state of a physical system is governed by the stochastic differential equation

$$\begin{cases} \mathrm{d}\boldsymbol{X}_t &= (\boldsymbol{A}\boldsymbol{X}_t + \boldsymbol{B}\boldsymbol{u}_t)\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}W_t, \quad t \geq 0, \\ \boldsymbol{X}_0 &= \boldsymbol{y}. \end{cases} \tag{39}$$

where we assume that $\boldsymbol{A} \in \mathbb{R}^{m\times m}$, $\boldsymbol{B} \in \mathbb{R}^{m\times k}$, $\boldsymbol{\sigma} \in \mathbb{R}^k$ are deterministic matrices, $(W_t)_{t\geq 0}$ is a one-dimensional Wiener process, and $\boldsymbol{y} \in \mathbb{R}^m$. We will consider two control problems - a control problem with discounted cost and a control problem with ergodic cost. Denote by $(\boldsymbol{X}_t^{\boldsymbol{y},\boldsymbol{u}})_{t\geq 0}$ the solution to equation (39) and define the following two cost functionals:

$$J_\alpha(\boldsymbol{y},\boldsymbol{u}) := \mathbb{E}\int_0^\infty \Big[\langle \boldsymbol{Q}\boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} + \langle \boldsymbol{R}\boldsymbol{u}_r, \boldsymbol{u}_r\rangle_{\mathbb{R}^k}\Big]\mathrm{e}^{-\alpha t}\,\mathrm{d}r \tag{DC}$$

for $\alpha > 0$ and

$$J_0(\boldsymbol{y},\boldsymbol{u}) := \limsup_{T\to\infty}\frac{1}{T}\,\mathbb{E}\int_0^T \Big[\langle \boldsymbol{Q}\boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_r^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} + \langle \boldsymbol{R}\boldsymbol{u}_r, \boldsymbol{u}_r\rangle_{\mathbb{R}^k}\Big]\,\mathrm{d}r \tag{EC}$$

where $\boldsymbol{Q} \in \mathbb{R}^{m\times m}$ satisfies $\boldsymbol{Q}^* = \boldsymbol{Q}$ and $\boldsymbol{Q} > 0$, and $\boldsymbol{R} \in \mathbb{R}^{k\times k}$ satisfies $\boldsymbol{R}^* = \boldsymbol{R}$ and $\boldsymbol{R} > 0$.

2.2.1 Intermezzo: Deterministic control theory

In order to formulate the results, we will need several notions from deterministic control theory. Let us begin by the definition of exponential stability.

Remark 8. Recall that for $\boldsymbol{A} \in \mathbb{R}^m$, the fundamental system to the equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

is given by the matrix exponential

$$\boldsymbol{\Phi}(t) = \mathrm{e}^{\boldsymbol{A}t}.$$

We say that $(\boldsymbol{\Phi}(t), t \geq 0)$ is *exponentially stable* if there exist constants $M > 0$ and $\omega > 0$ such that the estimate

$$\|\boldsymbol{\Phi}(t)\| \leq M\mathrm{e}^{-\omega t}$$

holds for every $t \geq 0$. Here, the norm $\|\cdot\|$ is any matrix norm (recall that these are all equivalent). It follows that $(\boldsymbol{\Phi}(t), t \geq 0)$ is exponentially stable if and only if all the eigenvalues of the matrix $\boldsymbol{A}$ have negative real part.

Definition 12. Consider the controlled deterministic differential equation

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \qquad t \geq 0, \\ \boldsymbol{x}(0) = \boldsymbol{y} \end{cases} \tag{40}$$

where $\boldsymbol{u} \in L^1_{\mathrm{loc}}(0,\infty;\mathbb{R}^k)$ and $\boldsymbol{y} \in \mathbb{R}^m$ and denote by $\boldsymbol{x}^{\boldsymbol{y},\boldsymbol{u}}$ its solution. The pair $(\boldsymbol{A}, \boldsymbol{B})$ as well as equation (40) is called

- *stabilizable* if there exists a matrix $\boldsymbol{K} \in \mathbb{R}^{k\times m}$ such that the exponential matrix $\mathrm{e}^{\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}}$ is exponentially stable;

- *controllable at time* $T > 0$, if for every initial condition $\boldsymbol{x} \in \mathbb{R}^m$ and every terminal point $\boldsymbol{z} \in \mathbb{R}^m$, there exists a control $\boldsymbol{u} \in L^1(0, T; \mathbb{R}^k)$ such that $\boldsymbol{x}^{\boldsymbol{y},\boldsymbol{u}}(T) = \boldsymbol{z}$.

Remark 9. If the fundamental system $(\mathrm{e}^{\boldsymbol{A}t}, t \geq 0)$ is exponentially stable, then the pair $(\boldsymbol{A}, \boldsymbol{B})$ is trivially stabilizable (simply choose $\boldsymbol{K} = 0$). On the other hand, if the system $(\mathrm{e}^{\boldsymbol{A}t}, t \geq 0)$ is not exponentially stable on its own and $\boldsymbol{B} = \boldsymbol{I}$, then we can set $\boldsymbol{K} = -\lambda \boldsymbol{I}$ and obtain exponential stability of the system $(\mathrm{e}^{(\boldsymbol{A}-\lambda \boldsymbol{I})t}, t \geq 0)$ for sufficiently large $\lambda > 0$. Thus, stabilizability of the pair $(\boldsymbol{A}, \boldsymbol{B})$ says that the matrix $\boldsymbol{B}$ is, in a certain sense, not degenerate.

The following result shows that controllability is stronger than stabilizability.

Theorem 12. *If the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable, then the pair $(\boldsymbol{A}, \boldsymbol{B})$ is also stabilizable.*

Proof. See [8]. □

We continue with characterization of controllability.

Theorem 13. *The following claims are equivalent:*

1. *There exists a $T > 0$ such that the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable at T.*
2. *For every $T > 0$, the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable at T.*
3. *Every $\boldsymbol{z} \in \mathbb{R}^m$ is attainable from $\boldsymbol{x} = 0$, that is, there exist a time $T > 0$ and a control $\bar{\boldsymbol{u}} \in L^1(0, T; \mathbb{R}^k)$ such that $\boldsymbol{y}^{0,\bar{\boldsymbol{u}}}(T) = \boldsymbol{z}$.*
4. *Kalman rank condition: The matrix*
$$(\boldsymbol{A}|\boldsymbol{B}) := (\boldsymbol{B}, \boldsymbol{A}\boldsymbol{B}, \dots, \boldsymbol{A}^{m-1}\boldsymbol{B})$$
has full rank, i.e.
$$\operatorname{rank}(\boldsymbol{A}|\boldsymbol{B}) = m.$$

Proof. See [8, Theorem 1.2]. □

As a consequence, we can obtain the following.

Proposition 5. *For $T > 0$, denote*[8]
$$\boldsymbol{Q}_T := \int_0^T \mathrm{e}^{\boldsymbol{A}t} \boldsymbol{B}\boldsymbol{B}^* \mathrm{e}^{\boldsymbol{A}^* t} \, \mathrm{d}t.$$
Then the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable at time T if and only if $\boldsymbol{Q}_T > 0$[9]*.*

Proof. We will only show that if $\boldsymbol{Q}_T > 0$, then the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable at T. Let $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^m$. Since $\boldsymbol{Q}_T > 0$, the matrix $\boldsymbol{Q}_T$ is invertible and we can define the control
$$\boldsymbol{u}_t := -\boldsymbol{B}^* \mathrm{e}^{\boldsymbol{A}^*(T-t)} \boldsymbol{Q}_T^{-1} \left(\mathrm{e}^{\boldsymbol{A}T} \boldsymbol{y} - \boldsymbol{z} \right), \quad t \in [0, T].$$
Then, using the variation of constants formula from Theorem 4, we have for the solution $\boldsymbol{x}^{\boldsymbol{y},\boldsymbol{u}}$ to equation (40) that
$$\boldsymbol{x}^{\boldsymbol{y},\boldsymbol{u}}(T) = \mathrm{e}^{\boldsymbol{A}T}\boldsymbol{y} - \left(\int_0^T \mathrm{e}^{\boldsymbol{A}(T-t)} \boldsymbol{B}\boldsymbol{B}^* \mathrm{e}^{\boldsymbol{A}^*(T-t)} \, \mathrm{d}t \right) \boldsymbol{Q}_T^{-1} (\mathrm{e}^{\boldsymbol{A}T}\boldsymbol{y} - \boldsymbol{z}) = \boldsymbol{z}.$$
In other words, we have found a control $\boldsymbol{u}$ that steers the solution to equation (42) from the initial value $\boldsymbol{y}$ to the point $\boldsymbol{z}$. □

[8]The matrix $\boldsymbol{Q}_T$ is usually called the *controllability matrix* or the *controllability Gramian.*
[9]Note that it follows from the definition of $\boldsymbol{Q}_T$ that $\boldsymbol{Q}_T \geq 0$.

2.2.2 Infinite horizon LQ optimal control

Let us now consider the following family of equations parametrized by $\alpha \geq 0$.

$$\boldsymbol{V}_\alpha \boldsymbol{A} + \boldsymbol{A}^* \boldsymbol{V}_\alpha - \boldsymbol{V}_\alpha \boldsymbol{B} \boldsymbol{R}^{-1} \boldsymbol{B}^* \boldsymbol{V}_\alpha + \boldsymbol{Q} = \alpha \boldsymbol{V}_\alpha \tag{ARE$_\alpha$}$$

The solution to equations (ARE$_\alpha$) will be sought in the following set

$$M := \{\boldsymbol{V} \in \mathbb{R}^{m\times m} : \boldsymbol{V}^* = \boldsymbol{V}, \boldsymbol{V} \geq 0\}.$$

Remark 10. In dimension one, equation (ARE$_\alpha$) is the quadratic equation

$$-\frac{B^2}{R} V_\alpha^2 + (2A - \alpha)V_\alpha + Q = 0.$$

Since both Q and R are positive, we have that the above equation has two solutions - one negative and one positive and we will only consider the positive one.

Theorem 14. *Assume that the pair $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable. Then the equation (ARE$_\alpha$) has a unique solution $\boldsymbol{V}_\alpha$ in the set M.*

Sketch of proof. We will only consider the case $\alpha = 0$ (for $\alpha > 0$, the proof is similar). The solution to equation (ARE$_\alpha$) will be sought as a limit of solution of a certain Riccati differential equation. Consider the equation

$$\begin{cases} \dot{\boldsymbol{V}}(t) &= \boldsymbol{V}(t)\boldsymbol{A} + \boldsymbol{A}^*\boldsymbol{V}(t) - \boldsymbol{V}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}(t) + \boldsymbol{Q}, \quad t \geq 0, \\ \boldsymbol{V}(0) &= 0. \end{cases} \tag{41}$$

Note that contrary to equation (RE), equation (41) has an initial, not a terminal condition.

Step 1: Equation (41) has a symmetric, non-negative solution $\boldsymbol{V}$ defined on the interval $[0, \infty)$. *Proof of Step 1:* Let $T > 0$ and $\boldsymbol{y} \in \mathbb{R}^m$ be arbitrary and consider the following (deterministic) control problem. The system output is modelled by a function $\boldsymbol{x}^{\boldsymbol{y},\boldsymbol{u}}$ that satisfies the equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{42}$$

on the time interval $[0, T]$ with the initial condition $\boldsymbol{x}(0) = \boldsymbol{y}$. The control $\boldsymbol{u}$ is sought in the set of admissible controls $L^2(0, T; \mathbb{R}^k)$ so that the cost

$$\tilde{J}_T(\boldsymbol{y}, \boldsymbol{u}) := \int_0^T \left[\langle \boldsymbol{Q}\boldsymbol{x}^{\boldsymbol{y},\boldsymbol{u}}(s), \boldsymbol{x}^{\boldsymbol{y},\boldsymbol{u}}(s)\rangle_{\mathbb{R}^m} + \langle \boldsymbol{R}\boldsymbol{u}(s), \boldsymbol{u}(s)\rangle_{\mathbb{R}^k} \right] \mathrm{d}s$$

is minimal. The existence of a symmetric non-negative solution $\boldsymbol{V}$ to (41) can be shown by a similar argument as in Corollary 2.

Step 2: For every $\boldsymbol{y} \in \mathbb{R}^m$, the following limit exists and is finite:

$$\lim_{T\to\infty} \langle \boldsymbol{V}(T)\boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m}.$$

Proof of Step 2: Let $T > 0$ and $\boldsymbol{y} \in \mathbb{R}^m$ be arbitrary. For the solution $\boldsymbol{V}$ to (41), we have that

$$0 \leq \langle \boldsymbol{V}(T)\boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m} = \min_{\boldsymbol{u}\in L^2(0,T;\mathbb{R}^k)} \tilde{J}_T(\boldsymbol{y}, \boldsymbol{u}).$$

Since the pair $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable, there exists a matrix $\boldsymbol{K} \in \mathbb{R}^{k\times m}$ such that the system $(\mathrm{e}^{(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})t}, t \geq 0)$ is exponentially stable. This implies that the solution $\bar{\boldsymbol{x}}^{\boldsymbol{y}}$ to the equation

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \boldsymbol{y},$$

(that is equation (42) with $\boldsymbol{u} = \bar{\boldsymbol{u}} := \boldsymbol{K}\boldsymbol{x}$) satisfies

$$\|\bar{\boldsymbol{x}}^{\boldsymbol{y}}(t)\|_{\mathbb{R}^m} \leq \left\|\mathrm{e}^{(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})t}\boldsymbol{y}\right\|_{\mathbb{R}^m} \leq M\mathrm{e}^{-\omega t}\|\boldsymbol{y}\|_{\mathbb{R}^m}$$

for every $t \in [0, T]$ where the constants M and ω come from the exponential stability of the fundamental system. Consequently, we have the following chain of estimates:

$$\begin{aligned}
\min_{\boldsymbol{u}\in L^2(0,T;\mathbb{R}^k)} \tilde{J}_T(\boldsymbol{y}, \boldsymbol{u}) &\leq \tilde{J}_T(\boldsymbol{y}, \bar{\boldsymbol{u}}) \\
&= \int_0^T \Big[\langle \boldsymbol{Q}\bar{\boldsymbol{x}}^{\boldsymbol{y}}(s), \bar{\boldsymbol{x}}^{\boldsymbol{y}}(s)\rangle_{\mathbb{R}^m} + \langle \boldsymbol{R}\bar{\boldsymbol{u}}(s), \bar{\boldsymbol{u}}(s)\rangle_{\mathbb{R}^k}\Big]\,\mathrm{d}s \\
&= \int_0^T \Big[\langle \boldsymbol{Q}\bar{\boldsymbol{x}}^{\boldsymbol{y}}(s), \bar{\boldsymbol{x}}^{\boldsymbol{y}}(s)\rangle_{\mathbb{R}^m} + \langle \boldsymbol{R}\boldsymbol{B}\bar{\boldsymbol{x}}^{\boldsymbol{y}}(s), \boldsymbol{B}\bar{\boldsymbol{x}}^{\boldsymbol{y}}(s)\rangle_{\mathbb{R}^k}\Big]\,\mathrm{d}s \\
&\leq \int_0^T K_1\|\bar{\boldsymbol{x}}^{\boldsymbol{y}}(s)\|^2_{\mathbb{R}^m}\,\mathrm{d}s + \int_0^T K_2\|\bar{\boldsymbol{x}}^{\boldsymbol{y}}(s)\|^2_{\mathbb{R}^m}\,\mathrm{d}s \\
&\leq K_3 \int_0^T M^2\mathrm{e}^{-2\omega t}\|\boldsymbol{y}\|^2_{\mathbb{R}^m}\,\mathrm{d}t \\
&\leq K_3 \int_0^\infty M^2\mathrm{e}^{-2\omega t}\|\boldsymbol{y}\|^2_{\mathbb{R}^m}\,\mathrm{d}t \\
&\leq K_4\|\boldsymbol{y}\|^2_{\mathbb{R}^m}
\end{aligned}$$

where the K_i's are positive constants that do not depend on T. Hence, we obtain that the estimate

$$0 \leq |\langle \boldsymbol{V}(T)\boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m}| \leq K_4\|\boldsymbol{y}\|^2_{\mathbb{R}^m}$$

holds. Therefore, the function $T \mapsto \langle \boldsymbol{V}(T)\boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m}$ is bounded on the interval $[0, \infty)$. By a similar argument, it can also be show that this function is non-decreasing and we obtain the claim.

Step 3: The limit

$$\boldsymbol{V}_0 := \lim_{T\to\infty} \boldsymbol{V}(T)$$

exists in the strong operator topology (i.e. in any matrix norm) and $\boldsymbol{V}_0$ satisfies equation (ARE_0).

Proof of Step 3: Let $T > 0$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ be arbitrary. By *Step 2*, we have that the expression

$$\langle \boldsymbol{V}(T)\boldsymbol{x}, \boldsymbol{y}\rangle_{\mathbb{R}^m} = \frac{1}{2}\Big(\langle \boldsymbol{V}(T)(\boldsymbol{x}+\boldsymbol{y}), \boldsymbol{x}+\boldsymbol{y}\rangle_{\mathbb{R}^m} - \langle \boldsymbol{V}(T)(\boldsymbol{x}-\boldsymbol{y}), \boldsymbol{x}-\boldsymbol{y}\rangle_{\mathbb{R}^m}\Big)$$

has a finite limit as $T \to \infty$. It follows that the matrix $\boldsymbol{V}(T)$ tends to some matrix $\boldsymbol{V}_0$ as $T \to \infty$ in any matrix norm. Moreover, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}T}\langle \boldsymbol{V}(T)\boldsymbol{x}, \boldsymbol{y}\rangle_{\mathrm{R}^m} = \langle \boldsymbol{V}(T)\boldsymbol{x}, \boldsymbol{A}\boldsymbol{y}\rangle_{\mathbb{R}^m} &+ \langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{V}(T)\boldsymbol{y}\rangle_{\mathbb{R}^m} \\
&- \langle \boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}(T)\boldsymbol{x}, \boldsymbol{B}^*\boldsymbol{V}(T)\boldsymbol{y}\rangle_{\mathbb{R}^m} + \langle \boldsymbol{Q}\boldsymbol{x}, \boldsymbol{y}\rangle_{\mathbb{R}^m}.
\end{aligned}$$

The right-hand side of the above equation is convergent with $T \to \infty$ to some $\langle \boldsymbol{N}\boldsymbol{x}, \boldsymbol{y}\rangle_{\mathbb{R}^m}$. On the other hand, the limit $\langle \boldsymbol{N}\boldsymbol{x}, \boldsymbol{y}\rangle_{\mathbb{R}^m}$ must be zero since otherwise, the function $T \mapsto \langle \boldsymbol{V}(T)\boldsymbol{x}, \boldsymbol{y}\rangle_{\mathbb{R}^m}$ could not be bounded. It follows that the equation

$$\langle \boldsymbol{V}_0\boldsymbol{x}, \boldsymbol{A}\boldsymbol{y}\rangle_{\mathbb{R}^m} + \langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{V}_0\boldsymbol{y}\rangle_{\mathbb{R}^m} - \langle \boldsymbol{V}_0\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}_0\boldsymbol{x}, \boldsymbol{y}\rangle_{\mathbb{R}^m} + \langle \boldsymbol{Q}\boldsymbol{x}, \boldsymbol{y}\rangle_{\mathbb{R}^m} = 0$$

holds and this concludes the proof. □

The following theorem gives the solution to the infinite-time horizon control problems.

Theorem 15. *Assume that the pair $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable. Then the feedback control*

$$\bar{\boldsymbol{u}}_t^\alpha := -\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}_\alpha\boldsymbol{X}_t$$

is a control for equation (39) that minimizes

(1) *(for $\alpha > 0$) the cost function (DC) over the set of admissible controls $\mathscr{U}_\alpha$. Here, $\mathscr{U}_\alpha$ denotes the set of $(\mathscr{F}_t)$-progressively measurable stochastic processes $\boldsymbol{u} \in L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \Omega; \mathbb{R}^k)$ such that*

$$\lim_{T\to\infty} \mathrm{e}^{-\alpha T}\, \mathbb{E}\langle \boldsymbol{V}_\alpha\boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} = 0.$$

In this case, the optimal cost is

$$J_\alpha(\boldsymbol{y}, \bar{\boldsymbol{u}}^\alpha) = \langle \boldsymbol{V}_\alpha\boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m} + \frac{1}{\alpha}\langle \boldsymbol{V}_\alpha\boldsymbol{\sigma}, \boldsymbol{\sigma}\rangle_{\mathbb{R}^m}.$$

(2) *(for $\alpha = 0$) the cost function (EC) over the set of admissible controls $\mathscr{U}_0$. Here, $\mathscr{U}_0$ denotes the set of $(\mathscr{F}_t)$-progressively measurable stochastic processes such that*

$$\lim_{T\to\infty} \frac{1}{T}\, \mathbb{E}\langle \boldsymbol{V}_0\boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} = 0.$$

In this case, the optimal cost is

$$J_0(\boldsymbol{y}, \bar{\boldsymbol{u}}^0) = \langle \boldsymbol{V}_0\boldsymbol{\sigma}, \boldsymbol{\sigma}\rangle_{\mathbb{R}^m}.$$

Before we prove the theorem, let us state and prove two lemmas that will be needed.

Lemma 1. *Let $\boldsymbol{M} \in \mathbb{R}^{m\times m}$. We have that the estimate*

$$\int_0^\infty \left\|\mathrm{e}^{\boldsymbol{M}t}\boldsymbol{x}\right\|^2_{\mathbb{R}^m}\, \mathrm{d}r \le C\|\boldsymbol{x}\|^2_{\mathbb{R}^m}$$

holds for every $\boldsymbol{x} \in \mathbb{R}^m$ with some finite positive constant C if and only if there exist finite positive constants $c, \omega > 0$ such that the estimate

$$\left\|\mathrm{e}^{\boldsymbol{M}t}\boldsymbol{x}\right\|^2_{\mathbb{R}^m} \le c\mathrm{e}^{-\omega t}\|\boldsymbol{x}\|_{\mathbb{R}^m}$$

holds for every $\boldsymbol{x} \in \mathbb{R}^m$ and every $t \ge 0$.

Lemma 2. *Let $\boldsymbol{x}^{\boldsymbol{y}}$ be the solution to the following differential equation:*

$$\begin{cases} \dot{\boldsymbol{x}}(t) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}_\alpha)\boldsymbol{x}(t), & t \ge 0, \\ \boldsymbol{x}(0) = \boldsymbol{y} \end{cases}$$

where $\boldsymbol{y} \in \mathbb{R}^m$. Then we have that there exists a finite positive constant C such that the estimate

$$\|\boldsymbol{x}^{\boldsymbol{y}}\|_{\mathbb{R}^m} \le C\|\boldsymbol{y}\|_{\mathbb{R}^m}$$

holds.

Proof of Theorem 15. Let $\alpha \geq 0$ and let $\boldsymbol{u} \in \mathscr{U}_\alpha$ be arbitrary. By the Itô formula used on the function $(t, \boldsymbol{x}) \mapsto \mathrm{e}^{-\alpha t}\langle \boldsymbol{V}_\alpha \boldsymbol{x}, \boldsymbol{x}\rangle_{\mathbb{R}^m}$, we have that

$$
\begin{aligned}
&\mathbb{E}\, \mathrm{e}^{-\alpha T}\langle \boldsymbol{V}_\alpha \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} - \langle \boldsymbol{V}_\alpha \boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m} = \\
&\qquad = \mathbb{E} \int_0^T \mathrm{e}^{-\alpha s}\Big[- \alpha \langle \boldsymbol{V}_\alpha \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} \\
&\qquad\qquad + \langle (\boldsymbol{A}^*\boldsymbol{V}_\alpha + \boldsymbol{V}_\alpha \boldsymbol{A})\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} \\
&\qquad\qquad + 2\langle \boldsymbol{B}^*\boldsymbol{V}_\alpha \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{u}_s\rangle_{\mathbb{R}^k} \\
&\qquad\qquad + \langle \boldsymbol{V}_\alpha \boldsymbol{\sigma}, \boldsymbol{\sigma}\rangle_{\mathbb{R}^m}\Big]\,\mathrm{d}s \\
&\qquad = \mathbb{E} \int_0^T \mathrm{e}^{-\alpha s}\Big[- \langle \boldsymbol{Q} \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} \\
&\qquad\qquad + \langle \boldsymbol{V}_\alpha \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}_\alpha \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} \\
&\qquad\qquad + 2\langle \boldsymbol{B}^*\boldsymbol{V}_\alpha \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{u}_s\rangle_{\mathbb{R}^k} \\
&\qquad\qquad + \langle \boldsymbol{V}_\alpha \boldsymbol{\sigma}, \boldsymbol{\sigma}\rangle_{\mathbb{R}^m}\Big]\,\mathrm{d}s.
\end{aligned}
$$

where we used that $\boldsymbol{V}_\alpha$ satisfies equation (ARE_α). By adding and subtracting the term $\langle \boldsymbol{R}\boldsymbol{u}_s, \boldsymbol{u}_s\rangle_{\mathbb{R}^k}$ to the right-hand side of the above equation, we obtain

$$
\begin{aligned}
&\mathbb{E} \int_0^T \mathrm{e}^{-\alpha s}\Big[\langle \boldsymbol{Q} \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} + \langle \boldsymbol{R}\boldsymbol{u}_s, \boldsymbol{u}_s\rangle_{\mathbb{R}^k}\Big]\,\mathrm{d}s = \\
&\qquad = \langle \boldsymbol{V}_\alpha \boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m} + \int_0^T \mathrm{e}^{-\alpha s}\langle \boldsymbol{V}_\alpha \boldsymbol{\sigma}, \boldsymbol{\sigma}\rangle_{\mathbb{R}^m} - \mathbb{E}\, \mathrm{e}^{-\alpha T}\langle \boldsymbol{V}_\alpha \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} \\
&\qquad\quad + \mathbb{E} \int_0^T \mathrm{e}^{-\alpha s}\left|\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{u}_s + \boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{B}^*\boldsymbol{V}_\alpha \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\right|^2\,\mathrm{d}s.
\end{aligned}
\tag{43}
$$

Now, if $\alpha > 0$, we obtain

$$
J_\alpha(\boldsymbol{y}, \boldsymbol{u}) \geq \langle \boldsymbol{V}_\alpha \boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m} + \frac{1}{\alpha}\langle \boldsymbol{V}_\alpha \boldsymbol{\sigma}, \boldsymbol{\sigma}\rangle_{\mathbb{R}^m}
\tag{44}
$$

by taking the limit $T \to \infty$ of equation (43) because the third term on the right-hand side of equation (43) tends to zero as $T \to \infty$ by the assumption that $\boldsymbol{u} \in \mathscr{U}_\alpha$ and the fourth term is non-negative. On the other hand, when $\alpha = 0$, we obtain, by dividing both sides of equation (43) by T and taking $\limsup_{T\to\infty}$, that

$$
\begin{aligned}
J_0(\boldsymbol{y}, \boldsymbol{u}) = \limsup_{T\to\infty} \frac{1}{T}\Big(&\langle \boldsymbol{V}_0 \boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m} + T\langle \boldsymbol{V}_0 \boldsymbol{\sigma}, \boldsymbol{\sigma}\rangle_{\mathbb{R}^m} - \mathbb{E}\langle \boldsymbol{V}_0 \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} \\
&+ \mathbb{E} \int_0^T \mathrm{e}^{-\alpha s}\left|\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{u}_s + \boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{B}^*\boldsymbol{V}_\alpha \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\right|^2\,\mathrm{d}s\Big)
\end{aligned}
$$

holds so that we have the estimate

$$
J_0(\boldsymbol{y}, \boldsymbol{u}) \geq \langle \boldsymbol{V}_0 \boldsymbol{\sigma}, \boldsymbol{\sigma}\rangle_{\mathbb{R}^m}.
\tag{45}
$$

Clearly, in inequalities (44) and (45), equality occurs if $\boldsymbol{u}_s = \bar{\boldsymbol{u}}_s^\alpha$ where

$$
\bar{\boldsymbol{u}}_s^\alpha = -\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}_\alpha \boldsymbol{X}_t;
$$

however, we still need to prove that this control is admissible, i.e. that $\bar{\boldsymbol{u}}_s^\alpha$ belongs to the set of admissible controls $\mathscr{U}_\alpha$. We will only prove this for $\alpha = 0$. The case $\alpha > 0$ is similar. For this purpose, let us consider the solution $\bar{\boldsymbol{X}}^{\boldsymbol{y}}$ to the closed-loop equation

$$\begin{cases} \mathrm{d}\bar{\boldsymbol{X}}_t = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}_0)\bar{\boldsymbol{X}}_t \,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}W_t, \quad t \in (0,T), \\ \bar{\boldsymbol{X}}_0 = \boldsymbol{y} \end{cases}$$

that is equation (39) controlled by $\bar{\boldsymbol{u}}^0$. By Theorem 5, we have that

$$\bar{\boldsymbol{X}}_t^{\boldsymbol{y}} = \mathrm{e}^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}_0)t}\boldsymbol{y} + \int_0^t \mathrm{e}^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}_0)(t-s)}\boldsymbol{\sigma}\,\mathrm{d}W_s.$$

This implies, by using the Itô isometry from Proposition 2 and Lemma 2, that

$$\begin{aligned} \mathbb{E}\left\|\bar{\boldsymbol{X}}_T^{\boldsymbol{y}}\right\|_{\mathbb{R}^m}^2 &\leq 2\left\|\mathrm{e}^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}_0)t}\boldsymbol{y}\right\|_{\mathbb{R}^m}^2 + 2\int_0^T \left\|\mathrm{e}^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}_0)(t-s)}\boldsymbol{\sigma}\right\|_{\mathbb{R}^m}^2 \mathrm{d}s \\ &\leq 2K_1\mathrm{e}^{-2K_2T}\|\boldsymbol{y}\|_{\mathbb{R}^m}^2 + 2K_1\|\boldsymbol{\sigma}\|_{\mathbb{R}^m}^2 \int_0^T \mathrm{e}^{-2K_2s}\,\mathrm{d}s \end{aligned}$$

where K_1 and K_2 are two positive constants that do not depend on T. Hence, there exists a finite positive constant C that is independent of T such that $\mathbb{E}\left\|\bar{\boldsymbol{X}}_T^{\boldsymbol{y}}\right\|_{\mathbb{R}^m}^2 \leq C$ and thus,

$$\lim_{T\to\infty} \frac{1}{T}\,\mathbb{E}\left\|\bar{\boldsymbol{X}}_T^{\boldsymbol{y}}\right\|_{\mathbb{R}^m}^2 = 0.$$

Hence, $\bar{\boldsymbol{u}}^0 \in \mathscr{U}_0$ which completes the proof. □

Remark 11. For simplicity, we assumed in the formulation of the infinite horizon LQ control problem that $\boldsymbol{Q} > 0$. Note, however, that Theorem 15 would also hold if we only assumed that $\boldsymbol{Q} \geq 0$ and that the pair $(\boldsymbol{A}, \boldsymbol{Q})$ is *detectable*. The pair $(\boldsymbol{A}, \boldsymbol{Q})$ is called detectable if there exists a matrix $\boldsymbol{L} \in \mathbb{R}^{m\times m}$ such that the system $(\mathrm{e}^{(\boldsymbol{A}-\boldsymbol{L}\sqrt{Q})t}, t \geq 0)$ is exponentially stable.

Corollary 3. *The control*

$$\bar{\boldsymbol{u}}_t^0 = -\boldsymbol{R}^{-1}\boldsymbol{B}^*\boldsymbol{V}_0\bar{\boldsymbol{X}}_t^{\boldsymbol{y}}$$

is an $\mathbb{P}$-almost sure minimizer of the pathwise ergodic cost

$$J_{p,0}(\boldsymbol{y},\boldsymbol{u}) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left[\langle \boldsymbol{Q}\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} + \langle \boldsymbol{R}\boldsymbol{u}_s, \boldsymbol{u}_s\rangle_{\mathbb{R}^k}\right] \mathrm{d}s$$

over the set $\mathscr{U}_{p,0}$ that consist of all $(\mathscr{F}_t)$-progressively measurable processes $\boldsymbol{u} \in L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \Omega; \mathbb{R}^k)$ for which it holds that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \|\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\|^2\,\mathrm{d}s < \infty \quad \textit{and} \quad \lim_{T\to\infty} \frac{1}{T}\langle \boldsymbol{V}_0\boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^k} = 0$$

$\mathbb{P}$-almost surely.

Proof. Let $\boldsymbol{u} \in \mathscr{U}_{p,0}$ be arbitrary. By using the Itô formula similarly as in the proof of Theorem 15, we obtain that the equality

$$\begin{aligned}\frac{1}{T}\int_0^T \Big[\langle \boldsymbol{Q}\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} + \langle \boldsymbol{R}\boldsymbol{u}_s, \boldsymbol{u}_s\rangle_{\mathbb{R}^k}\Big]\,\mathrm{d}s = \langle \boldsymbol{V}_0\boldsymbol{\sigma}, \boldsymbol{\sigma}\rangle_{\mathbb{R}^m} + \frac{1}{T}\langle \boldsymbol{V}_0\boldsymbol{y}, \boldsymbol{y}\rangle_{\mathbb{R}^m} \\ - \frac{1}{T}\langle \boldsymbol{V}_0\boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{X}_T^{\boldsymbol{y},\boldsymbol{u}}\rangle_{\mathbb{R}^m} \\ + \frac{1}{T}\int_0^T \left|\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{u}_s + \boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{B}^*\boldsymbol{V}_0\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}\right|^2\,\mathrm{d}s \\ + \frac{1}{T}M_T\end{aligned}$$

where

$$M_T := \int_0^T \langle \boldsymbol{V}_0\boldsymbol{X}^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{\sigma}\rangle_{\mathbb{R}^m}\,\mathrm{d}W_s.$$

holds $\mathbb{P}$-almost surely. Clearly, the corollary will be proved once we show that M_T/T tends to zero as $T \to \infty$ $\mathbb{P}$-almost surely. Since the process $(M_t)_{t\geq 0}$ is a continuous martingale, there exists a (standard) Wiener process $\tilde{W}$ such that the equality

$$M_T = \tilde{W}_{\langle M\rangle_T}$$

holds $\mathbb{P}$-almost surely with $\langle M\rangle$ being the quadratic variation of the process M. Using the Itô isometry, we have that this quadratic variation satisfies

$$\langle M\rangle_T = \int_0^T \langle \boldsymbol{V}_0\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}, \boldsymbol{\sigma}\rangle_{\mathbb{R}^m}^2\,\mathrm{d}s$$

$\mathbb{P}$-almost surely. Let us write

$$\frac{1}{T}M_T = \frac{\tilde{W}_{\langle M\rangle_T}}{\langle M\rangle_T}\cdot\frac{\langle M\rangle_T}{T}$$

and set

$$\Omega' := \{\omega \in \Omega : \exists C(\omega)\,\forall T \geq 0 : \langle M\rangle_T(\omega) \leq C(\omega)\}.$$

On Ω', we have that the limit $\tau := \lim_{T\to\infty}\langle M\rangle_T$ is finite and it holds that

$$\lim_{T\to\infty}\frac{\tilde{W}_{\langle M\rangle_T}}{\langle M\rangle_T} = \frac{\tilde{W}_\tau}{\tau} < \infty.$$

We therefore obtain that for almost every $\omega \in \Omega'$ it holds that

$$\lim_{T\to\infty}\frac{1}{T}M_T(\omega) = \lim_{T\to\infty}\frac{\tilde{W}_{\langle M\rangle_T}(\omega)}{\langle M\rangle_T(\omega)}\cdot\frac{\langle M\rangle_T(\omega)}{T} = 0.$$

On the other hand, for almost every $\omega \in \Omega\setminus\Omega'$, we have the convergence

$$\lim_{T\to\infty}\frac{\tilde{W}_{\langle M\rangle_T}(\omega)}{\langle M\rangle_T(\omega)} = 0$$

by the strong law of large numbers for the Wiener process and, furthermore, we have that there exist finite positive constants K_1 and $K_2(\omega)$, independent of T, such that the estimate

$$\frac{1}{T}\langle M\rangle_T(\omega) = \frac{1}{T}\int_0^T \langle \boldsymbol{V}_0\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}(\omega), \boldsymbol{\sigma}\rangle_{\mathbb{R}^m}^2\,\mathrm{d}s \leq K_1\frac{1}{T}\int_0^T \|\boldsymbol{X}_s^{\boldsymbol{y},\boldsymbol{u}}(\omega)\|_{\mathbb{R}^m}^2\,\mathrm{d}s \leq K_2(\omega)$$

holds (the last estimate follows from the fact that $\boldsymbol{u} \in \mathscr{U}_{p,0}$). Therefore, the convergence

$$\lim_{T\to\infty} \frac{1}{T} M_T(\omega) = \lim_{T\to\infty} \frac{\tilde{W}_{\langle M\rangle_T}(\omega)}{\langle M\rangle_T(\omega)} \cdot \frac{\langle M\rangle_T(\omega)}{T} = 0$$

holds also for almost every $\omega \in \Omega \setminus \Omega'$ which concludes the proof. □

2.3 Exercises

Exercise 8. Prove Proposition 4.

Exercise 9. Let $a > 0$, $b \in \mathbb{R}$, and $c < 0$. Let further $p_0 > 0$. Find the non-negative solution of the following deterministic Riccati differential equation:

$$\begin{cases} \dot{p}(t) &= a[p(t)]^2 + bp(t) + c, \quad t \in [0,T], \\ p(T) &= p_0. \end{cases}$$

Exercise 10. Let $T > 0$ and assume that the output of the modelled system is governed by the (bilinear) stochastic differential equation

$$\begin{cases} \mathrm{d}X_t &= [a(t)X_t + b(t)u_t]\,\mathrm{d}t + c(t)X_t\,\mathrm{d}W_t, \quad t \in (0,T), \\ X_0 &= y \end{cases}$$

where $a, b, c \in L^\infty(0,T)$ are deterministic functions and $y \in \mathbb{R}$. Let further $q, r \in L^\infty(0,T)$ be deterministic functions such that $q(t) \geq 0$ and $r(t) > 0$ for every $t \in (0,T)$ and let $m \geq 0$. Show that the control

$$u_t = -\frac{b(t)}{r(t)} p(t) X_t,$$

where p is the non-negative solution to the following Riccati equation

$$\begin{cases} \dot{p}(t) &= -\frac{b^2(t)}{r(t)} p^2(t) - [2a(t) + c^2(t)]p(t) - q(t), \quad t \in (0,T), \\ p(T) &= m, \end{cases}$$

minimizes the cost function

$$J_T(y,u) = \frac{1}{2}\mathbb{E}\int_0^T [q(s)X_s^2 + r(s)u_s^2]\,\mathrm{d}s + \frac{1}{2} m\,\mathbb{E}\,X_T^2$$

over the set of admissible controls $\mathscr{U} = L^2_{\mathscr{F}}(0,T)$.

Exercise 11. Let $T > 0$ and consider the equation of the controlled stochastic harmonic oscillator that is formally given by

$$\begin{cases} \ddot{x} + \omega^2 x = bu + \sigma\dot{W} \quad \text{on} \quad (0,T) \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$$

where $\sigma \in \mathbb{R}$ is such that $\sigma \neq 0$, $b \in \mathbb{R}$, $\omega > 0$ and x_0, v_0 are $\mathscr{F}_0$-measurable random variables. The equation is rigorously given meaning by the following system of stochastic differential equations

$$\begin{cases} \mathrm{d}X_t &= V_t\,\mathrm{d}t \\ \mathrm{d}V_t &= [-\omega^2 X_t + bu_t]\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \end{cases}$$

considered on $(0,T)$ with the initial conditions $X_0 = x_0$, $V_0 = v_0$. Rewrite the above system as a linear stochastic differential equation in $\mathbb{R}^2$ and show that the equation is controllable.

3 Filtering

3.1 Problem formulation

Let $T > 0$ and let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space. Denote by $\boldsymbol{X}^{\boldsymbol{x_0}}$ the solution to the stochastic differential equation

$$\begin{cases} \mathrm{d}\boldsymbol{X}_t &= \boldsymbol{A}(t)\boldsymbol{X}_t\,\mathrm{d}t + \boldsymbol{\sigma}(t)\,\mathrm{d}\boldsymbol{W}_t, \quad t \in (0,T), \\ \boldsymbol{X}_0 &= \boldsymbol{x_0} \end{cases} \tag{46}$$

where $\boldsymbol{A} \in L^\infty(0,T;\mathbb{R}^{m\times m})$, $\boldsymbol{\sigma} \in L^\infty(0,T;\mathbb{R}^{m\times n})$. The process $\boldsymbol{X}^{\boldsymbol{x_0}}$ will be called the *signal*. This signal is not observed directly but instead, we observe the stochastic process $\boldsymbol{Y}$ that has the stochastic differential

$$\begin{cases} \mathrm{d}\boldsymbol{Y}_t &= \boldsymbol{H}(t)\boldsymbol{X}_t\,\mathrm{d}t + \boldsymbol{\sigma_1}(t)\,\mathrm{d}\boldsymbol{V}_t, \quad t \in (0,T), \\ \boldsymbol{Y}_0 &= 0. \end{cases}$$

where $\boldsymbol{H} \in \mathscr{C}([0,T];\mathbb{R}^{k\times m})$ and $\boldsymbol{\sigma_1} \in \mathscr{C}([0,T];\mathbb{R}^{k\times k})$ is such that $\boldsymbol{\sigma_1}(t) > 0$ for every $t \in [0,T]$. The processes $(\boldsymbol{W}_t)_{t\in[0,T]}$ and $(\boldsymbol{V}_t)_{t\in[0,T]}$ are mutually independent n-dimensional $(\mathscr{F}_t)$-Wiener processes defined on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ and $\boldsymbol{x_0}$ is an $\mathscr{F}_0$-measurable Gaussian random variable.

Denote

$$\mathscr{G}_t := \sigma(\boldsymbol{Y}_s, 0 \le s \le t)$$

the filtration generated by the process $\boldsymbol{Y}$. In this section, we will try to find the best estimate $\hat{\boldsymbol{X}}$ (in the sense given a few lines below) of the signal $\boldsymbol{X}$ based on the history of the observed process $\boldsymbol{Y}$. More precisely, we will try to find a stochastic process $(\hat{\boldsymbol{X}}_t)_{t\in[0,T]}$ such that

- $\hat{\boldsymbol{X}}$ is $(\mathscr{G}_t)$-adapted,
- it holds that $\mathbb{E}\,\|\hat{\boldsymbol{X}}_t\|^2_{\mathbb{R}^m} < \infty$ for every $t \in [0,T]$,
- and such that for every $\boldsymbol{b} \in \mathbb{R}^m$ and every $t \in [0,T]$ it holds that

$$\mathbb{E}\langle \boldsymbol{b}, \boldsymbol{X}_t - \hat{\boldsymbol{X}}_t\rangle^2_{\mathbb{R}^m} = \inf_{\bar{\boldsymbol{X}}\in\mathscr{M}} \mathbb{E}\langle \boldsymbol{b}, \boldsymbol{X}_t - \bar{\boldsymbol{X}}_t\rangle^2_{\mathbb{R}^m}$$

  where $\mathscr{M}$ denotes the set of all $(\mathscr{G}_t)$-adapted stochastic processes $\boldsymbol{u} : [0,T]\times\Omega \to \mathbb{R}^m$ such that $\mathbb{E}\,\|\boldsymbol{u}_t\|^2_{\mathbb{R}^m} < \infty$ holds for every $t \in [0,T]$.

The process $\hat{\boldsymbol{X}}$ is called *filter*. Of course, we can immediately guess that the filter satisfies

$$\hat{\boldsymbol{X}}_t = \mathbb{E}[\boldsymbol{X}_t|\mathscr{G}_t] \tag{47}$$

for every $t \in [0,T]$ $\mathbb{P}$-almost surely. Indeed, let us take arbitrary $\boldsymbol{b} \in \mathbb{R}^m$, an arbitrary process $\bar{\boldsymbol{X}} \in \mathscr{M}$, and denote $\tilde{\boldsymbol{X}}_t := \mathbb{E}[\boldsymbol{X}_t|\mathscr{G}_t]$. Then we have that

$$\mathbb{E}\langle \boldsymbol{b}, \boldsymbol{X}_t - \bar{\boldsymbol{X}}_t\rangle^2_{\mathbb{R}^m} = \mathbb{E}\langle \boldsymbol{b}, \boldsymbol{X}_t - \tilde{\boldsymbol{X}}_t\rangle^2_{\mathbb{R}^m} + \mathbb{E}\langle \boldsymbol{b}, \tilde{\boldsymbol{X}}_t - \bar{\boldsymbol{X}}_t\rangle^2_{\mathbb{R}^m} + 2\,\mathbb{E}\langle \boldsymbol{b}, \boldsymbol{X}_t - \tilde{\boldsymbol{X}}_t\rangle_{\mathbb{R}^m}\langle \boldsymbol{b}, \tilde{\boldsymbol{X}}_t - \bar{\boldsymbol{X}}_t\rangle_{\mathbb{R}^m}$$

holds for every $t \in [0,T]$. The last term is shown to be zero by the law of total expectation and by the $\mathscr{G}_t$-measurability of $(\tilde{\boldsymbol{X}}_t - \bar{\boldsymbol{X}}_t)$. We thus obtain that the following holds:

$$\mathbb{E}\langle \boldsymbol{b}, \boldsymbol{X}_t - \bar{\boldsymbol{X}}_t\rangle^2_{\mathbb{R}^m} \ge \mathbb{E}\langle \boldsymbol{b}, \boldsymbol{X}_t - \bar{\boldsymbol{X}}_t\rangle^2_{\mathbb{R}^m}.$$

3.2 Kalman-Bucy filter

Theorem 16 (Kalman-Bucy filter)**.** *Denote* $\boldsymbol{a}(t) := \boldsymbol{\sigma}(t)\boldsymbol{\sigma}(t)^*$ *and* $\boldsymbol{a_1}(t) := \boldsymbol{\sigma_1}(t)\boldsymbol{\sigma_1}(t)^*$. *Under the assumptions made in this section, the following claims are true:*

1. *The filter* $\hat{\boldsymbol{X}}$ *satisfies the stochastic differential equation*

$$\begin{cases} \mathrm{d}\hat{\boldsymbol{X}}_t &= [\boldsymbol{A}(t) - \boldsymbol{F}(t)\boldsymbol{H}(t)]\hat{\boldsymbol{X}}_t \,\mathrm{d}t + \boldsymbol{F}(t)\,\mathrm{d}\boldsymbol{Y}_t, \quad t \in (0,T), \\ \hat{\boldsymbol{X}}_0 &= \mathbb{E}\,\boldsymbol{x_0} \end{cases} \tag{48}$$

   where

$$\boldsymbol{F}(t) := \boldsymbol{K}(t)\boldsymbol{H}(t)^*[\boldsymbol{a_1}(t)]^{-1}$$

   with $\boldsymbol{K}(t)$ *being the covariance matrix of the error* $\boldsymbol{e}_t := \boldsymbol{X}_t - \hat{\boldsymbol{X}}_t$, *i.e.* $\boldsymbol{K}(t) := \mathbb{E}\,\boldsymbol{e}_t\boldsymbol{e}_t^*$.

2. *For every* $t \in [0,T]$, *the error* $(\boldsymbol{e}_t)$ *is independent of the sigma field* $\mathscr{G}_t$ *and its covariance* $\boldsymbol{K}$ *satisfies the differential equation*

$$\begin{cases} \dot{\boldsymbol{K}}(t) &= \boldsymbol{A}(t)\boldsymbol{K}(t) + \boldsymbol{K}(t)\boldsymbol{A}(t)^* - \boldsymbol{K}(t)\boldsymbol{H}(t)^*\boldsymbol{a_1}(t)^{-1}\boldsymbol{H}(t)\boldsymbol{K}(t), \quad t \in (0,T), \\ \boldsymbol{K}(0) &= \mathbb{E}(\boldsymbol{x_0} - \mathbb{E}\,\boldsymbol{x_0})(\boldsymbol{x_0} - \mathbb{E}\,\boldsymbol{x_0})^* \end{cases} \tag{49}$$

3. *There exists a* k*-dimensional* $(\mathscr{G}_t)$*-Wiener process* $(\hat{\boldsymbol{W}}_t)_{t\in[0,T]}$ *such that it holds that*

$$\mathrm{d}\boldsymbol{Y}_t = \boldsymbol{H}(t)\hat{\boldsymbol{X}}_t\,\mathrm{d}t + \boldsymbol{\sigma_1}(t)\,\mathrm{d}\hat{\boldsymbol{W}}_t, \quad t \in (0,T). \tag{50}$$

Remark 12. Let us make two remarks.

(i) Equation (48) is in the so-called *implementable* form. This means that if we observe the process $\boldsymbol{Y}_t$, then we can immediately plug it in this equation to obtain $\hat{\boldsymbol{X}}_t$ because the function $\boldsymbol{K}$ is deterministic and we can compute it from equation (49) in advance.

(ii) The significance of the last claim of Theorem 16 is that it shows that the filter $\hat{\boldsymbol{X}}$ satisfies a stochastic differential equation that is of the same type as the signal $\boldsymbol{X}$ (although with different coefficients). Indeed, by equations (48) and (50), we have that

$$\begin{aligned} \mathrm{d}\hat{\boldsymbol{X}}_t &= \boldsymbol{A}(t)\hat{\boldsymbol{X}}_t\,\mathrm{d}t - \boldsymbol{F}(t)\boldsymbol{H}(t)\hat{\boldsymbol{X}}_t\,\mathrm{d}t + \boldsymbol{F}(t)\,\mathrm{d}\boldsymbol{Y}_t \\ &= \boldsymbol{A}(t)\hat{\boldsymbol{X}}_t\,\mathrm{d}t - \boldsymbol{F}(t)\boldsymbol{H}(t)\hat{\boldsymbol{X}}_t\,\mathrm{d}t + \boldsymbol{F}(t)\boldsymbol{H}(t)\hat{\boldsymbol{X}}_t\,\mathrm{d}t + \boldsymbol{F}(t)\boldsymbol{H}(t)\boldsymbol{K}(t)\,\mathrm{d}\hat{\boldsymbol{W}}_t \end{aligned}$$

and therefore, it holds that

$$\mathrm{d}\hat{\boldsymbol{X}}_t = \boldsymbol{A}(t)\hat{\boldsymbol{X}}_t\,\mathrm{d}t + \boldsymbol{K}(t)\boldsymbol{H}(t)^*[\boldsymbol{a_1}(t)]^{-1}\boldsymbol{H}(t)\boldsymbol{K}(t)\,\mathrm{d}\hat{\boldsymbol{W}}_t, \quad t \in (0,T).$$

Compare this equation to equation (46).

Sketch of proof of Theorem 16. For simplicity, we will only consider the one-dimensional case, we will assume that $x_0 = 0$, and we will assume that the coefficients A, σ, H, and σ_1 are constant.

Step 1: In the first step, we shall only look for the optimal filter $\hat{X}$ in the class of processes that take the form

$$\gamma(t) = \int_0^t v(t,\tau)\,\mathrm{d}Y_\tau, \quad t \in [0,T], \tag{51}$$

where the function v belongs to the space $\mathscr{C}([0,T]^2)$. The reason is that if we denote

$$\psi(t,s) := \exp\left\{\int_s^t \left(A - \frac{H^2}{\sigma_1^2}K(r)\right) \mathrm{d}r\right\},$$

then the filter $\hat{X}$ can be written (by the variation-of-constants formula) as

$$\hat{X}_t = \int_0^t \psi(t,s)\frac{H}{\sigma_1^2}K(s)\,\mathrm{d}Y_s.$$

Therefore, we see that the optimal filter $\hat{X}$ takes the form (51). The question is, however, whether it holds that

$$v_{\min}(t,s) = \psi(t,s)\frac{H}{\sigma_1^2}K(s).$$

Denote by $\Lambda(t,\tau)$ the solution to the equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\Lambda(t,\tau) & = -\Lambda(t,\tau)A + v(t,\tau)H, \quad t \in [\tau,T], \\ \Lambda(t,t) & = 1. \end{cases} \tag{52}$$

Then we have, by using integration by parts, the equality

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\Lambda(t,\tau)X_\tau\,\mathrm{d}\tau = [\Lambda(t,\tau)X_\tau]_{\tau=0}^t - \int_0^t \Lambda(t,\tau)AX_\tau\,\mathrm{d}\tau - \int_0^t \Lambda(t,\tau)\sigma\,\mathrm{d}W_\tau$$

that, by using (52) becomes the equation

$$-\int_0^t \Lambda(t,\tau)AX_\tau\,\mathrm{d}\tau + \int_0^t v(t,\tau)HX_\tau\,\mathrm{d}\tau = X_t - \int_0^t \Lambda(t,\tau)AX_\tau\,\mathrm{d}\tau - \int_0^t \Lambda(t,\tau)\sigma\,\mathrm{d}W_\tau.$$

Therefore, we have that the process X_t satisfies

$$X_t = \int_0^t \Lambda(t,\tau)\sigma\,\mathrm{d}W_\tau + \int_0^t v(t,\tau)HX_\tau\,\mathrm{d}\tau$$

so that it holds that

$$X_t - \gamma(t) = \int_0^t A(t,\tau)\sigma\,\mathrm{d}W_\tau - \int_0^t v(t,\tau)\sigma_1\,\mathrm{d}V_t.$$

Since the stochastic integrals on the right-hand side of the last equality are independent Gaussian random variables, we obtain for the second moment of the difference that

$$\mathbb{E}\,|X_t - \gamma(t)|^2 = \int_0^t \Lambda(t,\tau)^2\sigma^2\,\mathrm{d}\tau + \int_0^t v(t,\tau)^2\sigma_1^2\,\mathrm{d}\tau.$$

In this manner, we have reformulated the filtering problem as a deterministic linear-quadratic optimal control problem. To see this, denote $x(r) := \Lambda(t,t-r)$ and $u(r) := v(t,t-r)$. Then, using equation (52), we have that x satisfies the equation

$$\begin{cases} \dot{x}(r) & = Ax(r) - Hu(r) \\ x(0) & = 1 \end{cases}$$

and, we wish to find a control u such that the integral

$$\mathbb{E}\,|X_t - \gamma(t)|^2 = \int_0^t \left[\sigma^2 x^2(r) + \sigma_1^2 u^2(r)\right]\,\mathrm{d}r.$$

is minimal. By Theorem 11, we have that the optimal control is

$$\bar{u}(r) := \frac{H}{\sigma^2} P(r),$$

where P satisfies the Riccati equation

$$\begin{cases} \dot{P}(r) &= \frac{H^2}{\sigma_1^2} P^2(r) - 2AP(r) - \sigma^2, \quad r \in [0,t], \\ P(t) &= 0. \end{cases}$$

Defining K by $K(s) := P(t-s)$ we obtain the Riccati-type equation (49). Then, we also obtain that

$$\bar{u}(r) = \frac{H}{\sigma_1^2} K(t-r)$$

that corresponds to the optimal $\bar{v}$ given by

$$\bar{v}(t,\tau) := \frac{H}{\sigma^2} K(\tau)\bar{\Lambda}(t,\tau)$$

where $\bar{\Lambda}$ is the solution to the equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\bar{\Lambda}(t,\tau) &= \left(-A + \frac{H^2}{\sigma_1^2} K(\tau)\right)\bar{\Lambda}(t,\tau), \quad \tau \in [t,T], \\ \bar{\Lambda}(t,t) &= 1 \end{cases}$$

and the solution to this equation is unique, it must hold that

$$\bar{v}(t,\tau) = \frac{H}{\sigma_1^2} K(\tau)\psi(t,\tau).$$

Step 2: The second step is to show that we do not loose anything by restricting ourselves to the class of processes that take the form (51) in *Step 1.* In order to prove this, we first need to show that the error $e_r := X_t - \gamma_{\min}(t)$ is independent of $\mathscr{G}_t$. Once this part is proved, we have that

$$\hat{X}_t = \mathbb{E}[X_t|\mathscr{G}_t] = \mathbb{E}[X_t - \gamma_{\min}(t)|\mathscr{G}_t] + \mathbb{E}[\gamma_{\min}(t)|\mathscr{G}_t] = \mathbb{E}[X_t - \gamma_{\min}(t)] + \gamma_{\min}(t) = \gamma_{\min}(t).$$

where we used equality (47), independence of the error e_t of the σ-field $\mathscr{G}_t$, and $\mathscr{G}_t$-measurability of $\gamma_{\min}(t)$ successively.

Step 3: Finally, it is shown that the process $\hat{W}$ defined by

$$\hat{W}_t := \frac{1}{\sigma_1}\left(Y_t - \int_0^t H\hat{X}_r\,\mathrm{d}r\right)$$

is a $(\mathscr{G}_t)$-Wiener process using Lévy's characterization theorem, see e.g. [4, Theorem 3.16]. ☐

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